Type II Blowup of Solutions to a Simplified Keller-Segel System in Two Dimensional Domains

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Abstract

We consider a parabolic-elliptic system which is introduced as a simplified system of so called Keller-Segel system. In particular, we consider the system in a bounded domain of two dimensional Euclidean space. In that situation, we can find solutions to the system blowing up in finite time. Then, these solutions become the sum of a $L^1$-function and delta functions at the blowup time.

Concerning the blowup speed, there exists a radial solution whose blowup speed is faster than the one of backward self-similar solutions. We refer to that blowup as Type II blowup. In this paper, we investigate whether finite time blowup solutions to the system exhibit Type II blowup or not.

Keywords: parabolic-elliptic system; Keller-Segel system; Nagai system; Blowup; Type II

1 Introduction

In the present paper, we consider the system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\nabla u - u \nabla v) \quad \text{in } \Omega \times (0, T), \\
0 &= \Delta v - v + u \quad \text{in } \Omega \times (0, T), \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} \quad \text{on } \partial \Omega \times (0, T), \\
\left. u(\cdot, 0) \right|_{\Omega} &= u_0 \quad \text{in } \Omega \times (0, T).
\end{align*}
\]

Here, $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$, $\nu = \nu(x)$ is the unit normal outer vector on $\partial \Omega$, and $u_0$ is smooth and non-negative in $\Omega$.

Keller and Segel [6] introduced a system to describe that cellular slime molds aggregate, owing to the motion of the cells moving toward higher

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concentration of a chemical substance produced by themselves. Nanjundiah [9] introduced the following system

\[
\begin{align*}
    u_t &= \nabla \cdot \left( \nabla u - u \nabla v \right) \quad \text{in } \Omega \times (0, T), \\
    v_t &= \Delta v - v + u \quad \text{in } \Omega \times (0, T), \\
    \frac{\partial u}{\partial v} &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
    u(\cdot, 0) &= u_0, \quad v(\cdot, 0) = v_0 \quad \text{in } \Omega \times (0, T)
\end{align*}
\]

(2)

as a simplified system of the one introduced by Keller and Segel [6], where \( u_0 \) and \( v_0 \) are smooth and non-negative in \( \overline{\Omega} \). Nagai [7] also simplified the system (2) as a system (1).

We refer (1) and (2) as Nagai system and Keller-Segel system, respectively. In those systems, \( v(x, t) \) represents the concentration of the chemical substance, and \( u(x, t) \) represents the density of cells. We consider the blowup of these solutions as the aggregation of cells.

The systems (1) and (2) have unique classical solutions \((u, v)\) with some \( T \in (0, \infty) \), and these solutions \((u, v)\) are non-negative in \( \overline{\Omega} \times (0, T) \). Let \( T_{\text{max}} \) be the maximal existence time of the classical solution. Integrating the first equation of (1) or (2) over \( \Omega \times (0, t) \) and noticing \( u \geq 0 \), we have

\[
\| u(\cdot, t) \|_1 = \| u_0 \|_1 \equiv \lambda \quad \text{for } t \in (0, T_{\text{max}}),
\]

(3)

where \( \| \cdot \|_p \) denotes the \( L^p(\Omega) \) norm for \( p \in [1, \infty] \).

We say that a solution \((u, v)\) to (1) or (2) blows up at \((x, t) = (q, T)\), if \((u, v)\) is a classical solution in \( \Omega \times (0, T) \) and if there exists a sequence \( \{(x_n, t_n)\} \subset \overline{\Omega} \times (0, T) \) satisfying

\[
\lim_{n \to \infty} (x_n, t_n) = (q, T) \quad \text{and} \quad \lim_{n \to \infty} u(x_n, t_n) = \infty.
\]

Then, we say that \( q \) and \( T \) are a blowup point and the blowup time, respectively. If \( T_{\text{max}} < \infty \), we can show that the solution \((u, v)\) blows up at \( t = T_{\text{max}} \). That is to say, \( T_{\text{max}} \) is the blowup time if \( T_{\text{max}} < \infty \).

The systems (1) and (2) have blowup solutions (see [4, 7, 8, 11]). In particular, Herrero and Velázquez [4] found radial blowup solutions \((u, v)\) to (2) with \( \Omega = \{ x \in \mathbb{R}^2 \mid |x| < R \} \) and \( R \in (0, \infty) \) satisfying

\[
u(\cdot, t) \sim 8\pi \delta_0 + f \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad \text{as } t \to T,
\]

(4)

where \( \delta_0 \) is a delta function whose support is the origin, \( f \) is a nonnegative function in \( L^1(\Omega) \cap C(\overline{\Omega} \setminus \{0\}) \), and \( T \) is a positive constant. Moreover, the blowup solutions found in [4] satisfies that

\[
  u(x, t) = \frac{8(1 + o(1))}{R(t)^2 \left( 1 + \left( |x| / R(t) \right)^2 \right)^2} + O \left( \frac{1}{|x|^2} e^{-\sqrt{2|\log(T-t)|} |x|} \chi_{|x| \geq R(t)} (|x|) \right)
\]

(5)

as \( t \to T \).
uniformly on the sets \(|x| \leq C\sqrt{T-t}\), where \(C > 0\),

\[
R(t) = O \left( (T-t)^{1/2} \log(T-t) \right)^{1/4} e^{-\sqrt{\log(T-t)/2}}
\]

and

\[
\chi_{\{r \geq R(t)\}}(r) = \begin{cases} 
1 & \text{if } r \geq R(t), \\
0 & \text{if } r < R(t).
\end{cases}
\]

The author and Suzuki [10] showed under the assumption \(T_{\text{max}} < \infty\) that the solution \((u, v)\) to (1) satisfies

\[
u(x, t) \to \sum_{q \in \mathcal{B}} m(q) \delta_q + f \quad \text{in } \mathcal{M}(\Omega) \quad \text{as } t \to T_{\text{max}},
\]

where \(\mathcal{B}\) is the set of blowup points, \(\delta_q\) is the delta function whose support is the point \(q\),

\[
m(q) \geq m_s(q) \equiv \begin{cases} 
8\pi & \text{if } q \in \Omega, \\
4\pi & \text{if } q \in \partial \Omega,
\end{cases}
\]

and \(f\) is a nonnegative function in \(L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{B})\). By (3), (6) and (7), the number of blowup points is finite. In this situation, Suzuki [14] showed

\[
m(q) = m_s(q) \quad \text{for each } q \in \mathcal{B}.
\]

By these, if \(\Omega\) is a disc with radius \(R \in (0, \infty)\) at the center of the origin, a solution \((u, v)\) to (1) is radial, and if \(T_{\text{max}} < \infty\), \((u, v)\) satisfies (4). Therefore, we expect that (5) holds also for blowup solutions to (1), although the result in [4] says only the existence of blowup solutions to (2) satisfying (4) and (5).

However, it is difficult for us to get the estimate such as (5) for all blowup solutions to (1).

In the present paper, we investigate whether the blowup is Type I blowup or Type II blowup. Here, for a solution \((u, v)\) to (1) or (2) blowing up at \((q, T)\), we say that the blowup solution \((u, v)\) is of Type I and of Type II at \((q, T) \in \overline{\Omega} \times (0, \infty)\), if \((u, v)\) is a classical solution in \(\Omega \times (0, T)\) and if \((u, v)\) satisfies that

\[
\limsup_{t \to T} (T-t)^{1/2} \max_{|x-q| \leq L\sqrt{T-t}} u(x, t) < \infty \quad \text{for any } L > 0
\]

and that

\[
\limsup_{t \to T} (T-t)^{1/2} \max_{|x-q| \leq L\sqrt{T-t}} u(x, t) = \infty \quad \text{for some } L > 0,
\]

respectively.

Then, we get the following.
Theorem 1 If a solution \((u,v)\) to (1) blows up at \(t = T_{\text{max}}\) with \(T_{\text{max}} \in (0, \infty)\), then for any \(q \in \mathcal{B}\) the blowup solution \((u,v)\) is of Type II at \((q, T_{\text{max}})\), if \(m(q) = m_*(q)\), where \(\mathcal{B}\) is the blowup set in (6), and \(m(q)\) and \(m_*(q)\) are constants in (6) and (7), respectively.

As mentioned above, the assumption “\(m(q) = m_*(q)\)” in Theorem 1 was shown in [14]. In radial case, we can give the proof of (8), which is easier than the one in [14]. Then, we describe the following theorem and the proof.

Theorem 2 Suppose that \(\Omega = \{x \in \mathbb{R}^2 \mid |x| < R_0\}\) with \(R_0 \in (0, \infty)\), a solution \((u,v)\) to (1) is radial with respect to \(x\) and that \(T_{\text{max}} < \infty\). Then, it holds that \(\mathcal{B} = \{0\}\), \(m(0) = 8\pi\) and that the blowup solution \((u,v)\) is of Type II at \((0, T_{\text{max}})\), where \(\mathcal{B}\) and \(m(0)\) are the blowup set and the constant in (6), respectively. Moreover, for any \(\varepsilon > 0\) it holds that

\[
\lim_{t \to T_{\text{max}}} \int_{\Omega \cap \{|x| < \sqrt{T_{\text{max}} - t}\}} u(x,t) dx = 8\pi.
\]

We conclude this introduction by describing the plan of this paper.

In Section 2, we will investigate some properties of solutions to (1) and Green’s function of the second equation of (1). In Section 3, we will define rescaled solutions and investigate some properties of these rescaled solutions. Section 4 will be devoted to showing Theorems 1 and 2.

2 Properties of solutions and Green’s function

In this section, we shall prove some properties of solutions to (1) and Green’s function of the second equation of (1).

The following lemma was shown in [10, Lemma 8] or [14, Lemma 5.2].

Lemma 2.1 Let \(\phi \in C^2(\overline{\Omega})\) satisfying \(\partial \phi/\partial n = 0\) on \(\partial \Omega\). For solutions \((u,v)\) to (1), it holds that

\[
\left| \frac{d}{dt} \int_{\Omega} u(x,t) \phi(x) dx \right| \leq C(\lambda^2 + \lambda) \|\phi\|_{C^2(\overline{\Omega})} \quad \text{for } t \in (0, T_{\text{max}}),
\]

where \(\lambda\) is the constant in (3).

Here and henceforth, \(C\) denotes a positive constant depending only on \(\Omega\). Then, each \(C\) may be different from the other \(C\)’s.
Let $\overline{\phi} \in C^\infty(\mathbb{R}^2)$ be radial and satisfy that $0 \leq \overline{\phi} \leq 1$ in $\mathbb{R}^2$,

$$x \cdot \nabla \overline{\phi}(x) \leq 0 \quad \text{for } x \in \mathbb{R}^2$$

and that

$$\overline{\phi}(x) = \begin{cases} 
1 & \text{if } |x| \leq 3/4, \\
0 & \text{if } |x| \geq 4/3.
\end{cases}$$

For any $x_0 \in \partial \Omega$, we have a constant $R_0 \in (0, 1]$, which is independent of $x_0 \in \partial \Omega$, and a conformal mapping

$$X(\cdot; x_0) : B(x_0; R_0) \to \mathbb{R}^2$$

satisfying $X(x_0; x_0) = 0$,

$$\frac{\partial X}{\partial x}(x; x_0) = \text{id} + O(|x - x_0|) \quad \text{in } B(x_0; R_0),$$

$$\frac{4}{5}|x - x_0| \leq |X(x; x_0)| \leq \frac{6}{5}|x - x_0| \quad \text{in } B(x_0; R_0),$$

$X(\Omega \cap B(x_0; R_0); x_0) \subset \mathbb{R}^2_+$ and $X(\partial \Omega \cap B(x_0; R_0); x_0) \subset \partial \mathbb{R}^2_+$, where

$$B(x_0; R_0) = \{ x \in \mathbb{R}^2 | |x - x_0| < R_0 \}, \quad \mathbb{R}^2_+ = \{ x = (x_1, x_2) \in \mathbb{R}^2 | x_2 > 0 \}$$

and $\text{id}$ is the $2 \times 2$ unite matrix.

Put $\phi_R(\cdot) = \overline{\phi}(\cdot / R)^4$ and

$$\psi_{R,x_0}(\cdot) = \begin{cases} 
\phi_R(\cdot - x_0) & \text{for } x_0 \in \Omega \quad \text{and } \quad R \in (0, R_1/2), \\
\phi_R(X(\cdot; x_0)) & \text{for } x_0 \in \partial \Omega \quad \text{and } \quad R \in (0, R_1/2),
\end{cases}$$

where

$$R_1 = R_1(x_0) = \begin{cases} 
\min\{1, \text{dist}(x_0, \partial \Omega)\} & \text{if } x_0 \in \Omega, \\
R_0 & \text{if } x_0 \in \partial \Omega
\end{cases}$$

and $\text{dist}(x_0, \partial \Omega) = \min\{|x_0 - x| \mid x \in \partial \Omega\}$. It holds that $\partial \psi_{R,x_0}/\partial \nu = 0$ on $\partial \Omega$ and that

$$\psi_{R,x_0}(B(x_0; 5R/8)) = 1 \quad \text{and } \quad \psi_{R,x_0}(\Omega \setminus B(x_0; 5R/3)) = 0.$$

Here, we regards $\psi_{R,x_0}$ as $0$ in $\mathbb{R}^2 \setminus B(x_0; 2R)$.

The author and Suzuki [10] showed the first half of the following lemma, and the second half is a fundamental property of Green’s function. Then, we omit the proof of the following lemma.

Let $\xi^* = (\xi_1, -\xi_2)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. 

Here, we regards $\psi_{R,x_0}$ as $0$ in $\mathbb{R}^2 \setminus B(x_0; 2R)$. 

The author and Suzuki [10] showed the first half of the following lemma, and the second half is a fundamental property of Green’s function. Then, we omit the proof of the following lemma.

Let $\xi^* = (\xi_1, -\xi_2)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. 
Lemma 2.2 Let $G(\cdot, \cdot)$ be the Green’s function of $-\Delta + 1$ in $\Omega$ with the homogeneous Neumann boundary condition. Put

$$G_S(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x; x_0) - X(x'; x_0)|} + \frac{1}{2\pi} \log \frac{1}{|X(x; x_0) - X(x'; x_0)|}$$

and

$$G_S(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|}$$

if $x_0 \in \partial \Omega$ and $x_0 \in \Omega$, respectively. Then, it holds that

$$K = G - G_S \in C^{1+\theta}(B(x_0; R) \cap \Omega \times B(x_0; R) \cap \Omega)$$

(11)

for some $\theta \in (0, 1)$, where $R \in (0, R_1/2)$ and $R_1$ is the constant in (10).

Moreover, it holds that

$$|\nabla_x G(x, x')| \leq C \left( \frac{1}{|x - x'|} + 1 \right)$$

(12)

for $(x, x') \in \overline{\Omega} \times \overline{\Omega} \setminus \{(x, x) \mid x \in \overline{\Omega}\}$.

Let

$$\hat{X}(x; x_0) = \begin{cases} X(x; x_0) & \text{if } x_0 \in \partial \Omega, \\ x - x_0 & \text{if } x_0 \in \Omega, \end{cases}$$

(13)

and

$$\pi^* = \begin{cases} \pi & \text{if } x_0 \in \partial \Omega, \\ 2\pi & \text{if } x_0 \in \Omega. \end{cases}$$

(14)

Lemma 2.3 For $x_0 \in \overline{\Omega}$ and $R \in (0, R_1/12)$, it holds that

$$|\rho^{(2)}(x, x') - \frac{1}{\pi^*} \psi_{R,x_0}(x)\psi_{3R,x_0}(x')|$$

$$\leq \frac{c}{R}(|\hat{X}(x; x_0)| + |\hat{X}(x'; x_0)|)\psi_{R,x_0}(x)^{1/2}\psi_{3R,x_0}(x') + \frac{\hat{c}}{R}|\hat{X}(x'; x_0)|\psi_{3R,x_0}(x')$$

and that

$$|\rho^{(2k)}(x, x')| \leq \hat{c}k^2(|\hat{X}(x; x_0)|^{2k-2} + |\hat{X}(x'; x_0)|^{2k-2})\psi_{R,x_0}(x)^{1/2}\psi_{3R,x_0}(x')$$

$$+ \hat{c}k^2|\hat{X}(x'; x_0)|^{2k-2}\psi_{3R,x_0}(x')$$

for $k = 2, 3, 4, \cdots$,

where

$$\rho^{(i)}(x, x') = \rho^{(i)}_{R,x_0}(x, x') = |\frac{1}{2}\psi_{3R,x_0}(x')\nabla_x (\hat{X}(x; x_0)\psi_{R,x_0}(x)) \cdot \nabla_x G(x, x') + \frac{1}{2}\psi_{3R,x_0}(x)\nabla_{x'} (\hat{X}(x'; x_0)\psi_{R,x_0}(x')) \cdot \nabla_x G(x', x').$$
Here and henceforth, $\tilde{C}$ denotes a positive constant depending only on $\Omega$ and $\tilde{\phi}$. Then, each $\tilde{C}$ may be different from the other $\tilde{C}$s.

**Proof:** The first inequality in the case where $x_0 \in \partial \Omega$ is shown in [11, Lemma 3.2]. The proofs of the first and second inequalities in the case where $x_0 \in \partial \Omega$ are more complicated than the proofs in the case where $x_0 \in \Omega$. Then, we devote to the proof of the second inequality in the case where $x_0 \in \partial \Omega$.

We omit $x_0$ from symbols $X(x; x_0), \phi_{R,x_0}$ and $\psi_{R,x_0}$, and assume $x_0 = 0$, without loss of generality.

Let $X(x) = \xi, X(x') = \xi'$,

$$G_{S,1}(x, x') = e_1(\xi, \xi') = \frac{1}{2\pi} \log \frac{1}{|\xi - \xi'|},$$

$$G_{S,2}(x, x') = e_2(\xi, \xi') = \frac{1}{2\pi} \log \frac{1}{|\xi - \xi'|},$$

and

$$c(\xi) = \det \left( \frac{\partial X}{\partial x}(X^{-1}(\xi)) \right).$$

Since the conformality of $Z = X_1 + iX_2$ implies that

$$i \left( \frac{\partial X}{\partial x} \right) \left( \frac{\partial X}{\partial x} \right) = \det \left( \frac{\partial X}{\partial x} \right) \cdot \text{id},$$

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we have
\[
\rho_{S,1}^{(2k)}(x, x') = \frac{1}{2} \psi_3 R(x') \nabla_x |X(x)|^{2k} \psi_R(x) \cdot \nabla_x G_{S,1}(x, x') \\
+ \frac{1}{2} \psi_3 R(x) \nabla_{x'} |X(x')|^{2k} \psi_R(x') \cdot \nabla_{x'} G_{S,1}(x, x')
\]
\[
= \frac{1}{2} c(\xi) \phi_{3R}(\xi') \nabla_\xi (|\xi|^{2k} \phi_R(\xi)) \cdot \nabla_\xi v_1(\xi, \xi') \\
+ \frac{1}{2} c(\xi') \phi_{3R}(\xi) \nabla_{\xi'} (|\xi'|^{2k} \phi_R(\xi')) \cdot \nabla_{\xi'} v_1(\xi, \xi')
\]
\[
= -\frac{(\xi - \xi')}{4\pi |\xi - \xi'|^2} \cdot \left\{ c(\xi) \phi_{3R}(\xi') (2k \xi |\xi|^{2k-2} \phi_R(\xi) + |\xi|^{2k} \nabla_\xi \phi_R(\xi)) \\
- c(\xi') \phi_{3R}(\xi) (2k \xi' |\xi'|^{2k-2} \phi_R(\xi') + |\xi'|^{2k} \nabla_{\xi'} \phi_R(\xi')) \right\}
\]
\[
= -\frac{k}{2\pi |\xi - \xi'|^2} (|\xi|^{2k-2} |\xi| - |\xi'|^{2k-2} \xi') \cdot (\xi - \xi') c(\xi) \phi_R(\xi) \phi_{3R}(\xi')
\]
\[
- \frac{(\xi - \xi')}{4\pi |\xi - \xi'|^2} \cdot \left( \nabla_\xi \phi_R(\xi) c(\xi) \phi_{3R}(\xi') \right) |\xi|^{2k} - |\xi'|^{2k}
\]
\[
- \frac{(\xi - \xi')}{4\pi |\xi - \xi'|^2} \cdot \left( \nabla_{\xi'} \phi_R(\xi') c(\xi) \phi_{3R}(\xi') \right) |\xi'|^{2k}
\]
\[
= -\rho_{S,1,I}^{(2k)} - \rho_{S,1,II}^{(2k)} - \rho_{S,1,III}^{(2k)} - \rho_{S,1,IV}^{(2k)} - \rho_{S,1,V}^{(2k)} - \rho_{S,1,V'}^{(2k)}.
\]  

(15)

It follows from (9) and the smoothness of $X$ that
\[
|\nabla_\xi v(\xi)| = O(1) \quad \text{and that} \quad c(\xi) = c(X(x)) = 1 + O(|x|).
\]  

(16)

Combining this with
\[
(|\xi|^{2k-2} - |\xi'|^{2k-2}) \cdot (\xi - \xi')
\]
\[
= \int_0^1 \frac{d}{d\theta} |\theta \xi + (1 - \theta) \xi'|^{2k-2} (\theta \xi + (1 - \theta) \xi') \cdot (\xi - \xi') d\theta
\]
\[
= \int_0^1 |\theta \xi + (1 - \theta) \xi'|^{2k-2} d\theta |\xi - \xi'|^2
\]
\[
+ (2k - 2) \int_0^1 |\theta \xi + (1 - \theta) \xi'|^{2k-3} \{(\theta \xi + (1 - \theta) \xi') \cdot (\xi - \xi')\}^2 d\theta
\]

implies that
\[
|\rho_{S,1,I}^{(2k)}| \leq C k^2 (|X(x)|^{2k-2} + |X(x')|^{2k-2} \psi_R(x) \psi_{3R}(x')).
\]  

(17)
By (16), we have
\[
|\rho_{s,1,II}^{(2k)}| \leq \frac{k|\xi'|^{2k-1}}{2\pi|\xi - \xi'|}\left\{ |c(\xi) - c(\xi')|\phi_{3R}(\xi')\phi_{R}(\xi) + c(\xi')\phi_{3R}(\xi')\phi_{R}(\xi) - c(\xi)\phi_{3R}(\xi)\phi_{R}(\xi) + c(\xi')\phi_{3R}(\xi')\phi_{R}(\xi) + c(\xi)\phi_{3R}(\xi)\phi_{R}(\xi)\right\} \\
\leq \frac{C_{k}}{R}\|\phi\|_{C^{1}(\mathbb{R}^{2})}\|\xi'|^{2k-1}\phi_{3R}(\xi') \\
+ \frac{C_{k}}{R}\|\phi\|_{C^{1}(\mathbb{R}^{2})}\|\xi'|^{2k-1}\phi_{R}(\xi) \\
\leq \tilde{C}_{k}|X(x')|^{2k-2}(\psi_{3R}(x')\psi_{R}(x) + \psi_{3R}(x') + \psi_{R}(x')). \tag{18}
\]

For \(\alpha = (\alpha_{1}, \alpha_{2})\), let \(D^\alpha = \partial_{x_{1}}^\alpha\partial_{x_{2}}^\alpha\) and let \(|\alpha| = \alpha_{1} + \alpha_{2}\), where \(\alpha_{1}\) and \(\alpha_{2}\) are nonnegative integers.

Noticing \(\|D^\alpha\phi_{R}\|_{\infty} = O(R^{-|\alpha|})\) and using the similar argument as that to establish (17), it holds that
\[
|\rho_{s,1,III}^{(2k)}| \leq \frac{1}{4\pi|\xi - \xi'|}\left|\nabla_{\xi}\phi_{R}(\xi)|c(\xi)\phi_{3R}(\xi')||\int_{0}^{1} \frac{d}{d\theta}|\theta\xi + (1 - \theta)\xi'|^{2k} d\theta\right| \\
\leq \frac{C_{k}}{R}\|\phi\|_{C^{1}(\mathbb{R}^{2})}\|\xi'|^{2k-1}\phi_{3R}(\xi') \\
\leq \tilde{C}_{k}|X(x)|^{2k-2} + |X(x')|^{2k-2}\psi_{R}(x)^{3/4}\psi_{3R}(x'). \tag{19}
\]

By (16) and \(\|D^\alpha\phi_{R}\|_{\infty} = O(R^{-|\alpha|})\), we have
\[
|\rho_{s,1,IV}^{(2k)}| \leq \frac{C}{R^{2}}\left(\|\phi\|_{C^{1}(\mathbb{R}^{2})} + \|\phi\|_{C^{2}(\mathbb{R}^{2})}\right)\|\xi'|^{2k}\phi_{3R}(\xi') \\
\leq \frac{\tilde{C}_{k}}{R^{2}}|X(x')|^{2k}\psi_{3R}(x') \leq \tilde{C}_{k}|X(x')|^{2k-2}\psi_{3R}(x'),
\]
\[
|\rho_{s,1,V}^{(2k)}| \leq \frac{C}{R^{2}}\|\phi\|_{C^{1}(\mathbb{R}^{2})}\|\xi'|^{2k}\phi_{R}(\xi')^{3/4}\phi_{3R}(\xi') + \frac{C}{R^{2}}\|\phi\|_{C^{2}(\mathbb{R}^{2})}\|\xi'|^{2k}\phi_{R}(\xi')^{3/4} \\
\leq \frac{\tilde{C}_{k}}{R^{2}}|X(x')|^{2k}\psi_{R}(x')^{3/4} \leq \tilde{C}_{k}|X(x')|^{2k-2}\psi_{R}(x')^{3/4}.
\]

Using those, (15), (17), (18), (19), \(\psi_{3R} \geq \psi_{R}^{3/4}\) and \(4R\psi_{3R}(x) \geq |X(x)|\psi_{3R}(x)\), we obtain
\[
|\rho_{s,1}^{(2k)}(x, x')| \leq \tilde{C}k^{2}(|X(x)|^{2k-2} + |X(x')|^{2k-2})\psi_{R}(x)^{3/4}\psi_{3R}(x') \\
+ \tilde{C}k^{2}|X(x')|^{2k-2}\psi_{3R}(x'). \tag{20}
\]
Concerning the term on $G_{S,2}(x, x')$, we make use of $\partial \psi_R / \partial v |_{\partial \Omega} = 0$ and $\partial \psi_{3R} / \partial v |_{\partial \Omega} = 0$. Then, we have

$$
\rho_{S,2}^{(2k)}(x, x') = \frac{1}{2} \psi_{3R}(x') \nabla_x \left( |X(x)|^{2k} \psi_R(x) \right) \cdot \nabla x G_{S,2}(x, x') + \frac{1}{2} \psi_{3R}(x) \nabla_x \left( |X(x')|^{2k} \psi_R(x') \right) \cdot \nabla x' G_{S,2}(x, x')
$$

$$
= \frac{1}{2} c(\xi) \phi_{3R} (\xi') \nabla_x \left( |x|^{2k} \phi_R(\xi) \right) \cdot \nabla_x e_2(\xi, \xi') + \frac{1}{2} c(\xi') \phi_{3R}(\xi) \nabla_x \left( |x'|^{2k} \phi_R(\xi') \right) \cdot \nabla_x e_2(\xi, \xi')
$$

$$
= - \frac{k(\xi_1 - \xi_2)}{2\pi|\xi - \xi'|^2} \left\{ c(\xi) \xi_1 |x|^{2k-2} \phi_R(\xi) \phi_{3R}(\xi') - c(\xi') \xi_1' |x'|^{2k-2} \phi_R(\xi') \phi_{3R}(\xi) \right\}
$$

$$
- \frac{k(\xi - \xi_1)}{4\pi|\xi - \xi'|^2} \left\{ c(\xi) \xi_2 |x|^{2k-2} \phi_R(\xi) \phi_{3R}(\xi') - c(\xi') \xi_2' |x'|^{2k-2} \phi_R(\xi') \phi_{3R}(\xi) \right\}
$$

$$
- \frac{k(\xi_2 + \xi_2')}{2\pi|\xi - \xi'|^2} \left\{ c(\xi) \xi_2 |x|^{2k-2} \phi_R(\xi) \phi_{3R}(\xi') + c(\xi') \xi_2' |x'|^{2k-2} \phi_R(\xi') \phi_{3R}(\xi) \right\}
$$

By the similar argument as that to establish (20), we obtain

$$
|\rho_{S,2, I}^{(2k)} + \rho_{S,2, II}^{(2k)}| \leq \tilde{C} k^2 |X(x)|^{2k-2} + |X(x')|^{2k-2} \psi_R(x')^{3/4} \psi_{3R}(x') + \tilde{C} k^2 |X(x')|^{2k-2} \psi_{3R}(x').
$$

(22)

Noticing $\xi_2 \geq 0$ and $\xi_2' \geq 0$, we have

$$
|\rho_{S,2, III}^{(2k)}| \leq \frac{k(\xi_2 + \xi_2')^2 |\xi|^{2k-2}}{2\pi|\xi - \xi'|^2} c(\xi) \phi_R(\xi) \phi_{3R}(\xi')
$$

$$
+ \frac{k(\xi_2 + \xi_2')^2 |\xi'|^{2k-2}}{2\pi|\xi - \xi'|^2} c(\xi') \phi_R(\xi') \phi_{3R}(\xi)
$$

$$
\leq C k |X(x)|^{2k-2} \psi_R(x) \psi_{3R}(x') + C k |X(x')|^{2k-2} \psi_R(x') \psi_{3R}(x).
$$

(23)

By using $\overline{\phi}_{\xi_2} = 0$ on $\partial \Omega^2_+ \setminus \xi_2 \geq 0$ and $\xi_2' \geq 0$, we have

$$
\phi_{B_2}(\xi) = \frac{4}{R} \phi_R(\xi_1, \xi_2)^{3/4} \left\{ \overline{\phi}_{\xi_2}(\xi_1, \xi_2 / R) - \overline{\phi}_{\xi_2}(\xi_1, 0 / R) \right\}
$$

$$
= \frac{4}{R} \phi_R(\xi_1, \xi_2)^{3/4} \overline{\phi}_{\xi_2}(\xi_1, \theta_{\xi_2} / R) \xi_2 \quad \text{for} \ \xi \in \mathbb{R}^2_+
$$
with some \( \theta \in (0, 1) \). Then, we have

\[
|\phi_{R\xi_2}(\xi)| \leq \frac{\tilde{C}}{R^2} \phi_R(\xi) \frac{3}{4} \xi_2, \quad |\phi_{R\xi_2}(\xi')| \leq \frac{\tilde{C}}{R^2} \phi_R(\xi') \frac{3}{4} \xi_2,
\]

and hence

\[
|\rho_{S,2,RV}^{(2k)}| \leq \frac{\tilde{C}}{R^2} |X(x)|^{2k} \psi_R(\xi')^{3/4} \psi_3R(x') + \frac{\tilde{C}}{R^2} |X(x')|^{2k} \psi_R(\xi')^{3/4} \psi_3R(x)
\]

\[
\leq \tilde{C} |X(x)|^{2k-2} \psi_R(\xi')^{3/4} \psi_3R(x') + \tilde{C} |X(x')|^{2k-2} \psi_R(\xi')^{3/4} \psi_3R(x).
\]

Combining this with (21), (22) and (23) implies that

\[
|\rho_{S,2}^{(2k)}(x, x')| \leq \tilde{C} k^2 \left( |X(x)|^{2k} + |X(x')|^{2k} \right) \psi_R(x)^{3/4} \psi_3R(x')
\]

\[+ \tilde{C} k^2 |X(x')|^{2k-2} \psi_3R(x'). \quad (24)
\]

Combining (11) with

\[
|\nabla x \left( |X(x)|^{2k} \psi_R(x) \right)| \leq C k |X(x)|^{2k-1} \psi_R(x) + \frac{\tilde{C}}{R} |X(x)|^{2k} \psi_R(x)^{3/4}
\]

implies that

\[
|\psi_3R(x') \nabla x \left( |X(x)|^{2k} \psi_R(x) \right) \cdot \nabla x K(x, x')| \leq \tilde{C} k |X(x)|^{2k-1} \psi_R(x)^{3/4} \psi_3R(x'). \quad (25)
\]

By using an argument similar to the above one, we have

\[
|\psi_3R(x) \nabla x' \left( |X(x')|^{2k} \psi_R(x') \right) \cdot \nabla x' K(x', x)| \leq \tilde{C} k |X(x')|^{2k-1} \psi_R(x')^{3/4} \psi_3R(x).
\]

Combining this with Lemma 2.2, (20), (24), (25), \( \psi_3R \geq \psi_{R}^{1/2} \geq \psi_{R}^{3/4} \) and \( 2R \psi_R(x) \geq |X(x)| \psi_R(x) \) implies this lemma. \( \square \)

### 3 Definition and properties of rescaled solutions

We shall introduce rescaled solutions of finite time blowup solutions to (1). Supposing that a solution \((u, v)\) to (1) blows up at \((q, T_{max}) \in \mathcal{B} \times (0, \infty), \)
we define the rescaled solution \((z, w)\) of that solution \((u, v)\) around \((q, T_{\max})\) as
\[
z(y, \tau) = (T_{\max} - t)u(x, t) \quad \text{and} \quad w(y, \tau) = v(x, t)
\]
with
\[
y = (x - q)/\sqrt{T_{\max} - t} \quad \text{and} \quad \tau = -\log(T_{\max} - t).
\]
Then, it holds that
\[
\begin{cases}
    z_t = \nabla \cdot \left( \nabla z - z \nabla w - \frac{y}{2} \right) & \text{in } \mathcal{O}_q, \\
    0 = \Delta w + z - e^{-\tau} w & \text{in } \mathcal{O}_q, \\
    \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_q, \\
    z(\tau, \tau_0) = z_0 & \text{in } \mathcal{O}_q(\tau_0),
\end{cases}
\]  
(26)

where \(\tau_0 = -\log T_{\max}\), \(\mathcal{O}_q = \bigcup_{\tau > \tau_0} (\mathcal{O}_q(\tau) \times \{\tau\})\), \(\Gamma_q = \bigcup_{\tau > \tau_0} (\Gamma_q(\tau) \times \{\tau\})\), and \(z_0(y) = T_{\max} u_0(q + \sqrt{T_{\max} y})\) with \(\mathcal{O}_q(\tau) = e^{\tau/2}(\Omega - q) = \{y = e^{\tau/2}(x - q) \mid x \in \Omega\}\) and \(\Gamma_q(\tau) = e^{\tau/2}(\partial \Omega - q) = \{y = e^{\tau/2}(x - q) \mid x \in \partial \Omega\}\). The rescaled solution \((z, w)\) is a classical solution to (26) in \(\mathcal{O}_q\).

Let us put
\[
\mathcal{D} = \begin{cases} 
    H(q) = \{y \in \mathbb{R}^2 \mid y \cdot \nu(q) < 0\} & \text{if } q \in \partial \Omega, \\
    \mathbb{R}^2 & \text{if } q \in \Omega. 
\end{cases}
\]  
(27)

In the case where \(q \in \partial \Omega\), we put
\[
Y(y, \tau; q) = e^{\tau/2}X(e^{-\tau/2}y + q; q).
\]

Put
\[
\hat{Y}(y, \tau; q) = e^{\tau/2}\hat{X}(e^{-\tau/2}y + q; q),
\]
\[
\Psi_{L,q}(y, \tau) = \psi_{L,\exp(-\tau/2)}q(e^{-\tau/2}y + q).
\]

The following lemma is [14, Theorem 14.2], and the lemma can be shown by using the similar argument as that to establish [13, Theorem 1].

**Lemma 3.1** Let \(T_{\max} < \infty\), \(q \in \mathcal{B}\) and \((z, w)\) be the rescaled solution of the blowup solution to (1) around \((q, T_{\max})\). For any \(\{\tau_n\}\) with \(\lim_{n \to \infty} \tau_n = \infty\), there exists a subsequence \(\{\tau_{n'}\}\) satisfying
\[
z(\cdot, \tau_{n'}) \to \sum_{Q \in \mathcal{S}} M_s(Q) \delta_Q + F \quad \text{in } \mathcal{M}([0, T_{\max})] \quad \text{as } n' \to \infty,
\]
where
\[
M_s(Q) = \begin{cases} 
    \frac{8\pi}{4\pi} & \text{if } Q \in \mathcal{D}, \\
    \frac{8\pi}{4\pi} & \text{if } Q \in \partial \mathcal{D}. 
\end{cases}
\]
$F \in L^1(\mathbb{R}^2)$ is a nonnegative function,

$$
\mathcal{S} = \left\{ Q \in \overline{\mathcal{D}} \mid \text{there exist } \{\tau_n\} \text{ and } \{y_n\} \text{ such that } \right.
\lim_{n \to \infty} y_n = Q \text{ and } \lim_{n' \to \infty} z(y_n, \tau_n) = \infty \}
$$

and $z(\cdot, \tau)$ is regarded as $0$ in $\mathbb{R}^2 \setminus \mathcal{O}_q(\tau)$.

In Lemma 3.1, the set $\mathcal{S}$ may be empty. Then, Lemma 3.1 does not say whether the blowup solution $(u, v)$ is of Type II or not.

In this section, we fix a point $q \in \mathcal{B}$ and treat only a rescaled solution $(z, w)$ around $(q, T_{\max})$. Henceforth, we omit $q$ from symbols $\mathcal{O}_q, \mathcal{O}_q(\tau), \Gamma_q, \Gamma_q(\tau), X(\cdot; q), \hat{X}(\cdot; q), Y(\cdot, \cdot; q), \hat{Y}(\cdot, \cdot; q), \psi_{R_q}$ and $\Psi_{L,q}$ for simplicity, and assume $q = 0$ without loss of generality. Furthermore, in the case where $0 \in \partial \Omega$, we assume $\nu(0) = (0, -1)$, without loss of generality.

Putting

$$
\mathcal{G}(y, y', \tau) = G(e^{-\tau/2}y, e^{-\tau/2}y'), \quad \mathcal{G}_S(y, y', \tau) = G_S(e^{-\tau/2}\hat{Y}(y, \tau), e^{-\tau/2}\hat{Y}(y', \tau))
$$

and

$$
\mathcal{K}(y, y', \tau) = K(e^{-\tau/2}y, e^{-\tau/2}y'),
$$

we have

$$
\mathcal{G}(y, y', \tau) = \mathcal{G}_S(y, y', \tau) + \mathcal{K}(y, y', \tau) + \frac{\tau}{2\pi^*}
$$

for $y, y' \in \overline{B(0, e^{r/2}R) \cap \mathcal{O}(\tau)}$, and

$$
\mathcal{K}(\cdot, \cdot, \tau) \in C^{1+\theta}(\overline{B(0, e^{r/2}R) \cap \mathcal{O}(\tau)} \times \overline{B(0, e^{r/2}R) \cap \mathcal{O}(\tau)})
$$

for $\tau \geq \tau_0$ and $R \in (0, R_1/2)$, where $G_S$ and $K$ are functions in Lemma 2.2, and $\pi^*$ is the constant in (14).

We obtain that $\mathcal{G}(\cdot, \cdot, \tau)$ is the Green’s function of $-\Delta + e^{-\tau}$ in $\mathcal{O}(\tau)$ with $\partial \cdot / \partial \nu = 0$ on $\partial \mathcal{O}(\tau)$ and that we can get any estimates of $\hat{Y}$ and $\mathcal{G}$ by using the estimates of $\hat{X}$ and $G$. Then, we can get the following properties.

In the case where $0 \in \partial \Omega$, by (9), $X(0) = 0, \nu(0) = (0, -1)$ and the smoothness of $X$, we observe

$$
\left| \frac{\partial Y}{\partial y}(y, \tau) - \text{id} \right| \leq Ce^{-\tau/|y|},
$$

$$
\frac{\partial Y}{\partial \tau}(y, \tau) = \frac{\partial}{\partial \tau} e^{\tau/2}X(e^{-\tau/2}y) = \frac{1}{2} e^{\tau/2} X(e^{-\tau/2}y) - \frac{1}{2} \frac{\partial X}{\partial x}(e^{-\tau/2}y)y
$$

$$
= \frac{1}{2} e^{\tau/2} \left( X(e^{-\tau/2}y) - X(0) \right) - \frac{1}{2} \frac{\partial X}{\partial x}(e^{-\tau/2}y)y
$$

$$
= \frac{1}{2} \left( \frac{\partial X}{\partial x}(\theta e^{-\tau/2}y) - \frac{\partial X}{\partial x}(e^{-\tau/2}y) \right) y = O(e^{-\tau/2})O(|y|^2)
$$
and
\[ Y(y, \tau) = Y(y, \tau) - Y(0, \tau) = \frac{\partial Y}{\partial y}(\theta y, \tau)y = y + O(e^{-\tau/2} |y|)y, \]

where \( \theta \in (0, 1) \).

In the case where \( 0 \in \Omega \), it holds that \( \dot{Y}(y, \tau) = y \), \( \partial \dot{Y}/\partial y = \text{id} \) and that \( \partial \dot{Y}/\partial \tau = 0 \).

**Lemma 3.2** For \( L \geq 1 \), it holds that
\[ \int_{\mathcal{O}(\tau) \cap B(0,L)} |w(y, \tau)| \, dy \leq C(L^2 \log L + L^2)\lambda(1 + |\tau|) \text{ in } [\tau_0, \infty). \]

**Proof:** \( w(y, \tau) \) is represented by
\[ w(y, \tau) = \int_{\mathcal{O}(\tau)} G(y, y', \tau) z(y', \tau) \, dy'. \]

Since \( G \) in (28) is the Green’s function of \( -\Delta + e^{-\tau} \) in \( \mathcal{O}(\tau) \) with \( \partial^* / \partial \nu = 0 \) on \( \partial \mathcal{O}(\tau) \), it follows from (28) that
\[ \int_{\mathcal{O}(\tau) \cap B(0,L)} |w(y, \tau)| \, dy \leq C \int_{\mathcal{O}(\tau) \cap B(0,L)} \int_{\mathcal{O}(\tau)} \left( 1 + |\tau| + \left| \log \frac{1}{|y - y'|} \right| \right) z(y', \tau) \, dy' \, dy \leq C(L^2 \log L + L^2)\lambda(1 + |\tau|) \text{ in } [\tau_0, \infty). \]

Thus, we have this lemma. \( \square \)

**Proposition 3.1** For any \( L \geq 1 \) and \( k = 1, 2, 3, \ldots \), there exists a positive constant \( \tilde{C}(k) \) such that
\[ \int_{\mathcal{O}(\tau)} (1 + |\dot{Y}(y, \tau)|^{2k}) z(y, \tau) \Psi_L(y, \tau) \, dy \leq \tilde{C}(k)(\lambda^{k+1} + 1) \]
for \( \tau \geq -\log(T_{\text{max}} - T(L; k)) \), where
\[ T(L; k) = \inf \left\{ t \in (0, T_{\text{max}}) \mid \left. \sqrt{T_{\text{max}} - t} \leq R_1, \quad \int_{\Omega \cap B(0, 3^k L \sqrt{T_{\text{max}} - t})} f(x) \, dx \leq \frac{1}{L^{2k}} \right\} \quad (29) \]
and \( f \) is the function in (6).
Here and henceforth, \( \tilde{C}(k) \) denotes a positive constant depending only on \( k, \Omega \) and \( \bar{\phi} \). Then, each \( \tilde{C}(k) \) may be different from the other \( \tilde{C}(k) \)'s.

**Proof:** Take \( R_i > 0 \) smaller satisfying \( (B(0; 2R_i) \setminus \{0\}) \cap B = \emptyset \), if necessary. Then, we take \( R \in (0, R_i/(4 \cdot 3^k)) \). By noticing

\[
\frac{1}{2} \frac{\partial}{\partial \nu} \left( |\hat{X}(\cdot)|^{2i} \psi_R \right) = 0 \quad \text{in } \partial \Omega \quad \text{for } i = 0, 1, 2, \cdots ,
\]

it follows from (6) and Lemma 2.1 that

\[
\left| \int_{\Omega} |\hat{X}(x)|^2 u(x, t) \psi_R(x) dx - \int_{\Omega} |\hat{X}(x)|^2 f(x) \psi_R(x) dx \right| 
\leq C(\lambda^2 + \lambda) \| |\hat{X}(\cdot)|^2 \psi_R \|_{C^2(\bar{\Omega})}(T_{\max} - t).
\]

Since \( \psi_R \) satisfies that \( \| D^{\alpha} \psi_R \|_{\infty} \leq \tilde{C} R^{-|\alpha|} \), it holds that \( \| |\hat{X}(\cdot)|^2 \psi_R \|_{C^2(\bar{\Omega})} \leq \tilde{C} \). Then, we get

\[
\int_{\Omega} |\hat{X}(x)|^2 u(x, t) \psi_R(x) dx \leq \tilde{C}(\lambda^2 + 1)(T_{\max} - t) + \int_{\Omega} |\hat{X}(x)|^2 f(x) \psi_R(x) dx.
\]

For \( i \in \{1, 2, 3, \cdots, k - 1\} \), we assume

\[
\int_{\Omega} |\hat{X}(x)|^{2i} u(x, t) \psi_R(x) dx \leq \tilde{C}(i)(\lambda^{i+1} + 1) \left\{ (T_{\max} - t)^i + \sum_{j=1}^{i} (T_{\max} - t)^{i-j} \int_{\Omega} |\hat{X}(x)|^{2j} f(x) \psi_{2^{i-j} R}(x) dx \right\}.
\]

(30)

It holds that

\[
- \int_{\Omega} \nabla \cdot (u(x, t) \nabla v(x, t)) (|\hat{X}(x)|^{2i+2} \psi_R(x)) dx
\]

\[= \int \int_{\Omega \times \Omega} \left[ \nabla G(x, x') \cdot \nabla (|\hat{X}(x)|^{2i+2} \psi_R(x)) \right] u(x, t) u(x', t) dx dx'
\]

(31)

and that

\[
\int \int_{\Omega \times \Omega} \left[ \nabla G(x, x') \cdot \nabla (|\hat{X}(x)|^{2i+2} \psi_R(x)) \right] u(x, t) u(x', t) \psi_{2^{i+1} R}(x') dx dx'
\]

\[= \frac{1}{2} \int \int_{\Omega \times \Omega} \left[ \nabla_x G(x, x') \cdot \nabla_x (|\hat{X}(x)|^{2i+2} \psi_R(x)) \right] \psi_{2^{i+1} R}(x') u(x, t) u(x', t) dx dx'
\]

\[+ \frac{1}{2} \int \int_{\Omega \times \Omega} \left[ \nabla_{x'} G(x', x) \cdot \nabla_{x'} (|\hat{X}(x')|^{2i+2} \psi_R(x')) \right] \psi_{2^{i+1} R}(x) u(x', t) u(x, t) dx' dx
\]

\[= \int \int_{\Omega \times \Omega} \rho^{(2i+2)}(x, x') u(x, t) u(x', t) dx dx',
\]

(32)
where \( \rho^{(2i+2)} \) is the function in Lemma 2.3.

Multiplying the first equation of (1) by \( |\dot{X}(x)|^{2i+2}\psi_R(x) \), integrating over \( \Omega \), and using (31) and (32), we have

\[
\frac{d}{dt} \int_{\Omega} |\dot{X}(x)|^{2i+2} u(x, t) \psi_R(x) \, dx = \int_{\Omega} |\dot{X}(x)|^{2i+2} u_t(x, t) \psi_R(x) \, dx
\]

\[
= \int_{\Omega} \nabla \cdot (\nabla u - u \nabla v) (|\dot{X}(x)|^{2i+2} \psi_R(x)) \, dx
\]

\[
= \int_{\Omega} \Delta (|\dot{X}(x)|^{2i+2} \psi_R(x)) u(x, t) \, dx + \int_{\Omega \times \Omega} \rho^{(2i+2)}(x, x') u(x, t) u(x', t) \, dx \, dx'
\]

\[
+ \int_{\Omega \times \Omega} \kappa^{(2i+2)}(x, x') u(x, t) u(x', t) \, dx \, dx' = I + II + III,
\]

where

\[
\kappa^{(2i+2)}(x, x') = (1 - \psi_{3R}(x')) \nabla_x G(x, x') \cdot \nabla_x \left( |\dot{X}(x)|^{2i+2} \psi_R(x) \right).
\]

It follows from

\[
\left| \Delta \left( |\dot{X}(x)|^{2i+2} \psi_R(x) \right) \right| \leq \tilde{C} \cdot (i + 1)^2 |\dot{X}(x)|^{2i} \psi_R(x)^{1/2}
\]

that

\[
|I| \leq \tilde{C} \cdot (i + 1)^2 \int_{\Omega} |\dot{X}(x)|^{2i} u(x, t) \psi_R(x)^{1/2} \, dx.
\]  

(34)

By Lemma 2.3, we have

\[
|II| \leq \tilde{C} \cdot (i + 1)^2 \int_{\Omega \times \Omega} \left\{ (|\dot{X}(x)|^{2i} + |\dot{X}(x')|^{2i}) \psi_R(x')^{1/2} \psi_{3R}(x') \right. \\
+ |\dot{X}(x')|^{2i} \psi_{3R}(x') \right\} u(x, t) u(x', t) \, dx \, dx'
\]

\[
\leq \tilde{C} \cdot (i + 1)^2 \lambda \int_{\Omega} |\dot{X}(x)|^{2i} \psi_{3R}(x) u(x, t) \, dx.
\]  

(35)

Since it follows from (12) in Lemma 2.2 that

\[
|\kappa^{(2i+2)}(x, x')| \leq \tilde{C} (1 - \psi_{3R}(x')) \left( \frac{1}{|x - x'|} + 1 \right)
\]

\[
\cdot \left\{ (2i + 2) |\dot{X}(x)|^{2i+1} \psi_R(x) + \frac{1}{R} |\dot{X}(x)|^{2i+2} \psi_R(x)^{3/4} \right\}
\]

\[
\leq \frac{\tilde{C}}{R} \cdot (i + 1) \left( |\dot{X}(x)|^{2i+1} + \frac{1}{R} |\dot{X}(x)|^{2i+2} \right) \psi_R(x)^{3/4}
\]

\[
\leq \tilde{C} \cdot (i + 1) |\dot{X}(x)|^{2i} \psi_R(x)^{3/4},
\]

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we have

$$
|III| \leq \tilde{C} \cdot (i + 1) \int_{\Omega \times \Omega} |\dot{X}(x)|^2 u(x, t) u(x', t') \psi_R(x)^{3/4} \, dx \, dx'
$$

$$
\leq \tilde{C} \lambda (i + 1) \int_{\Omega} |\dot{X}(x)|^2 u(x, t) \psi_R(x)^{3/4} \, dx.
$$

Combining this with (30), (33), (34) and (35) implies that

$$
\left| \int_{\Omega} |\dot{X}(x)|^{2i+2} u(x, t) \psi_R(x) \, dx - \int_{\Omega} |\dot{X}(x)|^{2i+2} f(x) \psi_R(x) \, dx \right|
$$

$$
\leq \tilde{C} \cdot (\lambda + 1) (i + 1)^2 \int_{\Omega} \int_{\Omega} |\dot{X}(x)|^{2i} u(x, \tilde{t}) \psi_R(x) \, dx \, d\tilde{t}
$$

$$
\leq \tilde{C} \cdot (\lambda + 1) (i + 1)^2 \int_{\Omega} \int_{\Omega} \tilde{C}(i) (\lambda^{i+1} + 1) \left\{ (T_{\text{max}} - \tilde{t})^i \right. 
\left. + \sum_{j=1}^{i} (T_{\text{max}} - \tilde{t})^{i-j} \int_{\Omega} |\dot{X}(x)|^{2j} f(x) \psi_{3^{j+1-2} R(x)} \, dx \right\} d\tilde{t}
$$

$$
\leq \tilde{C}(i + 1) \cdot (\lambda^{i+2} + 1) \left\{ (T_{\text{max}} - t)^{i+1} \right. 
\left. + \sum_{j=1}^{i+1} (T_{\text{max}} - t)^{i+1-j} \int_{\Omega} |\dot{X}(x)|^{2j} f(x) \psi_{3^{j+1-1} R(x)} \, dx \right\}.
$$

Then, we get

$$
\int_{\Omega} |\dot{X}(x)|^{2i+2} u(x, t) \psi_R(x) \, dx \leq \tilde{C}(i + 1) \cdot (\lambda^{i+2} + 1) \left\{ (T_{\text{max}} - t)^{i+1} 
\right. 
\left. + \sum_{j=1}^{i+1} (T_{\text{max}} - t)^{i+1-j} \int_{\Omega} |\dot{X}(x)|^{2j} f(x) \psi_{3^{j+1-1} R(x)} \, dx \right\}.
$$

Then, we get (30) with $i = k$.

For $L \geq 1$, putting $R = L\sqrt{T_{\text{max}} - \tilde{t}}, y = x/\sqrt{T_{\text{max}} - \tilde{t}}$ and $\tau = -\log(T_{\text{max}} - t)$, we have

$$
\int_{\Omega(\tau)} |\dot{Y}(y, \tau)|^{2k} z(y, \tau) \Psi_L(y, \tau) \, dy
$$

$$
\leq \tilde{C}(k)(\lambda^{k+1} + 1) \left\{ 1 + \sum_{j=1}^{k} \int_{\Omega} \left[ \frac{|\dot{X}(x)|^{2j}}{\sqrt{T_{\text{max}} - \tilde{t}}} \right] f(x) \psi_{3^{j} L \sqrt{T_{\text{max}} - \tilde{t}}}(x) \, dx \right\}
$$

$$
\leq \tilde{C}(k)(\lambda^{k+1} + 1) \left\{ 1 + \int_{\Omega} \left[ 1 + \left[ \frac{|\dot{X}(x)|^{2j}}{\sqrt{T_{\text{max}} - \tilde{t}}} \right] \right] f(x) \psi_{3^{j} L \sqrt{T_{\text{max}} - \tilde{t}}}(x) \, dx \right\}
$$

for $\tau \geq -\log(T_{\text{max}} - T(L; k))$. 

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Combining this with (29) and \( \| \cdot \phi \|_\infty \leq \tilde{C}(k) \) implies that
\[
\int_{\mathcal{O}(\tau)} |\tilde{Y}(y, \tau)|^{2k} z(y, \tau) \Psi_L(y, \tau) dy \\
\leq \tilde{C}(k)(\lambda^{k+1} + 1) \left\{ 1 \\
+ \int_{\Omega} \left( 1 + \frac{2^k}{3^k \lambda^k T_{\max} - t} \right)^{2k} f(x) \psi_{3^k L \sqrt{T_{\max} - t}}(x) dx \\
\leq \tilde{C}(k)(\lambda^{k+1} + 1) \left( 1 + L^{2k} \int_{\Omega \cap B(0, 2^k L \sqrt{T_{\max} - t})} f(x) dx \right) \\
\leq \tilde{C}(k)(\lambda^{k+1} + 1) \quad \text{for } L \geq 1 \text{ and } \tau \geq -\log(T_{\max} - T(L; k)).
\]

Combining this with
\[
\int_{\mathcal{O}(\tau)} z(y, \tau) \Psi_L(y, \tau) dy \leq \int_{\mathcal{O}(\tau)} z(y, \tau) dy = \int_{\Omega} u(x, t) dx = \lambda
\]
implies that
\[
\int_{\mathcal{O}(\tau)} (1 + |\tilde{Y}(y, \tau)|^{2k}) z(y, \tau) \Psi_L(y, \tau) dy \leq \tilde{C}(k)(\lambda^{k+1} + 1) \\
\quad \text{for } L \geq 1 \text{ and } \tau \geq -\log(T_{\max} - T(L; k)).
\]

Thus, we have this proposition. \( \square \)

**Lemma 3.3** For any \( \varepsilon > 0 \), there exists \( L(\varepsilon) \geq 1 \) such that
\[
\left| \int_{\mathcal{O}(\tau)} z(y, \tau) \Psi_L(y, \tau) dx - m(0) \right| \leq \varepsilon
\]  
(36)

for \( L \geq L(\varepsilon) \) and \( \tau \in [-\log(T_{\max} - T(L; 1)), \infty) \), where \( T(L; 1) \) is the constant in (29) with \( k = 1 \).

**Proof:** It follows from (6) and Lemma 2.1 that
\[
\left| \int_{\Omega} u(x, t) \psi_R(x) dx - m(0) - \int_{\Omega} f(x) \psi_R(x) dx \right| \\
\leq C_{37}(\lambda^2 + \lambda) \| \phi \|_{C^2(\mathbb{R}^2)} \frac{T_{\max} - t}{R^2}
\]  
(37)

for \( t \in (0, T_{\max}) \) and \( R \in (0, R_1 / 2) \) with \( B(0; 2R) \cap \mathcal{B} = \{0\} \), where \( C_{37} \) is a positive constant depending only on \( \Omega \).
For any \( \varepsilon > 0 \) we take \( L(\varepsilon) \geq 1 \) such that
\[
1 + C_{37}(\lambda^2 + \lambda \| \theta \|_{C^2(\Omega)}) \leq \frac{\varepsilon}{2} L(\varepsilon)^2,
\]
where \( C_{37} \) is the positive constant in (37).

Then, for \( L \geq L(\varepsilon) \), we have
\[
\int_{\Omega} f(x) \psi_L(\sqrt{T_{\text{max}} - T(L;1)}(x)) dx \leq \int_{\Omega \cap B(0;2L) \sqrt{T_{\text{max}} - T(L;1))}} f(x) dx \leq \frac{\varepsilon}{2}.
\]
Combining this with (37) and
\[
\int_{\mathcal{O}(\tau)} z(y, \tau) \Psi_L(y, \tau) dy = \int_{\Omega} u(x, t) \psi_L(\sqrt{T_{\text{max}} - T(x)}(x)) dx
\]
implies this lemma. \( \square \)

4 Proof of main results

In this section, we shall prove Theorems 1 and 2.

In this section, we omit \( q \) from symbols \( \mathcal{O}_q, \mathcal{O}_q(\tau), \Gamma_q, \Gamma_q(\tau), \hat{X}(\cdot; q), \hat{X}(\cdot, \cdot; q), \hat{Y}(\cdot, \cdot; q), \psi_{R,q} \) and \( \Psi_{L,q} \) for simplicity, and assume \( q = 0 \) without loss of generality. Furthermore, in the case where \( 0 \in \partial \Omega \), we assume \( \nu(0) = (0, -1) \), without loss of generality. Then, in the case where \( 0 \in \partial \Omega \), \( \mathcal{D} \) in (27) is \( \mathbb{R}^2 \).

Let \( \{\tau_n\} \subset [0, \infty) \) be a sequence satisfying \( \lim_{n \to \infty} \tau_n = \infty \). For \( n \geq 1 \), let
\[
z_n(y, \tau) = z(y, \tau + \tau_n), \quad w_n(y, \tau) = w(y, \tau + \tau_n), \quad \mathcal{O}_n(\tau) = \mathcal{O}(\tau + \tau_n), \quad Y_n(y, \tau) = Y(y, \tau + \tau_n), \quad \hat{Y}_n(y, \tau) = \hat{Y}(y, \tau + \tau_n)
\]
and \( \Psi_{n,L}(y, \tau) = \Psi_L(y, \tau + \tau_n) \).

Proof of Theorem 1: We assume that a blowup solution \( (u, v) \) is of Type I at \((0, T_{\text{max}})\). Then, we have \( \|z\|_{L^\infty(\mathcal{O})} < \infty \).

By this and the standard bootstrap argument, it holds that
\[
\|z\|_{C^{2+\theta,1+\theta/2}([0;T_{\text{max}}] \times [\tau_0 + 1, \infty))} < \infty \quad \text{for} \quad L \geq 1
\]
with some \( \theta \in (0, 1) \). Therefore, there exist a subsequence \( \{z_{n'}\} \) of \( \{z_n\} \) and a function \( z_\infty \in C^{2,1}(\mathcal{D} \times \mathbb{R}) \) satisfying
\[
z_{n'}(\mathcal{Y}^{-1}_{n'}, \cdot) \to z_\infty(\mathcal{Y}^{-1}_\infty, \cdot) \quad \text{in} \quad C^{2,1}(\mathcal{D} \cap B(0; L) \times [\tau, T]) \quad \text{as} \quad n' \to \infty
\]

for any $L \geq 1$ and $T > 0$.

Here, $\tilde{Y}_\infty(y) = \lim_{n \to \infty} \tilde{Y}_n(y, \tau)$. Since we assume $\nu(0) = (0, -1)$ in the
case where $0 \in \partial \Omega$, it holds that $\tilde{Y}_\infty(y) = y$. Then, in the case where $0 \in \partial \Omega$,
it holds that

\begin{align*}
\begin{cases}
\partial_{\tau} z_{\infty} = \nabla \cdot \left( \nabla z_{\infty} - z_{\infty} \mathcal{H}_{\infty} \right) + \frac{y}{2} z_{\infty} & \text{in } \mathbb{R}^2_+ \times \mathbb{R}, \\
\mathcal{H}_{\infty}(y, \tau) = \int_{\mathbb{R}^2_+} \nabla_y \tilde{G}_{\infty}(y, y') z_{\infty}(y', \tau) dy' & \text{for } (y, \tau) \in \mathbb{R}^2_+ \times \mathbb{R}, \\
\frac{\partial z_{\infty}}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}^2_+ \times \mathbb{R},
\end{cases}
\end{align*}

where

\[ \tilde{G}_{\infty}(y, y') = \frac{1}{2\pi} \log \frac{1}{|y - y'|} + \frac{1}{2\pi} \log \frac{1}{|y - \eta'|}. \]

We regard $z_{\infty}$ as the even extension with respect to $\partial \mathbb{R}^2_+$. Then, it holds that

\begin{align*}
\begin{cases}
\partial_{\tau} z_{\infty} = \nabla \cdot \left( \nabla z_{\infty} - z_{\infty} \mathcal{H}_{\infty} \right) + \frac{y}{2} z_{\infty} & \text{in } \mathbb{R}^2 \times \mathbb{R}, \\
\mathcal{H}_{\infty}(y, \tau) = -\int_{\mathbb{R}^2} \frac{(y - y')}{2\pi|y - y'|^2} z_{\infty}(y', \tau) dy' & \text{for } (y, \tau) \in \mathbb{R}^2 \times \mathbb{R}.
\end{cases}
\end{align*}

(38)

In the case where $0 \in \Omega$, we also see that (38) holds.

Put $\Phi_L(y) = \overline{\varphi}(y/L)^4$. By Lemma 3.3, we have

\[ \left| \int_{\mathbb{R}^2} z_{\infty}(y, \tau) \Phi_L(y) dy - \overline{m}(0) \right| < \varepsilon \quad \text{for } L \geq L(\varepsilon) \text{ and } \tau \in \mathbb{R}, \]

where

\[ \overline{m}(0) = \begin{cases} 2m(0) & \text{if } 0 \in \partial \Omega, \\
m(0) & \text{if } 0 \in \Omega. \end{cases} \]

It follows from this and (7) that

\[ \int_{\mathbb{R}^2} z_{\infty}(y, \tau) dy = \overline{m}(0) \geq 8\pi. \]

(39)

In Proposition 3.1 with $k = 2$, replacing $\tau$ by $\tau + \tau_{n'}$ and taking $n'$ infinite, we get

\[ \int_{\mathbb{R}^2} |y|^2 z_{\infty}(y, \tau) \Phi_L(y) dy \leq \tilde{C}(\lambda^3 + 1) \quad \text{for } L \geq 1 \text{ and } \tau \in \mathbb{R}. \]

Combining this with (39), $\overline{m}(0) \leq 2\lambda$ and $z \geq 0$ implies that

\[ 0 \leq \int_{\mathbb{R}^2} (1 + |y|^2) z_{\infty}(y, \tau) dy \leq \tilde{C}(\lambda^3 + 1). \]

(40)
Multiplying $|y|^2 \Phi_L(y)$ by the first equation of (38) and integrating over $\mathbb{R}^2$, we have

$$
\frac{d}{d\tau} \int_{\mathbb{R}^2} z_\infty(y, \tau) |y|^2 \Phi_L(y) dy = \int_{\mathbb{R}^2} z_\infty(y, \tau) \Delta(|y|^2 \Phi_L(y)) dy
$$

$$
- \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(y - y') \cdot \nabla_y(|y|^2 \Phi_L(y))}{2\pi |y - y'|^2} z_\infty(y, \tau) z_\infty(y', \tau) dy' dy
$$

$$
+ \int_{\mathbb{R}^2} \left( \frac{y \cdot \nabla_y(|y|^2 \Phi_L(y))}{2} \right) z_\infty(y, \tau) dy = IV_L - V_L + VI_L. \quad (41)
$$

By noticing $\|D^\alpha \Phi_L\|_\infty = O(L^{-|\alpha|})$, it follows from (40) that

$$
\left| IV_L - 4 \int_{\mathbb{R}^2} z_\infty(y, \tau) dy \right| \leq 4 \int_{\mathbb{R}^2} z_\infty(y, \tau) (1 - \Phi_L(y)) dy
$$

$$
+ \int_{\mathbb{R}^2} \left( 4 |y| |\nabla \Phi_L(y)| + |y|^2 |\Delta \Phi_L(y)| \right) z_\infty(y, \tau) dy
$$

$$
\leq \tilde{C} \int_{\mathbb{R}^2} \left( \frac{|y|}{L} + \frac{|y|^2}{L^2} \right) z_\infty(y, \tau) dy \to 0 \quad \text{as} \quad L \to \infty \quad (42)
$$

and that

$$
\left| VI_L - \int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) dy \right| \leq \int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) (1 - \Phi_L(y)) dy
$$

$$
+ \int_{\mathbb{R}^2} \frac{|y|^3}{2} |\nabla \Phi_L(y)| z_\infty(y, \tau) dy
$$

$$
\leq \tilde{C} \int_{\mathbb{R}^2} \frac{|y|^3}{L} z_\infty(y, \tau) dy \to 0 \quad \text{as} \quad L \to \infty. \quad (43)
$$

Since it holds that

$$
V_L = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\nabla_y(|y|^2 \Phi_L(y)) - \nabla_y(|y'|^2 \Phi_L(y')) \cdot (y - y')}{4\pi |y - y'|^2} z_\infty(y, \tau) z_\infty(y', \tau) dy dy' dy' dy
$$

$$
= \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} z_\infty(y, \tau) z_\infty(y', \tau) \Phi_L(y) dy dy'
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(\Phi_L(y) - \Phi_L(y')) y' \cdot (y - y')}{2\pi |y - y'|^2} z_\infty(y, \tau) z_\infty(y', \tau) dy dy'
$$

$$
+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|y|^2 \nabla_y \Phi_L(y) - |y'|^2 \nabla_y \Phi_L(y') \cdot (y - y')}{4\pi |y - y'|^2} z_\infty(y, \tau) z_\infty(y', \tau) dy dy',
$$

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we have
\[
\left| V_L - \frac{1}{2\pi} \left( \int_{\mathbb{R}^2} z_\infty(y, \tau) \, dy \right)^2 \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} z_\infty(y, \tau) z_\infty(y', \tau) (1 - \Phi_L(y)) \, dy dy' \\
+ \hat{C} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|y'|}{L} z_\infty(y, \tau) z_\infty(y', \tau) \, dy dy' \\
+ \hat{C} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \frac{|y| + |y'|}{L} + \frac{|y'|^2}{L^2} \right) z_\infty(y, \tau) z_\infty(y', \tau) \, dy dy' \to 0
\]
as $L \to \infty$,

by using (40) and $\| D^a \Phi_L \|_\infty = O(L^{-|a|})$.

Combining this with (39), (41), (42) and (43) implies that

\[
\frac{d}{d\tau} \int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) \, dy = 4\overline{m}(0) - \overline{m}(0)^2 \frac{2\pi}{\overline{m}(0)^2} + \int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) \, dy \quad \text{for } \tau \in \mathbb{R}.
\]

(44)

Since we assume $m(0) = m_*(0)$, then it holds that $4\overline{m}(0) - \overline{m}(0)^2/(2\pi) = 0$. Then, it follows from this and (44) that

\[
\int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) \, dy = e^\tau \int_{\mathbb{R}^2} |y|^2 z_\infty(y, 0) \, dy \quad \text{in } \mathbb{R}.
\]

By this and (40), we have

\[
\int_{\mathbb{R}^2} |y|^2 z_\infty(y, \tau) \, dy = 0 \quad \text{in } \mathbb{R}.
\]

Since $z_\infty$ is nonnegative and continuous in $\mathbb{R}^2 \times \mathbb{R}$, then it holds that $z = 0$ in $\mathbb{R}^2 \times \mathbb{R}$. It contradicts (39). Thus, the solution $(u, v)$ does not exhibit Type I blowup at $(0, T_{\text{max}})$. That is to say, every blowup solution $(u, v)$ is of Type II at $(0, T_{\text{max}})$, if $m(0) = m_*(0)$. Thus, we have Theorem 1. \hfill \Box

Next, we shall prove Theorem 2.

**Proof of Theorem 2:** Since the solution $(u, v)$ is radial, the blowup point appears only at the center of the domain $\Omega$, by (3) and (6). That is to say, it holds that $B = \{ \text{the center of the domain} \}$. Then, we put $\zeta = |y|$, $\overline{z}_n(\zeta, \tau) = z_n(y, \tau)$ and $\overline{w}_n(\zeta, \tau) = w_n(y, \tau)$. Since $\overline{Y}(y, \tau) = y$ and $\overline{\Psi}_{n,L}(y, \tau) = \overline{\sigma}(y/L)^4$, we put $\overline{\Phi}_L(\zeta) = \overline{\psi}_{n,L}(y, \tau)$.

For $L \geq 1$, put $T_L = \max(\tau_0, 2\log(2L/R_*))$. 

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Multiplying \(|y|^{2k}\Phi_L(|y|)\) by the first equation of (26) and integrating over \(O_n(\tau)\), we have

\[
\frac{d}{d\tau} \int_{O_n(\tau)} |y|^{2k} z_n(y, \tau) \Phi_L(|y|) dy = \int_{O_n(\tau)} z_n(y, \tau) \Delta(|y|^{2k}\Phi_L(|y|)) dy \\
+ \int_{O_n(\tau)} z_n(y, \tau) \nabla(|y|^{2k}\Phi_L(|y|)) \cdot \nabla w_n(y, \tau) dy \\
+ \int_{O_n(\tau)} \frac{y}{2} \nabla(|y|^{2k}\Phi_L(|y|)) dy = VII + VIII + IX
\]

for \(\tau \geq T - \tau_n\). 

(45)

It follows from \(\|d^r \Phi_L / d\zeta^r\|_{L^\infty([0,\infty))} = O(L^{-i})\) that

\[
\left| VII - (2k)^2 \pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k-1} \Phi_L(\zeta) d\zeta \right| \\
\leq 8\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) |\Phi_L(\zeta)| d\zeta \\
+ 2\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) |\Phi_L'(\zeta)| d\zeta \\
\leq \tilde{C} \int_{-\infty}^{\infty} \left( \zeta^{2k} + \frac{\zeta^{2k+2}}{L^2} \right) \varpi_n(\zeta, \tau) \Phi_L(\zeta) \zeta d\zeta
\]

and that

\[
\left| IX - 2\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k+2} \Phi_L(\zeta) d\zeta \right| \leq \tilde{C} \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k+2} |\Phi_L(\zeta)| d\zeta \\
\leq \frac{\tilde{C}}{L} \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k+2} \Phi_L(\zeta) \zeta^{2k+2} d\zeta.
\]

(46)

(47)

Since it holds that

\[
\zeta \frac{\partial}{\partial \zeta} \varpi_n(\zeta, \tau) = e^{-(\tau + \tau_n)} \int_{\zeta}^{\infty} \varpi_n(\tilde{\zeta}, \zeta) \tilde{\zeta} d\tilde{\zeta} - \int_{0}^{\zeta} \varpi_n(\zeta, \tau) \tilde{\zeta} d\tilde{\zeta},
\]

we have

\[
\text{VIII} = 2\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \left( 2k \zeta^{2k-1} \Phi_L(\zeta) + \zeta^{2k+1} \Phi_L'(\zeta) \right) \left( \frac{\partial}{\partial \zeta} \varpi_n(\zeta, \tau) \right) d\zeta
\]

\[
= -4\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k-1} \Phi_L(\zeta) \int_{\zeta}^{\infty} \varpi_n(\tilde{\zeta}, \zeta) \tilde{\zeta} d\tilde{\zeta} d\zeta \\
+ 4\pi e^{-(\tau + \tau_n)} \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k-1} \Phi_L(\zeta) \int_{0}^{\zeta} \varpi_n(\zeta, \tau) \tilde{\zeta} d\tilde{\zeta} d\zeta \\
+ 2\pi \int_{-\infty}^{\infty} \varpi_n(\zeta, \tau) \zeta^{2k+1} \Phi_L'(\zeta) \left( e^{-(\tau + \tau_n)} \int_{0}^{\zeta} \varpi_n(\tilde{\zeta}, \tau) \tilde{\zeta} d\tilde{\zeta} - \int_{0}^{\zeta} \varpi_n(\zeta, \tau) \tilde{\zeta} d\tilde{\zeta} \right) d\zeta.
\]

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Combining this with $\Phi_L = 0$ in $[2L, \infty)$ and Lemma 3.2 implies that

$$\left| V I I I + 4\pi k \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^2 \Phi_L(\zeta) \int_0^\zeta \bar{z}_n(\zeta', \tau) \zeta' d\zeta' d\zeta \right|$$

$$\leq 4k \pi e^{-\frac{1}{4}(r + \tau_n)} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k} \Phi_L(\zeta) d\zeta \cdot \int_0^{2L} \bar{w}_n(\zeta, \tau) \zeta d\zeta$$

$$+ \frac{\tilde{C}}{L} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k} \Phi_L(\zeta) e^{-1/4(\tau + \tau_n)} \int_0^{2L} \bar{w}_n(\zeta, \tau) \zeta d\zeta + \lambda$$

$$\leq \tilde{C} e^{-\frac{1}{4}(r + \tau_n)} (L^2 \log L + L^2) \lambda (1 + |\tau + \tau_n|) \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k} \Phi_L(\zeta) d\zeta$$

$$+ \frac{\tilde{C}}{L} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k} \Phi_L(\zeta) e^{-1/4(\tau + \tau_n)} d\zeta.$$ 

By this, (45), (46), (47) and Proposition 3.1, we obtain

$$\left| \frac{d}{d\tau} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k+1} \Phi_L(\zeta) d\zeta - (2k)^2 \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k-1} \Phi_L(\zeta) d\zeta \right.$$ 

$$+ \tilde{C} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k+1} \Phi_L(\zeta) \int_0^\zeta \bar{z}_n(\zeta', \tau) \zeta' d\zeta' d\zeta$$

$$- k \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k+1} \Phi_L(\zeta) d\zeta$$

$$\leq \frac{\tilde{C}}{L} k(\lambda + 1) \int_0^\infty \bar{z}_n(\zeta, \tau)(1 + \zeta^{2k+2}) \Phi_L(\zeta)^{1/2} d\zeta$$

$$+ \tilde{C} e^{-\frac{1}{4}(r + \tau_n)} (L^2 \log L + L^2)(\lambda + 1)(1 + |\tau + \tau_n|) \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k} \Phi_L(\zeta)^{3/4} d\zeta$$

$$\leq \tilde{C}(k + 1) \cdot (\lambda^{k+3} + 1) \left\{ \frac{1}{L} + (L^2 \log L + L^2) e^{-\frac{1}{4}(r + \tau_n)} (1 + |\tau + \tau_n|) \right\}$$

$$\text{for } \tau \geq T_L - \tau_n. \quad (48)$$

Combining this with Proposition 3.1 implies that

$$\left| \frac{d}{d\tau} \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k+1} \Phi_L(\zeta) d\zeta \right| \leq \tilde{C}(k + 1) (\lambda^{k+3} + 1) \left\{ 1 + \frac{1}{L} \right.$$ 

$$+(L^2 \log L + L^2) e^{-\frac{1}{4}(r + \tau_n)} (1 + |\tau + \tau_n|) \right\}. \quad (49)$$

For $L \geq 1$, $k \geq 1$ and $n \geq 1$, put

$$\Lambda^{(k)}(\tau, n, L) = \int_0^\infty \bar{z}_n(\zeta, \tau) \zeta^{2k+1} \Phi_L(\zeta) d\zeta.$$
For \( k \geq 1, n \geq 1, L_2 > L_1 \geq 1 \) and \( \tau \geq T_{L_2} - \tau_n \), it follows from \( \varepsilon \geq 0 \) that
\[
0 \leq \Lambda^{(k)}(\tau, n, L_1) \leq \Lambda^{(k)}(\tau, n, L_2) \leq \hat{C}(k)(\lambda^{k+1} + 1). \tag{50}
\]

Let \( Q \subset \mathbb{R} \) be a countable set satisfying \( \overline{Q} = \mathbb{R} \). There exists a subsequence \( \{\tau_{n'}\} \) of \( \{\tau_n\} \) and \( \{z_{\infty}(\tau, \tau')\}_{\tau \in Q \cup \{0\}} \subset \mathcal{M}(\mathbb{R}^2) \) satisfying
\[
z_{n'}(\tau, \tau') \rightarrow z_{\infty}(\tau, \tau) \quad \text{in} \quad \mathcal{M}(\mathbb{R}^2) \quad \text{as} \quad n' \rightarrow \infty \tag{51}
\]
for \( \tau \in Q \cup \{0\} \). By this and (49), we obtain the existence of limits
\[
\Lambda^{(k)}(\tau, L) = \lim_{n' \rightarrow \infty} \Lambda^{(k)}(\tau, n', L)
\]
for any \( k \geq 1, \tau \in \mathbb{R} \) and \( L \geq 1 \). For \( L \geq 1 \) and \( k \geq 1 \), we have that \( \Lambda^{(k)}(\tau, L) \) is Lipschitz continuous in \( \mathbb{R} \), by (49). Combining this with (48), (49) and (50) implies that
\[
\frac{d}{d\tau} \Lambda^{(k)}(\tau, L) \geq (2k)^2 \Lambda^{(k-1)}(\tau, L) - \frac{k\lambda}{\pi} \Lambda^{(k-1)}(\tau, L) + k\Lambda^{(k)}(\tau, L)
\]
\[
- \frac{\hat{C}(k)}{L}(\lambda^{k+3} + 1) \quad \text{for any} \quad L \geq 1 \quad \text{and a.e.} \quad \tau \in \mathbb{R}. \tag{52}
\]

\[
\left| \frac{d}{d\tau} \Lambda^{(k)}(\tau, L) \right| \leq \hat{C}(k)(\lambda^{k+3} + 1) \left\{ 1 + \frac{1}{L} \right\}
\]
for any \( L \geq 1 \) and a.e. \( \tau \in \mathbb{R} \) \tag{53}

and that
\[
0 \leq \Lambda^{(k)}(\tau, L_1) \leq \Lambda^{(k)}(\tau, L_2) \leq \hat{C}(k)(\lambda^{k+1} + 1)
\]
for any \( k \geq 1, L_2 \geq L_1 \geq 1 \) and \( \tau \in \mathbb{R}. \tag{54}
\]

By (54), we have limits
\[
\Lambda^{(k)}(\tau) = \lim_{L \rightarrow \infty} \Lambda^{(k)}(\tau, L) \quad \text{for any} \quad k \geq 1 \quad \text{and} \quad \tau \in \mathbb{R}.
\]

Combining this with (52), (53) and (54) implies that
\[
\frac{d}{d\tau} \Lambda^{(k)}(\tau) \geq (2k)^2 \Lambda^{(k-1)}(\tau) - \frac{k\lambda}{\pi} \Lambda^{(k-1)}(\tau) + k\Lambda^{(k)}(\tau)
\]
\[
\quad \text{for any} \quad k \geq 1 \quad \text{and a.e.} \quad \tau \in \mathbb{R}, \tag{55}
\]

\[
\left| \frac{d}{d\tau} \Lambda^{(k)}(\tau) \right| \leq \hat{C}(k)(\lambda^{k+3} + 1) \quad \text{for any} \quad k \geq 1 \quad \text{and a.e.} \quad \tau \in \mathbb{R}
\]

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and that

\[ 0 \leq \Lambda^{(k)}(\tau) \leq \tilde{C}(k)(\lambda^{k+1} + 1) \quad \text{for any } k \geq 1 \text{ and } \tau \in \mathbb{R}. \quad (56) \]

It holds that

\[ 0 \leq \Lambda^{(0)}(0, n, L_1) \leq \Lambda^{(0)}(0, n, L_2) \leq \lambda \]

for any \( n \geq 1 \) and \( L_2 \geq L_1 \geq 1 \) with \( \tau_n \geq T_{L_2}. \quad (57) \]

By (51), we have limits

\[ \Lambda^{(0)}(0, L) = \lim_{n' \to \infty} \Lambda^{(0)}(0, n', L) \quad \text{for any } L \geq 1. \]

and by (57) and Lemma 3.3

\[ \frac{1}{2\pi} (m(0) - \varepsilon) \leq \Lambda^{(0)}(0, L_1) \leq \Lambda^{(0)}(0, L_2) \leq \frac{1}{2\pi} (m(0) + \varepsilon) \]

for any \( \varepsilon > 0 \) and \( L_2 \geq L_1 \geq L(\varepsilon). \)

By this, we have a limit \( \Lambda^{(0)}(0) = \lim_{L \to \infty} \Lambda^{(0)}(0, L) \) and that

\[ \Lambda^{(0)}(0) = \frac{m(0)}{2\pi} \geq 4. \quad (58) \]

Take \( k > \lambda/(4\pi) \). It follows from (55) and \( \Lambda^{(k-1)} \geq 0 \) that

\[ \Lambda^{(k)}(\tau) \geq e^{k(\tau - T)} \Lambda^{(k)}(T) \quad \text{for } \tau \in [T, \infty) \text{ and } T \in \mathbb{R}. \]

By this and (56), we have that \( \Lambda^{(k)} = 0 \) in \( \mathbb{R} \). For any \( k > \lambda/(4\pi) \), it follows from this and (54) that

\[ 0 \leq \Lambda^{(k)}(0, L) = \lim_{n' \to \infty} \Lambda^{(k)}(0, n', L) \leq \Lambda^{(k)}(0) = 0. \]

Combining this with Lemma 3.1 and (58) implies that

\[ z(\cdot, \tau_n')\chi_{O(\tau_n')} \to 8\pi\delta_0 \quad \text{in } \mathcal{M}(\mathbb{R}^2) \quad \text{as } n' \to \infty. \]

Combining this with (58) implies that

\[ \frac{m(0)}{2\pi} = \Lambda^{(0)}(0) = \lim_{L \to \infty, n' \to \infty} \Lambda^{(0)}(0, n', L) = 4. \]

Then, \( m(0) = 8\pi \) and the blowup solution is of Type II at \( (0, T_{\max}) \).

Since \( \{\tau_n\} \subset [0, \infty) \) is an arbitrary sequence satisfying \( \lim_{n \to \infty} \tau_n = \infty \), we have

\[ z(\cdot, \tau)\chi_{O(\tau)} \to 8\pi\delta_0 \quad \text{in } \mathcal{M}(\mathbb{R}^2) \quad \text{as } \tau \to \infty. \]

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By this, for any $\varepsilon > 0$ and $j = \pm 1$ we have
\[ \int_{\Omega} u(x, t) \psi_{2^j \varepsilon \sqrt{t_{\text{max}} - t}}(x) dx = \int_{\mathcal{O}(\tau)} z(y, \tau) \Phi_{2^j \varepsilon}(y) dy \to 8\pi. \]

Combining this with
\[
\int_{\Omega} u(x, t) \psi_{(\varepsilon/2) \sqrt{t_{\text{max}} - t}}(x) dx < \int_{\Omega \cap B(0, \varepsilon \sqrt{t_{\text{max}} - t})} u(x, t) dx \\
< \int_{\Omega} u(x, t) \psi_{2^j \varepsilon \sqrt{t_{\text{max}} - t}}(x) dx
\]
implies that
\[
\int_{\Omega \cap B(0, \varepsilon \sqrt{t_{\text{max}} - t})} u(x, t) dx = \int_{\mathcal{O}(\tau) \cap B(0, \varepsilon)} z(y, \tau) dy \to 8\pi \quad \text{as } t \to T_{\text{max}}.
\]

Thus, we have Theorem 2. \qed

References


