INFINITE-DIMENSIONAL LIE ALGEBRAS WITH NULL JACOBSON RADICAL

By

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0. Introduction

For a Lie algebra $L$ the Jacobson radical of $L$ is defined to be the intersection of all maximal ideals of $L$ ($L$ has no maximal ideal of $L$). The properties of the Jacobson radicals of finite-dimensional Lie algebras have been investigated by Marshall [6] and he has shown the following

**Theorem 0.1.** If a finite-dimensional Lie algebra $L$ has a Levi decomposition $L = S + \sigma(L)$, then the Jacobson radical of $L$ equals to $[L, \sigma(L)]$, where $\sigma(L)$ is the solvable radical of $L$.

For the infinite-dimensional case, Kamiya [2] shows

**Theorem 0.2.** If a Lie algebra $L$ generated by finite-dimensional local subideals of $L$, then the Jacobson radical of $L$ equals to $[L, \sigma(L)]$, where $\sigma(L)$ is the maximal locally solvable ideal of $L$.

These results are not true for the general infinite-dimensional case, even for the locally finite Lie algebras. This will be seen in §4. Taking a look at the Lie algebras given in §4 it seems to be difficult to find the characterization of the Jacobson radicals of infinite-dimensional Lie algebras by the well-known radicals. In this paper, to investigate the Jacobson radical of Lie algebras, we study the Lie algebras whose Jacobson radical is zero.

In §2 we will prove a local property that if $H$ is an ascendant subalgebra of a Lie algebra $L$ then the Jacobson radical of $H$ is contained in that of $L$. This goes as well for a serial subalgebra $H$ of a locally finite Lie algebra $L$.

The main result of §3 is that a locally finite Lie algebra $L$ with null Jacobson radical is a direct sum of a semisimple ideal of $L$, whose Jacobson radical is zero, and the center of $L$.

In §4 we give the two examples of Lie algebras. These Lie algebras tell us that some results about the Jacobson radical of finite-dimensional Lie algebras are not true in the infinite-dimensional Lie algebras.
1. Preliminaries

Throughout this paper we always consider not necessarily finite dimensional Lie algebras over a field of characteristic zero unless otherwise specified. Notation and terminology are mainly based on Amayo and Stewart [1]. In particular "\(\triangleleft\)" , "si" , "asc" , "lsi" and "ser" denote the relations "ideal" , "subideal" , "ascendant subalgebra" , "local subideal" and "serial subalgebra" respectively. For example, a subalgebra \(H\) of a Lie algebra \(L\) is ascendant, this is denoted by \(H\ asc L\), if there is an ordinal \(\sigma\) and a series of subalgebras \(\{L_\alpha\}_{\alpha \leq \sigma}\) such that

\[
L_0 = H, \quad L_\sigma = L, \\
L_\alpha \triangleleft L_{\alpha + 1} \quad \text{for all } \alpha < \sigma, \\
L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha \quad \text{for all limit ordinals } \lambda \leq \sigma.
\]

If \(\sigma\) is finite then \(H\) is a subideal of \(L\).

Let \(L\) be a Lie algebra and \(H\) a subalgebra of \(L\). We denote the center of \(L\) by \(\zeta(L)\) and \(C_L(H) = \{x \in L | [H, x] = 0\}\). Triangular brackets \(\langle \rangle\) denote the subalgebra generated by their contents. We also denote by \(ad_L(x)(y) = [y, x]\) for any \(x, y \in L\). Let \(S\) be a non-empty subset of \(L\). The ideal closure \(S^L\) is defined by

\[
S^L = \sum_{n=0}^{\infty} [S, n L]
\]

where \([S, n L]\) is inductively defined as follow: \([S, 0 L] = S, [S, 1 L] = [S, L], [S, n+1 L] = [[[S, n L], L], L]\). This is the smallest ideal of \(L\) containing \(S\).

Let \(K\) be a Lie algebra and \(I\) a subalgebra of the derivation algebra \(\text{Der}(K)\) of \(K\). Consider the direct sum

\[
L = K + I
\]

of vectorspaces \(K\) and \(I\). Then \(L\) is a Lie algebra with the product: \([h + i, k + j] = [h, k] + i(k) - j(h) + [i, j]\) \((h, k \in K, i, j \in I)\). This Lie algebra \(L\) is called the split extension of \(K\) by \(I\).

Let \(L\) be a Lie algebra. If every finite subset of \(L\) is contained in a finite-dimensional subalgebra (resp. solvable subalgebra) of \(L\) then \(L\) is called locally finite (resp. locally solvable). For a locally finite Lie algebra \(L\), \(\sigma(L)\) is the maximal locally solvable ideal of \(L\) and \(L\) is said to be the semisimple if \(\sigma(L) = 0\). Then we have the following

**Lemma 1.1 ([1; Theorem 13.3.7]).** Let \(L\) be a locally finite Lie algebra. If \(H\ ser L\) then \(\sigma(H) \subseteq \sigma(L)\).

Since a locally finite Lie algebra satisfying the minimal condition for two step subideals is finite-dimensional and solvable ([1; Corollary 8.5.5]), the following lemma
is immediate.

**Lemma 1.2.** The locally solvable simple Lie algebra is 1-dimensional.

We always put \( L^\omega = \bigcap_{n=1}^\infty L^n \).

**Lemma 1.3** ([1; Lemmas 1.3.2,1.3.4,13.2.3]). (1) If \( H \) is a non-trivial ascendant subalgebra of \( L \) then \( H^\omega < L \).

(2) If \( H \) is a finite-dimensional ascendant subalgebra of \( L \) then \( H^\omega < L \).

For a Lie algebra \( L \) let \( A(L) \), \( B(L) \) be the subalgebra generated by all finite-dimensional ascendant subalgebras of \( L \), the subalgebra generated by all finite-dimensional local subideals of \( L \) respectively. We set

\[
F_{a}(L) = \{ H \text{ asc } L | H \text{ is finite-dimensional} \},
\]

\[
F_{b}(L) = \{ H \text{ lsi } L | H \text{ is finite-dimensional} \}.
\]

It is known that the class of the finite-dimensional Lie algebras is ascendantly coalescent and lsi-coalescent ([1; Theorems 3.2.5, Corollary 13.2.2]). In other words if \( H, K \in F_{a}(L) \) (resp. \( F_{b}(L) \)) then \( \langle H, K \rangle \in F_{a}(L) \) (resp. \( F_{b}(L) \)). Therefore we can write

\[
A(L) = \bigcup_{H \in F_{a}(L)} H, \quad B(L) = \bigcup_{H \in F_{b}(L)} H.
\]

For these radicals of \( L \), Kubo [4] shows

**Lemma 1.4.** If \( L \) is a locally finite Lie algebra then \( A(L) \) and \( B(L) \) are ideals of \( L \).

### 2. Local properties of \( L \) with \( J(L) = 0 \)

For a Lie algebra \( L \) we denote by \( J(L) \) the Jacobson radical of \( L \).

**Lemma 2.1.** (1) (Levič [5]) A simple Lie algebra can no non-trivial ascendant subalgebra.

(2) (Stewart [8]) A locally finite simple Lie algebra can no non-trivial serial subalgebra.

We state the following key lemma.

**Lemma 2.2.** (1) If \( H \) is an ascendant subalgebra of a Lie algebra \( L \) then \( J(H) \subseteq J(L) \).

(2) If \( L \) is locally finite and \( H \) is a serial subalgebra of \( L \) then \( J(H) \subseteq J(L) \).

**Proof.** Let \( H \) be an ascendant subalgebra (resp. a serial subalgebra in the case that \( L \) is locally finite) of \( L \) and \( M \) a maximal ideal of \( L \). For a homomorphism \( f \) of \( L \) it is obvious that if \( H \text{ asc } L \) then \( f(H) \text{ asc } f(L) \). If \( L \) is locally finite and \( H \text{ ser } L \) then \( f(L) \text{ ser } f(L) \) by [1; Proposition 13.2.4]. Hence \( (H + M)/M \text{ asc } L/M \) (resp. \( (H + M)/M \text{ ser } L/M) \). Therefore \( H \subseteq M \) or \( H + M = L \) by Lemma 2.1. If \( H + M = L \) then \( H \cap M \) is
We shall see the structure of finite-dimensional ascendant (serial) subalgebras of a Lie algebra with null Jacobson radical.

**Theorem 2.3.** If \( L \) is a Lie algebra with \( J(L) = 0 \) then every finite-dimensional ascendant subalgebra \( H \) of \( L \) has a unique Levi decomposition

\[
H = H^2 \oplus \zeta(H)
\]

where \( H^2 \) is a semisimple ideal of \( L \). Moreover \( L = H^2 \oplus C_L(H^2) \).

**Proof.** By Theorem 0.1 and Lemma 2.1 we immediately have

\[
[\sigma(H), H] = J(H) \subseteq J(L) = 0.
\]

Hence \( \zeta(H) = \sigma(H) \). This shows that \( H \) has a unique Levi factor \( H^2 \). By Lemma 1.3 we have \( H^2 = H^\omega < L \). Let \( x \in L \). Since \( H^2 \) is a finite-dimensional semisimple ideal of \( L \), \( \text{ad}_L x|_{H^2} \) is an inner derivation of \( H^2 \). Hence there exists \( z \in H^2 \) such that \( \text{ad}_L x|_{H^2} = \text{ad}_L z|_{H^2} \). Therefore \( x - z \in C_L(H^2) \) and \( L = H^2 \oplus C_L(H^2) \).

**Theorem 2.4.** If \( L \) is a Lie algebra with \( J(L) = 0 \) then every finite-dimensional local subideal \( H \) of \( L \) has a unique Levi decomposition

\[
H = H^\omega \oplus \sigma(H)
\]

where \( H^\omega \) is a semisimple ideal of \( L \). Moreover \( L = H^\omega \oplus C_L(H^\omega) \).

**Proof.** Since \( H \) is finite-dimensional and \( J(L) = 0 \), we can choose maximal ideals \( M_1, \ldots, M_r \) of \( L \) such that \( H \cap (M_1 \cap \cdots \cap M_i) = 0 \) and \( M_1 \cap \cdots \cap M_i \cap M_{i+1} \cap \cdots \cap M_r \neq M_i \) for \( i = 1, \ldots, r \). We write

\[
L/(\cap_{i=1}^r M_i) = Z \oplus (\oplus_{i=1}^r S_i)
\]

where \( Z = \zeta(L/(\cap_{i=1}^r M_i)) \) and the \( S_i \)'s are non-abelian simple. Considered \( H \) as a local subideal of \( Z \oplus (\oplus_{i=1}^r S_i) \), we have \( H^\omega \triangleleft \oplus_{i=1}^r S_i = (Z \oplus (\oplus_{i=1}^r S_i))^\omega \) by Lemma 1.3. Hence \( H^\omega \) is semisimple. Let \( S \) be a Levi factor of \( H \). Then \( S = S^\omega \subseteq H^\omega \) and so \( H^\omega = S \). The last assertion is similarly proved as in Theorem 2.3.

We shall describe the structures of the radicals \( A(L), B(L) \) of \( L \) given in §1 when \( J(L) = 0 \), as follow.

**Theorem 2.5.** If \( L \) is a Lie algebra with \( J(L) = 0 \) then

\[
A(L) = \bigoplus_{\lambda \in A} S_\lambda \oplus \zeta(A(L)), \quad B(L) = \bigoplus_{\lambda \in A} S_\lambda \oplus \sigma(B(L))
\]

where \( \{S_\lambda \mid \lambda \in A\} \) is the set of all finite-dimensional non-abelian simple ideals of \( L \).

**Proof.** We already saw in §1 that \( A(L) = \bigcup_{\lambda \in A} H_\lambda \) and \( B(L) = \bigcup_{\lambda \in A} H_\lambda \). We
write

\[ S_A = \bigcup_{t \in F_a} H^L, \quad Z_A = \bigcup_{t \in F_a} \sigma(H), \]
\[ S_B = \bigcup_{t \in F_b} H^L, \quad Z_B = \bigcup_{t \in F_b} \sigma(H). \]

Then by Theorems 2.3, 2.4, \( S_A \) and \( S_B \) are direct sums of finite-dimensional non-abelian simple ideals of \( L \). Hence \( S_A = S_B = \bigoplus_{\lambda \in A} S_\lambda \) where \( S_\lambda \) runs over all finite-dimensional non-abelian simple ideals of \( L \).

Let \( H, K \in F_a(L) \) (resp. \( F_b(L) \)). Since \( H \) is \( \langle H, K \rangle \in F_a(L) \) (resp. \( F_b(L) \)), \( \sigma(H) \subseteq \sigma(\langle H, K \rangle) \) by Lemma 1.1. Hence \( [\sigma(H), K] \subseteq [\sigma(\langle H, K \rangle), \langle H, K \rangle] \subseteq \sigma(\langle H, K \rangle) \subseteq Z_A \) (resp. \( Z_B \)). Therefore \( Z_A \) (resp. \( Z_B \)) is an ideal of \( A(L) \) (resp. \( B(L) \)).

Obviously \( A(L) = S_A \oplus Z_A \) and \( B(L) = S_B \oplus Z_B \). Take any element \( x \in Z_A \) for some \( H \in F_a(L) \). For any \( K \in F_a(L) \) we have \( [K, x] \subseteq [\langle H, K \rangle, \sigma(H)] \subseteq [\langle H, K \rangle, \sigma(\langle H, K \rangle)] = 0 \) by Theorem 2.3. This implies that \( Z_A \subseteq \zeta(A(L)) \). Therefore \( \zeta(A(L)) = Z_A + S_A \cap \zeta(A(L)) = Z_A \). By the definition of \( Z_B \) and the fact that \( Z_B \triangleleft B(L) \), we have \( Z_B \subseteq \sigma(B(L)) \). Therefore \( \sigma(B(L)) = Z_B + S_B \cap \sigma(B(L)) = Z_B \).

3. Locally finite Lie algebras with null Jacobson radical

Let \( L \) be a finite-dimensional Lie algebra. By Theorem 0.1 we can easily derive that \( J(L) = 0 \) if and only if \( L = S \oplus \zeta(L) \) where \( S \) is a semisimple ideal of \( L \) with \( J(S) = 0 \). In this section we will extend this results to locally finite Lie algebras in the following theorem 3.3. Of course for a finite-dimensional Lie algebras we can drop the condition that \( J(S) = 0 \). But we can not drop it for locally finite Lie algebras, because there is a locally finite semisimple Lie algebra \( L \) with \( J(L) \neq 0 \). Such a Lie algebra \( L \) will be given in the next section.

**Lemma 3.1.** Let \( L \) be a locally finite Lie algebra. If \( \sigma(L) \cap L^2 = 0 \) then \( L \) has a Levi decomposition \( S \oplus \zeta(L) \) where \( S \) is a semisimple ideal of \( L \).

**Proof.** Let \( S \) be a subspace of \( L \) such that \( L = S \oplus \sigma(L) \) (the direct sum of vector spaces) and \( L^2 \subseteq S \). Then \( S \) is a semisimple ideal of \( L \). Since \( [\sigma(L), L] \cap L^2 = 0 \), we have \( \sigma(L) = \zeta(L) \).

**Lemma 3.2.** If \( L = \bigoplus_{\lambda \in A} L_\lambda \) then \( J(L) = \bigoplus_{\lambda \in A} J(L_\lambda) \).

**Proof.** It follows from Lemma 2.2 that \( \bigoplus_{\lambda \in A} J(L_\lambda) \subseteq J(L) \). If \( M_\lambda \) is a maximal ideal of \( L_\lambda \) then \( M_\lambda \oplus (\bigoplus_{\tau \neq \lambda} L_\tau) \) is a maximal ideal of \( L \). Hence \( J(L) \subseteq \bigcap_{\lambda \in A} (J(L_\lambda) \oplus (\bigoplus_{\tau \neq \lambda} L_\tau)) = \bigoplus_{\lambda \in A} J(L_\lambda) \).

**Theorem 3.3.** Let \( L \) be a locally finite Lie algebra. Then \( J(L) = 0 \) if and only if \( L \) has a Levi decomposition \( L = S \oplus \zeta(L) \) where \( S \) is a semisimple ideal of \( L \) with \( J(S) = 0 \).

**Proof.** Let \( M \) be a maximal ideal of \( L \). Assume that \( \sigma(L) \nsubseteq M \). Since \( L = M \)
If $J(L) = 0$ then by Lemma 3.1 $L$ has a Levi decomposition $L = S \oplus \zeta(L)$ where $S$ is a semisimple ideal of $L$. By Lemma 2.2 $J(S) = 0$.

The other implication is obvious by Lemma 3.2.

**Corollary 3.5.** Let $L$ be a locally finite Lie algebra with $J(L) = 0$. Then

$$B(L) = (\bigoplus_{\lambda \in A} S_{\lambda}) \oplus \zeta(L)$$

where $\{S_{\lambda} | \lambda \in A\}$ is the set of all finite-dimensional non-abelian simple ideals of $L$. Hence $A(L) = B(L)$.

**Proof.** By Theorem 2.5 we write $B(L) = (\bigoplus_{\lambda \in A} S_{\lambda}) \oplus \sigma(B(L))$. Since $B(L) \triangleleft L$ by Lemma 1.4, $\sigma(B(L)) \subseteq \sigma(L)$. We have $\sigma(L) = \zeta(L)$ by Theorem 3.3. Hence

$$\sigma(B(L)) \subseteq \zeta(L) \subseteq B(L)$$

and so $\sigma(B(L)) = \zeta(L)$.

Since $S_{\lambda} \subseteq A(L)$ ($\lambda \in A$) and $\zeta(L) \subseteq A(L)$, we have $B(L) \subseteq A(L)$. Hence $B(L) = A(L)$. \)

**4. Examples**

**Example 1.** Let $V$ be a vector space of infinite dimension over a field of characteristic zero. Let $S$ the set of all linear transformations of $V$, regarded as a Lie algebra under the usual Lie multiplication $[s, t] = st - ts$ ($s, t \in S$). Let $F$ be the set of elements of $S$ of finite rank and $A$ the set of elements of $F$ of trace zero (in the sense of §4). It is shown in [7] that $A$ is infinite-dimensional simple. It is easy to see that $A = F^2$ and $F$ is locally finite. Moreover the only ideals of $F$ are $0$, $A$ and $F$. Hence $\sigma(F) = 0$ and $J(F) = 0$.

**Proposition 4.1.** There is a locally finite Lie algebra $L$ such that $L$ is semisimple and $J(L) \neq 0$.

**Example 2.** We slightly change the construction of the Lie algebra given in Kubo [3]. For any positive integer $i$, let $S_i$ be the 3-dimensional split simple Lie algebra over $k$ of characteristic zero with basis $\{x_i, y_i, h_i\}$ and multiplications $[x_i, y_i] = h_i$, $[x_i, h_i] = 2x_i$, $[y_i, h_i] = -2y_i$. Write $K = \bigoplus_{i=1}^{\infty} S_i$. Take derivations

$$x = \sum_{i=1}^{\infty} \text{ad } x_i, \quad h = \sum_{i=1}^{\infty} \text{ad } h_i$$

Additional details and explanations may be necessary for a complete understanding, but this provides a clear and structured representation of the natural text.
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of $K$. Then $[x, h] = \Sigma \text{ad } [x_i, h_i] = \Sigma 2\text{ad } x_i = 2x$. Hence $\langle x, h \rangle = kx + kh$ is a subalgebra of $\text{Der}(K)$. Consider the split extension

$L = K \oplus \langle x, h \rangle$

of $K$ by $\langle x, h \rangle$.

**Lemma 4.2.** Every nonzero ideal of $L$ contains $S_i$ for some $i$.

**Proof.** Let $H$ be a non-zero ideal of $L$. We assume that $H \not\subseteq K$. We can take a non-zero element $w$ of $H$ such that $w = \sum_{i=1}^{n} v_i + \alpha x + \beta h$ ($v_i \in S_i, \alpha, \beta \in k$) where $x \neq 0$ or $\beta \neq 0$. Since $[w, S_{n+1}] = [\alpha x_{n+1} + \beta h_{n+1}, S_{n+1}] \neq 0$, this is a non-zero subspace of $S_{n+1}$.

Hence

$S_{n+1} \subseteq [w, S_{n+1}] \subseteq H$.

**Theorem 4.3.** (1) $L$ is locally finite and semisimple.
(2) $L$ is not a direct sum of non-abelian simple ideals of $L$.
(3) $J(L) = 0$.

**Proof.** The semisimplicity of $L$ follows from Lemma 4.2. Since $L^2 \neq L$, we have the assertion (2).

(3): Since $S_n$ is finite-dimensional and simple, $L/C_L(S_n) \cong \text{Der}(S_n) \cong S_n$. Hence $C_L(S_n)$ is a maximal ideal of $L$. Put $H = \bigcap_{n=1}^{\infty} C_L(S_n)$. If $H \neq 0$ then $S_i \subseteq H$ for some $i$ by Lemma 4.2, a contradiction. Hence $H = 0$ and $J(L) \subseteq H = 0$.

We describe all ideals of $L$ as follows.

**Proposition 4.4.** An ideal $H$ of $L$ is $0$, $L$, $K$, $K + \langle x \rangle$, or of the forms

$H = (\bigoplus_{p \in P} S_p) + k(x - \sum_{q \in Q} x_q) + k\beta(h - \sum_{q \in Q} h_q)$

where $P$ is a non-empty subset of $N$, $Q$ is a non-empty finite subset of $N$ with $Q \cap P = \phi$ and $\beta \in k$.

**Proof.** Let $H$ be an ideal which is not one of the first four types listed above. Let $P = \{p \in N | [S_p, H] \neq 0\}$ and $Q = \{q \in N | [S_q, H] = 0\}$. By Lemma 4.2 $P \neq \phi$. If $P = N$ then $K \subseteq H$, contradiction. Hence $Q \neq \phi$. Obviously $H = \bigoplus_{p \in P} S_p + (\bigoplus_{q \in Q} S_q + \langle x, h \rangle) \cap H$.

Any non-zero element $w$ of $(\bigoplus_{q \in Q} S_q + \langle x, h \rangle) \cap H \setminus \bigoplus_{q \in Q} S_q$ can be of the form

$w = \sum_{t \in T} w_t + \alpha x + \beta h$

$(\alpha, \beta \in k, w_t \in S_t, T \subseteq Q)$ where $\alpha \neq 0$ or $\beta \neq 0$. We may assume that $w_t \neq 0$ for $t \in T$, and
choose \( w \) in such a way that \(|T|\) is as small as possible. Write \( w_i = a_i x_i + b_i y_i + c_i h_i \) \((a_i, b_i, c_i \in k)\). By \([x_i, w] = [h_i, w] = 0\) we have \( a_i = -\alpha, b_i = 0, c_i = -\beta \) and so

\[
w = \alpha(x - \sum_{i \in T} x_i) + \beta(h - \sum_{i \in T} h_i).
\]

Assume that \( Q \neq T \). Then \([x_q, w] = 2\beta x_q\) and \([h_q, w] = -2\alpha x_q\) for \( q \in Q \setminus T \). Hence \( S_q \subseteq H \), a contradiction. Therefore \( T = Q \) and \( Q \) is a finite set. Since \([w, x] = -2\beta(x - \sum_{i \in T} x_i)\in H\), \( H \) is of the form required. □

Let \( U \) be a finite-dimensional Lie algebra and \( V \) its ideal. If \( J(U) = 0 \) then \( J(U/V) = 0 \) by Theorem 0.1. But this is not true for our Lie algebra \( L \). Because \( J(L) = 0 \) but \( J(L/K) = \langle x \rangle + K \).

**Theorem 4.5.** There is a locally finite Lie algebra \( L \) and its ideal \( K \) such that \( J(L) = 0 \) but \( J(L/K) \neq 0 \).

**References**


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