Multi-colored rooted tree analysis of the weak order conditions of a stochastic Runge-Kutta family

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Abstract

The aim of the present paper is to give a tractable way of seeking weak order conditions of a stochastic Runge-Kutta family for stochastic differential equations with a multi-dimensional Wiener process. This is accomplished by an extension of the rooted tree analysis for ordinary Runge-Kutta methods. As a result, weak order conditions can be obtained directly from diagrams of multi-colored rooted trees. This new methodology will be of benefit when high weak order conditions need to be sought.

Key words: Stratonovich stochastic differential equations; Multi-dimensional Wiener Process; Numerical schemes

1 Introduction

The importance of numerical schemes for stochastic differential equations (SDEs) has increased significantly as SDEs have been used for mathematical modeling in many fields [19]. This is because SDEs are analytically unsolvable in many cases.

Corresponding to the meaning of approximation, there are two kinds of numerical schemes for SDEs, that is, strong schemes and weak schemes [10]. Strong schemes give an approximate solution in the mean square sense [2,4,6,15]. Weak schemes give an approximation to the moment of an exact solution [1,9,12,14,17,18,20–22]. In either type of scheme we have to seek order conditions and solve them in order to obtain high order schemes.

Generally speaking, it is hard to derive order conditions. Fortunately, however, the rooted tree analysis, invented by Butcher [5], has opened the way to derive order conditions of Runge-Kutta schemes for ordinary differential equations (ODEs) in a transparent manner [7,8], and it has been extended to be applicable to the order conditions of schemes for SDEs. In fact, Burrage and Burrage [2] have given the rooted tree analysis of strong schemes for SDEs with a scalar Wiener process and they [3] have also extended it for SDEs with a multi-dimensional Wiener process. Komori, Mitsui and Sugiura [13] have extended the rooted tree analysis for ODEs to that of weak schemes for SDEs with a scalar Wiener process.

The aim of the present paper is to further extend this to the analysis of weak order conditions of a stochastic Runge-Kutta family for SDEs with a multi-dimensional Wiener process.

Rößler [17] also has developed the rooted tree analysis of week order conditions for another Runge-Kutta family. The following are the main differences between his analysis and the present one.

- Rößler's analysis gives a unified methodology to derive weak order conditions of his Runge-Kutta family for Stratonovich SDEs and Itô SDEs.
- On the other hand, the present analysis has a highly diagram-based approach to obtain weak order conditions of our Runge-Kutta family for Stratonovich SDEs.

The organization of the present paper is as follows. In the next section we will introduce basic notations and definitions. In Section 3 we will first express the Stratonovich-Taylor expansion of the solution of an SDE by a function of multi-colored rooted trees, and second give a detailed expression of the function by introducing the notions of the elementary integral, differential and weight. In Section 4 we will first express, with a function of labeled multi-colored rooted trees, the Taylor expansion of an approximate solution given by a stochastic Runge-Kutta family, and second give a detailed expression of the function by the elementary differential, numerical integral and weight. In Section 5 we will express the order conditions of the stochastic Runge-Kutta family in the weak sense by only the expectations of elementary weights and numerical weights. In addition, we will give a tractable way of seeking the expectations with multi-colored rooted trees. In Section 6 we will give the summary and remarks. In the appendix, we will show the expectations of elementary weights and numerical weights for weak order 2, and a stochastic Runge-Kutta scheme in the multi-dimensional Wiener process case.

2 Preliminaries

We introduce some notations and concepts dealt with in the paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\boldsymbol{W}(t) = [W_1(t), \ldots, W_m(t)]$ an m-dimensional Wiener process defined on the probability space. We mainly consider the following d-dimensional stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{g}_0(\boldsymbol{y}(s))\, \mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s), \quad 0 \le t \le T_{end}, \tag{1}$$

where $\boldsymbol{g}_j$ $(j = 0, 1, \ldots, m)$ are d-vector valued functions and $\circ$ means the Stratonovich formulation. The equation can be expressed, in differential form, by the SDE

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{g}_0(\boldsymbol{y}(t))\, \mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t)) \circ \mathrm{d}W_j(t), \quad \boldsymbol{y}(0) = \boldsymbol{x}_0, \quad 0 \le t \le T_{end}.$$

We give equidistant grid points $\tau_n \stackrel{\text{def}}{=} nh$ $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and consider discrete approximations $\boldsymbol{y}_n$ to $\boldsymbol{y}(\tau_n)$. Let $C_P^L(\boldsymbol{R}^d, \boldsymbol{R})$ denote the totality of L times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to L have polynomial growth. Now we can give the following definition [3].

Definition 1 *Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_M$ be the discrete approximations given by a certain scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $C > 0$ (independent of h) and $\delta > 0$ exist such that*

$$|E[G(\boldsymbol{y}(\tau_M)] - E[G(\boldsymbol{y}_M)]| \le Ch^q, \quad h \in (0, \delta).$$

3 The Stratonovich-Taylor expansion for the stochastic differential equation solution

The goal of this section is to represent the Stratonovich-Taylor expansion of the solution $\boldsymbol{y}(t)$ of

$$\boldsymbol{y}(\tau_{n+1}) = \boldsymbol{y}_n + \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_0(\boldsymbol{y}(s))\, \mathrm{d}s + \sum_{j=1}^{m} \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s) \tag{2}$$

by functions on the set of rooted trees.

At the first step we define the integral operator J_j: for any integrable function H of $\boldsymbol{y}$ and $s > \tau_n$,

$$J_0(H)(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1))\,\mathrm{d}s_1, \qquad J_j(H)(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1)) \circ \mathrm{d}W_j(s_1)$$

($1 \leq j \leq m$). Suppose that any component of $\boldsymbol{g}_j$ belongs to $C_P^{\kappa}(\boldsymbol{R}^d, \boldsymbol{R})$ ($0 \leq j \leq m$), and set $\boldsymbol{z}(\boldsymbol{y}(s)) \stackrel{\text{def}}{=} \boldsymbol{y}(s) - \boldsymbol{y}_n$ as the increment of $\boldsymbol{y}$ from time τ_n to s. Then, the expansion of $\boldsymbol{g}_j(\boldsymbol{y}_n + \boldsymbol{z})$ about $\boldsymbol{y}_n$ by Taylor's theorem yields the formal series:

$$\begin{aligned}
&J_j(\boldsymbol{g}_j)(s) \\
= \;& J_j\Big(\boldsymbol{g}_j(\boldsymbol{y}_n + \boldsymbol{z})\Big)(s) \\
= \;& J_j\Bigg(\boldsymbol{g}_j(\boldsymbol{y}_n) + \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{z}] + \cdots + \frac{1}{(\kappa-1)!}\boldsymbol{g}_j^{(\kappa-1)}(\boldsymbol{y}_n)[\underbrace{\boldsymbol{z}, \ldots, \boldsymbol{z}}_{\kappa - 1 \text{ times}}] \\
& + \frac{1}{\kappa!}\boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\underbrace{\boldsymbol{z}, \ldots, \boldsymbol{z}}_{\kappa \text{ times}}]\Bigg)(s) \\
= \;& J_j\Big(\boldsymbol{g}_j(\boldsymbol{y}_n)\Big)(s) + J_j\left(\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{z}]\right)(s) + \cdots \\
& + \frac{1}{(\kappa-1)!} J_j\left(\boldsymbol{g}_j^{(\kappa-1)}(\boldsymbol{y}_n)[\boldsymbol{z}, \ldots, \boldsymbol{z}]\right)(s) \\
& + \frac{1}{\kappa!} J_j\left(\boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\boldsymbol{z}, \ldots, \boldsymbol{z}]\right)(s),
\end{aligned} \tag{3}$$

where $0 < \theta_j < 1$. Here, the vector $\boldsymbol{z}(\boldsymbol{y}(s))$ is represented by $\boldsymbol{z}$ for simplicity and $\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)$, $\boldsymbol{g}_j^{(\kappa-1)}(\boldsymbol{y}_n)$ and $\boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n)$ denote the 1st, $(\kappa-1)$-st and κ-th order derivatives of $\boldsymbol{g}_j$, respectively. Since the linear operator $\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)$ is represented in a matrix form, for example, $\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{z}]$ means the product of the matrix $\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)$ and the vector $\boldsymbol{z}(\boldsymbol{y}(s))$. Similarly, $\boldsymbol{g}_j^{(\kappa-1)}(\boldsymbol{y}_n)[\boldsymbol{z}, \ldots, \boldsymbol{z}]$ and $\boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n)[\boldsymbol{z}, \ldots, \boldsymbol{z}]$ are evaluated as a result of multilinear operators on arguments $\boldsymbol{z}, \ldots, \boldsymbol{z}$ ([5], p. 132).

By introducing the notations

$$\mathrm{P}_j^{(0)}(s) \stackrel{\text{def}}{=} J_j(\boldsymbol{g}_j(\boldsymbol{y}_n))(s),$$

$$\mathrm{P}_j^{(k)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k](s) \stackrel{\text{def}}{=} \frac{1}{k!} J_j\left(\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k]\right)(s)$$

for $k \geq 1$ and

$$\mathrm{R}_j^{(\kappa)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_\kappa](s) \stackrel{\text{def}}{=} \frac{1}{\kappa!} J_j \left(\boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_\kappa] \right)(s),$$

from Eqs. (2) and (3) we have

$$\begin{aligned} \boldsymbol{z}(\boldsymbol{y}(s)) = & \sum_{j=0}^{m} \mathrm{P}_j^{(0)}(s) + \sum_{j=0}^{m} \mathrm{P}_j^{(1)}[\boldsymbol{z}](s) + \cdots + \sum_{j=0}^{m} \mathrm{P}_j^{(\kappa-1)}[\boldsymbol{z}, \ldots, \boldsymbol{z}](s) \\ & + \sum_{j=0}^{m} \mathrm{R}_j^{(\kappa)}[\boldsymbol{z}, \ldots, \boldsymbol{z}](s). \end{aligned} \tag{4}$$

We can use this to obtain the expansion of $\boldsymbol{y}(\tau_{n+1}) - \boldsymbol{y}_n$. In relation to the second term in the right-hand side, for example, one application of (4) yields

$$\begin{aligned} \sum_{j=0}^{m} P_j^{(1)}[\boldsymbol{z}](\tau_{n+1}) = & \sum_{j=0}^{m} P_j^{(1)}[\sum_{l=0}^{m} \mathrm{P}_l^{(0)}](\tau_{n+1}) + \sum_{j=0}^{m} P_j^{(1)}\Big[\sum_{l=0}^{m} \mathrm{P}_l^{(1)}[\boldsymbol{z}]\Big](\tau_{n+1}) \\ & + \cdots + \sum_{j=0}^{m} P_j^{(1)}\Big[\sum_{l=0}^{m} \mathrm{P}_l^{(\kappa-1)}[\boldsymbol{z}, \ldots, \boldsymbol{z}]\Big](\tau_{n+1}) \\ & + \sum_{j=0}^{m} P_j^{(1)}\Big[\sum_{l=0}^{m} \mathrm{R}_l^{(\kappa)}[\boldsymbol{z}, \ldots, \boldsymbol{z}]\Big](\tau_{n+1}). \end{aligned}$$

In the right-hand side, only the first term has a multiple integral with a constant integrand and its multiplicity is 2. When (4) is applied once again, the second term yields another multiple integral with a constant integrand, whose multiplicity is 3. Like this, by repeatedly applying (4) we obtain the expansion of $\boldsymbol{y}(\tau_{n+1}) - \boldsymbol{y}_n$ expressed by the sum of the multiple integrals with constant integrands whose multiplicity is at most κ and those with nonconstant integrands whose multiplicity is greater than κ:

$$\begin{aligned} \boldsymbol{y}(\tau_{n+1}) - \boldsymbol{y}_n = & \sum_{j=0}^{m} \mathrm{P}_j^{(0)}(\tau_{n+1}) + \sum_{j=0}^{m} \mathrm{P}_j^{(1)}[\sum_{l=0}^{m} \mathrm{P}_l^{(0)}](\tau_{n+1}) + \cdots \\ & + \sum_{j=0}^{m} \mathrm{P}_j^{(\kappa-1)}[\sum_{l=0}^{m} \mathrm{P}_l^{(0)}, \ldots, \sum_{l=0}^{m} \mathrm{P}_l^{(0)}](\tau_{n+1}) + \cdots \\ & + \sum_{j=0}^{m} \mathrm{R}_j^{(\kappa)}[\boldsymbol{z}, \ldots, \boldsymbol{z}](\tau_{n+1}). \end{aligned} \tag{5}$$

In the right-hand side of (5), τ_{n+1} only stands for the upper bound of the integral interval. In the sequel we omit this symbol as far as it does not cause a confusion.

Let N be a finite set of consecutive natural numbers, $\#S$ the cardinal number of a set S, and $V(N)$ the set of all possible partitions of N. That is, if $p \in V(N)$ and $p = \{p_1, \dots, p_{\#p}\}$ hold, then $p_1, \dots, p_{\#p}$ are non-empty pairwise-disjoint subsets of N, the equation $N = \bigcup_{i=1}^{\#p} p_i$ holds, and the elements of p_i are consecutive. For example,

$$\begin{aligned} N &= \{1,2,3\}, \\ V(N) &= \{\{N\}, \{\{1,2\},\{3\}\}, \{\{1\},\{2,3\}\}, \{\{1\},\{2\},\{3\}\}\}. \end{aligned}$$

Let $\mathrm{Q}^{(n)}$ be the sum of all products of n terms of $\mathrm{P}^{(\cdot)}_{\cdot}$ appearing in the right-hand side of (5). Then the following Lemma readily follows from (5).

Lemma 2 $\mathrm{Q}^{(n)}$ *can be given recursively as follows:*

$$\mathrm{Q}^{(1)} = \sum_{j=0}^{m} \mathrm{P}_j^{(0)}, \qquad \mathrm{Q}^{(\#N+1)} = \sum_{p \in V(N)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)}[\mathrm{Q}^{(\#p_1)}, \dots, \mathrm{Q}^{(\#p_{\#p})}],$$

where $1 \le \#N \le \kappa - 1$.

For a combinatorial description of the above expansion, we introduce multi-colored rooted trees (MRTs).

Definition 3 (Multi-colored rooted tree (MRT)) *A multi-colored rooted tree with a root* ⓙ *(colored with a label* j *from* 0 *to* m*) is a tree recursively defined in the following manner:*

1) $\tau^{(j)}$ *is the primitive tree having only a vertex* ⓙ*.*
2) If $t_1, \dots, t_k$ *are multi-colored rooted trees, then* $[t_1, \dots, t_k]^{(j)}$ *is also a multi-colored rooted tree with the root* ⓙ*.*

The totality of MRTs is denoted by T*.*

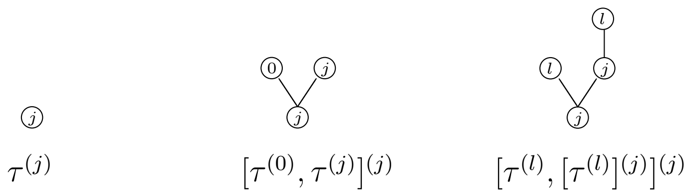

Fig. 1. Examples of MRTs

For an expression of the Stratonovich-Taylor expansion upon MRTs, we introduce the elementary integral.

Definition 4 (Elementary integral $\mathbf{\Psi}(t)$ on T) *An elementary integral* $\mathbf{\Psi}(t)$ *for* $t \in T$ *is a function recursively given in the following manner:*

$$\mathbf{\Psi}(\tau^{(j)}) \stackrel{\text{def}}{=} \mathrm{P}_j^{(0)}, \qquad \mathbf{\Psi}(t) \stackrel{\text{def}}{=} \mathrm{P}_j^{(k)}[\mathbf{\Psi}(t_1), \dots, \mathbf{\Psi}(t_k)] \quad \text{for } t = [t_1, \dots, t_k]^{(j)}.$$

For example,

$$\boldsymbol{\Psi}\Big([\tau^{(l)}, [\tau^{(l)}]^{(j)}]^{(j)}\Big) = P_j^{(2)}\Big[P_l^{(0)}, P_j^{(1)}[P_l^{(0)}]\Big].$$

For the set N, denote by T_N the totality of MRTs with $\#N$ vertices numbered with the elements of N in the following manner.

- Along each outwardly directed arc the numbers increase.
- Vertices of a subtree are consecutively numbered. This rule is also applied to subtrees of the subtree recursively.
- Isomorphic trees are regarded to be identical.

Fig. 2. Examples of trees in T_N

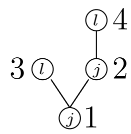

Fig. 3. An example of a tree not in T_N

For $u \in T_N$, let $|\ u\ |$ stand for an MRT from which the numbers are removed.

Lemma 5 *For the set N satisfying $\#N \leq \kappa$ the following equation holds:*

$$\mathrm{Q}^{(\#N)} = \sum_{u \in T_N} \boldsymbol{\Psi}(|\ u\ |).$$

PROOF. We will prove this by a mathematical induction. When $\#N = 1$, Lemma 2 and Definition 4 imply

$$\mathrm{Q}^{(1)} = \sum_{j=0}^{m} \mathrm{P}_j^{(0)} = \sum_{j=0}^{m} \boldsymbol{\Psi}(\tau^{(j)}).$$

When N_0 is a finite set of consecutive natural numbers satisfying $\min(N_0) \geq 2$, suppose that the equation

$$\mathrm{Q}^{(\#N)} = \sum_{u \in T_N} \boldsymbol{\Psi}(|u|)$$

holds for any set N $(\subseteq N_0)$ of consecutive natural numbers. Then, for $N = \{\min(N_0) - 1\} \cup N_0$, from Lemma 2 we can see

$$\begin{aligned}\mathrm{Q}^{(\#N)} &= \mathrm{Q}^{(\#N_0+1)} \\ &= \sum_{p \in V(N_0)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)}[\mathrm{Q}^{(\#p_1)}, \ldots, \mathrm{Q}^{(\#p_{\#p})}]\end{aligned}$$

$$= \sum_{p \in V(N_0)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)} [\sum_{u_1 \in T_{p_1}} \boldsymbol{\Psi}(|u_1|), \ldots, \sum_{u_{\#p} \in T_{p_{\#p}}} \boldsymbol{\Psi}(|u_{\#p}|)]$$
$$= \sum_{u \in T_N} \boldsymbol{\Psi}(|u|),$$

where $u \in T_N$ whose root has the number $\min(N_0) - 1$, and which consists of subtrees $u_1, \ldots, u_{\#p}$. This completes the proof. $\square$

Let $\nu(t)$ $(t \in T)$ be the number of different ways of numbering on t. That is, $\nu(t) = \#\{u \in T_N : |u| = t\}$. Furthermore denote by $\rho(t)$ the number of vertices of $t \in T$. From Lemma 5, we readily have the following lemma:

Lemma 6 *The identity*

$$\mathrm{Q}^{(\#N)} = \sum_{\substack{\rho(t)=\#N \\ t \in T}} \nu(t) \boldsymbol{\Psi}(t) \qquad for \quad \#N \leq \kappa$$

holds.

For any multiple stochastic integral x, let $\lambda(x)$ be the multiplicity of integrals with respect to a time variable or Wiener processes, and $\sigma(x)$ the multiplicity of integrals with respect to a time variable.

From Lemma 6, all terms x appearing in the expansion (5) with $\lambda(x) \leq \kappa$ can be expressed with $\boldsymbol{\Psi}(t)$. Actually, let $\boldsymbol{y}_\kappa(\tau_{n+1})$ denote the truncated expansion of $\boldsymbol{y}(\tau_{n+1})$ satisfying $\lambda(x) + \sigma(x) \leq \kappa$. Then from Lemma 6, we have one of the main results.

Theorem 7 *The finitely truncated expansion has the following expression:*

$$\boldsymbol{y}_\kappa(\tau_{n+1}) = \boldsymbol{y}_n + \sum_{i=1}^{\kappa} \sum_{\substack{\rho(t)+r(t)=i \\ t \in T}} \nu(t) \boldsymbol{\Psi}(t),$$

where $r(t)$ is the number of vertices of t with the color 0.

Moreover, $\boldsymbol{\Psi}(t)$ has a more concise representation given by the elementary weight, the elementary differential and the elementary coefficient:

Definition 8 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows [3]:*

$$\Phi(\tau^{(j)}) \stackrel{\text{def}}{=} J_j(1), \qquad \Phi(t) \stackrel{\text{def}}{=} J_j \left(\prod_{i=1}^{k} \Phi(t_i) \right) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

Definition 9 (Elementary differential $\boldsymbol{F}(t)$ on T) *An elementary differential is a possibly multilinear operator recursively given as follows [3]:*

$$\boldsymbol{F}(\tau^{(j)}) \stackrel{\text{def}}{=} \boldsymbol{g}_j(\boldsymbol{y}_n),$$
$$\boldsymbol{F}(t) \stackrel{\text{def}}{=} \boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

Definition 10 (Elementary coefficient $\beta(t)$ on T) *The index $\beta(t)$ ($t \in T$) is defined recursively [3]:*

$$\beta(\tau^{(j)}) \stackrel{\text{def}}{=} 1, \qquad \beta(t) \stackrel{\text{def}}{=} \frac{1}{k!}\prod_{i=1}^{k}\beta(t_i) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

If $t = [\tau^{(l)}, [\tau^{(l)}]^{(j)}]^{(j)}$, for example, $\beta(t) = 1/2$,

$$\Phi(t) = J_j\Big(J_l(1)J_j(J_l(1))\Big), \quad \boldsymbol{F}(t) = \boldsymbol{g}_j^{(2)}(\boldsymbol{y}_n)\Big[\boldsymbol{g}_l(\boldsymbol{y}_n), \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{g}_l(\boldsymbol{y}_n)]\Big].$$

The following is the main goal of this section.

Theorem 11 *For any $t \in T$ we have the identity:*

$$\boldsymbol{\Psi}(t) = \beta(t)\boldsymbol{F}(t)\Phi(t).$$

PROOF. We carry out the proof by a mathematical induction. When $\rho(t) = 1$, a simple interpretation gives

$$\boldsymbol{\Psi}(\tau^{(j)}) = \mathrm{P}_j^{(0)} = J_j\Big(\boldsymbol{g}_j(\boldsymbol{y}_n)\Big) = \boldsymbol{g}_j(\boldsymbol{y}_n)J_j(1) = \beta(\tau^{(j)})\boldsymbol{F}(\tau^{(j)})\Phi(\tau^{(j)}).$$

Suppose that the statement is valid for all trees with $\rho(t) \le k'$. If t has a root colored with j such as $t = [t_1, \ldots, t_k]^{(j)}$ ($\rho(t) = k' + 1$), then the definition of the elementary integral implies

$$\begin{aligned}
\boldsymbol{\Psi}(t) &= \mathrm{P}_j^{(k)}[\boldsymbol{\Psi}(t_1), \ldots, \boldsymbol{\Psi}(t_k)] \\
&= \frac{1}{k!}J_j\left(\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\beta(t_1)\boldsymbol{F}(t_1)\Phi(t_1), \ldots, \beta(t_k)\boldsymbol{F}(t_k)\Phi(t_k)]\right) \\
&= \frac{1}{k!}J_j\left(\prod_{i=1}^{k}\beta(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)]\prod_{i=1}^{k}\Phi(t_i)\right) \\
&= \frac{1}{k!}\prod_{i=1}^{k}\beta(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)] \cdot J_j\left(\prod_{i=1}^{k}\Phi(t_i)\right) \\
&= \beta(t)\boldsymbol{F}(t)\Phi(t).
\end{aligned}$$

Thus the statement holds in this case. $\square$

4 The Taylor expansion for a stochastic Runge-Kutta family

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ to the solution $\boldsymbol{y}(t_{n+1})$ of (2), we consider the stochastic Runge-Kutta family given by

$$\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \sum_{j_a,j_b=0}^{m} c_i^{(j_a,j_b)} \boldsymbol{Y}_i^{(j_a,j_b)}, \\
\boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \eta_{i_a}^{(j_a,j_b)} \left\{ b_{i_a}^{(j_a,j_b)} \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \right. \\
&\qquad \left. + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \gamma_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)} \right\}
\end{aligned} \tag{6}$$

$(1 \le i_a \le s,\ 0 \le j_a, j_b \le m)$, where each $\eta_{i_a}^{(j_a,j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\eta_{i_a}^{(j_a,j_b)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j_b = 0), \\ K_2 h^k & (j_b \ne 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. Here, $\boldsymbol{g}_{j_b}^{(1)}$ denotes the derivative of $\boldsymbol{g}_{j_b}$. If $b_{i_a}^{(j_a,j_b)} \ne 0$, by setting $\tilde{\eta}_{i_a}^{(j_a,j_b)} \stackrel{\text{def}}{=} \eta_{i_a}^{(j_a,j_b)} b_{i_a}^{(j_a,j_b)}$ and $\tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \stackrel{\text{def}}{=} \gamma_{i_a i_b}^{(j_a,j_b,j_c,j_d)} / b_{i_a}^{(j_a,j_b)}$ we can rewrite this in the simpler form:

$$\begin{aligned}
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \sum_{j_a,j_b=0}^{m} c_i^{(j_a,j_b)} \boldsymbol{Y}_i^{(j_a,j_b)}, \\
\boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \tilde{\eta}_{i_a}^{(j_a,j_b)} \left\{ \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \right. \\
&\qquad \left. + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)} \right\}.
\end{aligned} \tag{7}$$

These formulations include stochastic Runge-Kutta and Rosenbrock-Wanner methods [3,12].

In this section we deal with the simple formulation.

For a transparent analysis later on, we adopt nominal symbols $\tilde{\eta}_{s+1}^{(j_a,j_b)}$, $\alpha_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and $\tilde{\gamma}_{s+1,i_b}^{(j_a,j_b,j_c,j_d)}$ and define $\alpha_{s+1,i_b}^{(0,0,j_c,j_d)}$ and $\boldsymbol{Y}_{s+1}^{(j_a,j_b)}$ by

$$\alpha_{s+1,i_b}^{(0,0,j_c,j_d)} \stackrel{\text{def}}{=} c_{i_b}^{(j_c,j_d)} \qquad (i_b = 1, \ldots, s),$$

$$\boldsymbol{Y}_{s+1}^{(j_a,j_b)} \stackrel{\text{def}}{=} \tilde{\eta}_{s+1}^{(j_a,j_b)} \left\{ \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{s+1,i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \right.$$
$$\left. + \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n) \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \tilde{\gamma}_{s+1,i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)} \right\}.$$

The Taylor-series expansion of $\boldsymbol{Y}_{i_a}^{(j_a,j_b)}$ at $\boldsymbol{y}_n$ implies the formal series:

$$\begin{aligned}
\boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \tilde{\eta}_{i_a}^{(j_a,j_b)} \boldsymbol{g}_{j_b}(\boldsymbol{y}_n) \\
&+ \tilde{\eta}_{i_a}^{(j_a,j_b)} \boldsymbol{g}_{j_b}^{(1)}(\boldsymbol{y}_n)[\sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} (\alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} + \tilde{\gamma}_{i_a i_b}^{(j_a,j_b,j_c,j_d)}) \boldsymbol{Y}_{i_b}^{(j_c,j_d)}] \\
&+ \frac{\tilde{\eta}_{i_a}^{(j_a,j_b)}}{2} \boldsymbol{g}_{j_b}^{(2)}(\boldsymbol{y}_n)[\sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}, \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}] \\
&+ \cdots \\
&+ \frac{\tilde{\eta}_{i_a}^{(j_a,j_b)}}{\kappa!} \boldsymbol{g}_{j_b}^{(\kappa)}(\boldsymbol{y}_n + \theta_{j_b} \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \\
&\quad \times [\sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}, \ldots, \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}],
\end{aligned} \tag{8}$$

where $0 < \theta_{j_b} < 1$. For an expression concerning (8), we introduce some further symbols. For a rectangular matrix $\bar{\boldsymbol{z}}_i = [\boldsymbol{z}_{10}^i, \ldots, \boldsymbol{z}_{1m}^i, \ldots, \boldsymbol{z}_{s+1,0}^i, \ldots, \boldsymbol{z}_{s+1,m}^i]$ of $(m+1)(s+1)$ columns, where $\boldsymbol{z}_{\cdot,\cdot}^i$ stands for a d-dimensional column vector, the multilinear operator of k-th derivative of $\boldsymbol{g}_j$ induces

$$\begin{aligned}
\bar{\boldsymbol{g}}_j^{(0)} &\stackrel{\text{def}}{=} [\underbrace{\boldsymbol{g}_j(\boldsymbol{y}_n),\ \ldots,\ \boldsymbol{g}_j(\boldsymbol{y}_n)}_{(m+1)(s+1)\ \text{times}}], \\
\bar{\boldsymbol{g}}_j^{(k)}[\bar{\boldsymbol{z}}_1, \ldots, \bar{\boldsymbol{z}}_k] &\stackrel{\text{def}}{=} \left[\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{z}_{10}^1, \ldots, \boldsymbol{z}_{10}^k], \ldots, \boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{z}_{s+1,m}^1, \ldots, \boldsymbol{z}_{s+1,m}^k]\right]
\end{aligned}$$

for $k \geq 1$ and

$$\begin{aligned}
\bar{\boldsymbol{r}}_j^{(\kappa)}[\bar{\boldsymbol{z}}_1, \ldots, \bar{\boldsymbol{z}}_\kappa] \stackrel{\text{def}}{=} \Bigg[& \boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n + \theta_j \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{1 i_b}^{(0,j,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)})[\boldsymbol{z}_{10}^1, \ldots, \boldsymbol{z}_{10}^\kappa], \ldots, \\
& \boldsymbol{g}_j^{(\kappa)}(\boldsymbol{y}_n + \theta_j \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{s+1,i_b}^{(m,j,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)})[\boldsymbol{z}_{s+1,m}^1, \ldots, \boldsymbol{z}_{s+1,m}^\kappa] \Bigg]
\end{aligned}$$

as the remainder term.

Next, we introduce several matrices related to the formula parameters of the family (7). Let us define

$$A^{(j,j')} \stackrel{\text{def}}{=} \begin{bmatrix} \alpha_{11}^{(0,j,0,j')} & \cdots & \alpha_{11}^{(m,j,0,j')} & \cdots & \alpha_{s+1,1}^{(0,j,0,j')} & \cdots & \alpha_{s+1,1}^{(m,j,0,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{11}^{(0,j,m,j')} & \cdots & \alpha_{11}^{(m,j,m,j')} & \cdots & \alpha_{s+1,1}^{(0,j,m,j')} & \cdots & \alpha_{s+1,1}^{(m,j,m,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{1s}^{(0,j,0,j')} & \cdots & \alpha_{1s}^{(m,j,0,j')} & \cdots & \alpha_{s+1,s}^{(0,j,0,j')} & \cdots & \alpha_{s+1,s}^{(m,j,0,j')} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{1s}^{(0,j,m,j')} & \cdots & \alpha_{1s}^{(m,j,m,j')} & \cdots & \alpha_{s+1,s}^{(0,j,m,j')} & \cdots & \alpha_{s+1,s}^{(m,j,m,j')} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

for $\alpha_{i_a i_b}^{(j_a,j,j_c,j')}$, where $\mathbf{0}$ stands for an $m+1$-dimensional column vector of 0's. Similarly, define the matrix $\tilde{\Gamma}^{(j,j')}$ for $\tilde{\gamma}_{i_a i_b}^{(j_a,j,j_c,j')}$, and set $\tilde{A}^{(j,j')} \stackrel{\text{def}}{=} A^{(j,j')} + \tilde{\Gamma}^{(j,j')}$. In addition, define the $(m+1)(s+1) \times (m+1)(s+1)$ diagonal matrix $D^{(j)}$ by

$$D^{(j)} \stackrel{\text{def}}{=} \operatorname{diag}(\tilde{\eta}_1^{(0,j)}, \ldots, \tilde{\eta}_1^{(m,j)}, \ldots, \tilde{\eta}_{s+1}^{(0,j)}, \ldots, \tilde{\eta}_{s+1}^{(m,j)}).$$

Finally let us denote the rectangular matrix composed with $\boldsymbol{Y}_i^{(j_a,j_b)}$'s by

$$\bar{\boldsymbol{Y}}^{(j)} \stackrel{\text{def}}{=} [\boldsymbol{Y}_1^{(0,j)}, \ldots, \boldsymbol{Y}_1^{(m,j)}, \ldots, \boldsymbol{Y}_{s+1}^{(0,j)}, \ldots, \boldsymbol{Y}_{s+1}^{(m,j)}].$$

Applying these symbols, we obtain the following expression from (8):

$$\begin{aligned} \bar{\boldsymbol{Y}}^{(j)} = \quad & \bar{\boldsymbol{g}}_j^{(0)} D^{(j)} + \bar{\boldsymbol{g}}_j^{(1)} [\sum_{j_1=0}^{m} \bar{\boldsymbol{Y}}^{(j_1)} \tilde{A}^{(j,j_1)}] D^{(j)} \\ & + \frac{1}{2} \bar{\boldsymbol{g}}_j^{(2)} [\sum_{j_1=0}^{m} \bar{\boldsymbol{Y}}^{(j_1)} A^{(j,j_1)}, \sum_{j_1=0}^{m} \bar{\boldsymbol{Y}}^{(j_1)} A^{(j,j_1)}] D^{(j)} \\ & + \cdots + \frac{1}{\kappa!} \bar{\boldsymbol{r}}_j^{(\kappa)} [\sum_{j_1=0}^{m} \bar{\boldsymbol{Y}}^{(j_1)} A^{(j,j_1)}, \ldots, \sum_{j_1=0}^{m} \bar{\boldsymbol{Y}}^{(j_1)} A^{(j,j_1)}] D^{(j)}. \end{aligned} \tag{9}$$

Similarly to $\mathrm{P}_j^{(k)}$ and $\mathrm{R}_j^{(\kappa)}$ in the previous section, we adopt the following notations by using a label $X^{(j)} \in \{A^{(j)}, \tilde{A}^{(j)}\}$ as well as a matrix $X^{(j,j')} \in \{A^{(j,j')}, \tilde{A}^{(j,j')}\}$:

$$\mathrm{P}_{j',X^{(j)}}^{(0)} \stackrel{\text{def}}{=} \bar{\boldsymbol{g}}_{j'}^{(0)} D^{(j')} X^{(j,j')},$$

$$\mathrm{P}^{(k)}_{j',X^{(j)}}[\bar{\boldsymbol{z}}_1,\ldots,\bar{\boldsymbol{z}}_k] \stackrel{\mathrm{def}}{=} \frac{1}{k!}\bar{\boldsymbol{g}}^{(k)}_{j'}[\bar{\boldsymbol{z}}_1,\ldots,\bar{\boldsymbol{z}}_k]D^{(j')}X^{(j,j')}$$

for $k \geq 1$ and

$$\mathrm{R}^{(\kappa)}_{j',X^{(j)}}[\bar{\boldsymbol{z}}_1,\ldots,\bar{\boldsymbol{z}}_\kappa] \stackrel{\mathrm{def}}{=} \frac{1}{\kappa!}\bar{\boldsymbol{r}}^{(\kappa)}_{j'}[\bar{\boldsymbol{z}}_1,\ldots,\bar{\boldsymbol{z}}_\kappa]D^{(j')}X^{(j,j')}.$$

Equation (9) then yields

$$\begin{aligned}\sum_{j_1=0}^{m}\bar{\boldsymbol{Y}}^{(j_1)}X^{(j,j_1)} &= \sum_{j_1=0}^{m}\mathrm{P}^{(0)}_{j_1,X^{(j)}} + \sum_{j_1=0}^{m}\mathrm{P}^{(1)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}\tilde{A}^{(j_1,j_2)}] \\ &+ \sum_{j_1=0}^{m}\mathrm{P}^{(2)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)},\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)}] + \cdots \\ &+ \sum_{j_1=0}^{m}\mathrm{R}^{(\kappa)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)},\ldots,\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)}]. \end{aligned} \tag{10}$$

Repeated substitution of (10) into itself implies the formal expression similarly to (5):

$$\begin{aligned}\sum_{j_1=0}^{m}\bar{\boldsymbol{Y}}^{(j_1)}X^{(j,j_1)} &= \sum_{j_1=0}^{m}\mathrm{P}^{(0)}_{j_1,X^{(j)}} + \sum_{j_1=0}^{m}\mathrm{P}^{(1)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\mathrm{P}^{(0)}_{j_2,\tilde{A}^{(j_1)}}] + \cdots \\ &+ \sum_{j_1=0}^{m}\mathrm{P}^{(\kappa-1)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\mathrm{P}^{(0)}_{j_2,A^{(j_1)}},\ldots,\sum_{j_2=0}^{m}\mathrm{P}^{(0)}_{j_2,A^{(j_1)}}] + \cdots \\ &+ \sum_{j_1=0}^{m}\mathrm{R}^{(\kappa)}_{j_1,X^{(j)}}[\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)},\ldots,\sum_{j_2=0}^{m}\bar{\boldsymbol{Y}}^{(j_2)}A^{(j_1,j_2)}]. \end{aligned} \tag{11}$$

The above equations can be verified by using the multilinearity of operator $\mathrm{P}^{(k)}_{j',X^{(j)}}[\bar{\boldsymbol{z}}_1,\ldots,\bar{\boldsymbol{z}}_k]$. The right-hand side of (11) has the following features:

- The case of $k = 1$
  The argument of $\mathrm{P}^{(1)}_{j',X^{(j)}}[\cdot]$ is always labeled by $\tilde{A}^{(j')}$. That is, it is formed by $\mathrm{P}^{(\cdot)}_{\cdot,\tilde{A}^{(j')}}[\cdots]$.
- The case of $k \geq 2$
  The arguments of $\mathrm{P}^{(k)}_{j',X^{(j)}}[\cdots]$ are always labeled by $A^{(j')}$. That is, they are formed by $\mathrm{P}^{(\cdot)}_{\cdot,A^{(j')}}[\cdots]$.

This observation suggests the introduction of rooted trees with labels $A^{(j)}$ or $\tilde{A}^{(j)}$. In the sequel, let us denote $A^{(\cdot)}$ or $\tilde{A}^{(\cdot)}$ simply by X for ease of notation as far as it does not cause a confusion. Similarly to $\mathrm{Q}^{(n)}$ in the previous section, we denote by $\mathrm{Q}^{(n)}_X$ the sum of all products of n terms of $\mathrm{P}^{(\cdot)}_{\cdot,\cdot}$ in the right-hand side of (11). Then the following lemma readily follows from (11).

Lemma 12 $\mathrm{Q}_X^{(n)}$ *can be given recursively as follows: for* $1 \leq \#N \leq \kappa - 1$

$$\mathrm{Q}_X^{(1)} = \sum_{j=0}^{m} \mathrm{P}_{j,X}^{(0)},$$
$$\mathrm{Q}_X^{(\#N+1)} = \sum_{j=0}^{m} \mathrm{P}_{j,X}^{(1)}[\mathrm{Q}_{\tilde{A}^{(j)}}^{(\#N)}] + \sum_{p \in V(N)-\{N\}} \sum_{j=0}^{m} \mathrm{P}_{j,X}^{(\#p)}[\mathrm{Q}_{A^{(j)}}^{\#p_1}, \ldots, \mathrm{Q}_{A^{(j)}}^{\#p_{\#p}}].$$

Note that $\mathrm{Q}_X^{(n)}$ has the same structure as $\mathrm{Q}^{(n)}$ in Lemma 2 except with the labels.

Definition 13 (Multi-colored rooted tree with labels (MRTL)) *A multi-colored rooted tree with labels, denoted by* t_X*, is one attached by labels according to the following rules:*

1) *The label of the root is* X*.*
2) *The label of the other vertices is decided by the number of branches and the color of the parent vertex:*
   - *the label is* $\tilde{A}^{(j)}$ *if the parent vertex has a single branch and it is colored with* j*;*
   - *the label is* $A^{(j)}$ *if the parent vertex has more than one branch and it is colored with* j*.*

The totality of MRTL's whose label of the root is X, is denoted by $\mathcal{T}_X$. For example, some MRTL's are listed in Fig. 4.

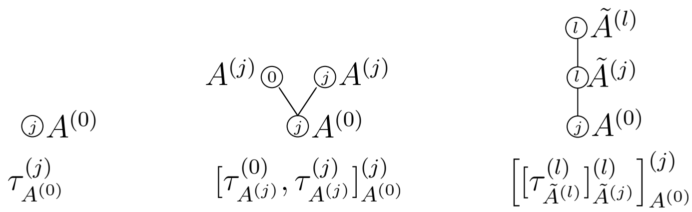

Fig. 4. Examples of trees in $\mathcal{T}_{A^{(0)}}$

Definition 14 (Elementary numerical integral $\bar{\mathbf{\Psi}}(t)$ on $\mathcal{T}_X$) *An elementary numerical integral corresponding to an MRTL is defined recursively as follows:*

$$\bar{\mathbf{\Psi}}(\tau_X^{(j)}) \stackrel{\text{def}}{=} \mathrm{P}_{j,X}^{(0)}, \qquad \bar{\mathbf{\Psi}}(t) \stackrel{\text{def}}{=} \mathrm{P}_{j,X}^{(k)}[\bar{\mathbf{\Psi}}(t_1), \ldots, \bar{\mathbf{\Psi}}(t_k)] \quad \text{for } t = [t_1, \ldots, t_k]_X^{(j)},$$

where $\tau_X^{(j)}$ *and* $[t_1, \ldots, t_k]_X^{(j)}$ *express MRTL's whose roots are labeled by* X*.*

From Definitions 4 and 14, we can see that both structures of $\bar{\boldsymbol{\Psi}}(t)$ and $\boldsymbol{\Psi}(t)$ are the same except with the labels as well as those of $Q_X^{(n)}$ and $Q^{(n)}$. Furthermore, if we define $\hat{t}$ as an MRT obtained by removing all labels from $t \in \mathcal{T}_X$, $\Theta : \mathcal{T}_X \ni t \mapsto \hat{t} \in T$ is a one to one correspondence from $\mathcal{T}_X$ onto T. Therefore, we can obtain the lemma similar to Lemma 6:

Lemma 15 *The identity*

$$\mathrm{Q}_X^{(\#N)} = \sum_{\substack{r(\hat{t})=\#N \\ t\in\mathcal{T}_X}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}(t) \qquad \textit{for} \quad \#N \leq \kappa$$

holds.

For any monomial x of $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$, let $\bar{\lambda}(x)$ be the multiplicity of products with respect to $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$, and $\bar{\sigma}(x)$ the multiplicity of products with respect to $\tilde{\eta}_{i_a}^{(\cdot,\cdot)}$ except $\tilde{\eta}_{i_a}^{(\cdot,0)}$. From Lemma 15, all terms x appearing in the expansion of $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ with $\bar{\lambda}(x) \leq \kappa$ can be expressed by $((m+1)s+1)$-st element of $\bar{\boldsymbol{\Psi}}(t)$. Actually, let $\boldsymbol{y}_{n+1,\kappa}$ denote the truncated expansion of $\boldsymbol{y}_{n+1}$ satisfying $\bar{\lambda}(x) + \bar{\sigma}(x) \leq \kappa$. Then, we have one of our main results:

Theorem 16 *The finitely truncated expansion of the numerical solution by the stochastic Runge-Kutta family has the following expression:*

$$\boldsymbol{y}_{n+1,\kappa} = \boldsymbol{y}_n + \sum_{i=1}^{\kappa} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t\in\mathcal{T}_{A^{(0)}}}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}_{(m+1)s+1}(t),$$

where $\bar{\boldsymbol{\Psi}}_{(m+1)s+1}(t)$ denotes the $((m+1)s+1)$-st element of $\bar{\boldsymbol{\Psi}}(t)$.

The above theorem holds because $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ is equal to the $((m+1)s+1)$-st element of $\sum_{j=0}^{m} \bar{\boldsymbol{Y}}^{(j)} A^{(0,j)}$, all terms of the expression of $\sum_{j=0}^{m} \bar{\boldsymbol{Y}}^{(j)} A^{(0,j)}$ are started from $\mathrm{P}^{(\cdot)}_{\cdot,A^{(0)}}$, and Lemma 15 holds.

Definition 17 (Elementary numerical weight $\bar{\Phi}(t)$ on $\mathcal{T}_{X^{(j)}}$) *An elementary numerical weight of $t \in \mathcal{T}_{X^{(j)}}$ is given recursively as follows:*

$$\begin{aligned} &\bar{\Phi}(\tau_{X^{(j)}}^{(j')}) \overset{\text{def}}{=} \mathbf{1}D^{(j')}X^{(j,j')}, \\ &\bar{\Phi}(t) \overset{\text{def}}{=} (\prod_{i=1}^{k}{}^{\cdot}\, \bar{\Phi}(t_i))D^{(j')}X^{(j,j')} \quad \text{for } t = [t_1, \ldots, t_k]_{X^{(j)}}^{(j')} \end{aligned}$$

$(0 \leq j, j' \leq m)$*, where $\mathbf{1}$ stands for an $(m+1)(s+1)$-dimensional row vector of 1's, and $\prod_{i=1}^{k}{}^{\cdot}\, \bar{\Phi}(t_i)$ means the elementwise product of row vectors $\bar{\Phi}(t_i)$.*

The following is the goal of this section. The proof can also be obtained in a similar way to that in Theorem 11 [13].

Theorem 18 *For any $t \in \mathcal{T}_X$ we have the identity:*

$$\bar{\boldsymbol{\Psi}}(t) = \beta(\hat{t})\boldsymbol{F}(\hat{t})\bar{\Phi}(t).$$

In Theorem 16 we obtained the Taylor expression of the approximate solution $\boldsymbol{y}_{n+1}$ in the same form as that in Theorem 7 except with labels. Furthermore, we showed that the elementary numerical integral $\bar{\boldsymbol{\Psi}}(t)$ can be decomposed into $\beta(\hat{t})$, $\boldsymbol{F}(\hat{t})$ and $\bar{\Phi}(t)$. This decomposition also has the same form as that in Theorem 11. In the next section it will be seen that this fact, that is, keeping the same form, leads to an advantage to seek the order conditions of the stochastic Runge-Kutta family (7).

5 Order conditions of the stochastic Runge-Kutta family

In this section we will show the transparent way of seeking the order conditions by utilizing the rooted tree analysis in Sections 3 and 4.

5.1 Order conditions

First of all, we introduce an important theorem in relation to weak order, which Platen [10,16] has presented.

Theorem 19 ([10], p. 474) *Suppose that $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ have the same probability law with all moments finite, where $\boldsymbol{y}_0$ is an initial random variable of a scheme when it is applied to (1). In addition, suppose that any component of $\boldsymbol{g}_j$ belongs to $C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$ $(0 \leq j \leq m)$ and satisfies Lipschitz conditions and linear growth bounds. Furthermore, suppose that for each $p = 1, 2, \ldots$ there exist constants $K < \infty$ and $r' \in \{1, 2, \ldots\}$ independent of h such that for each $q' \in \{1, \ldots, p\}$*

$$E\left[\max_{0\leq n\leq M} |\boldsymbol{y}_n|^{2q'} \middle| \mathcal{F}_0\right] \leq K\left(1 + |\boldsymbol{y}_0|^{2r'}\right) \qquad \text{(w.p.1)} \qquad (12)$$

and

$$E\left[|\boldsymbol{y}_{n+1} - \boldsymbol{y}_n|^{2q'} \middle| \mathcal{F}_n\right] \leq K\left(1 + \max_{0\leq k\leq n} |\boldsymbol{y}_k|^{2r'}\right) h^{q'} \qquad \text{(w.p.1)} \qquad (13)$$

for $n = 0, 1, \ldots, M-1$, *and such that*

$$\left| E\left[\prod_{j=1}^{L} (\boldsymbol{y}_{n+1} - \boldsymbol{y}_n)_{p_j} - \prod_{j=1}^{L} (\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n))_{p_j} \middle| \mathcal{F}_n \right] \right| \\ \leq K(1 + \max_{0 \leq k \leq n} |\boldsymbol{y}_k|^{2r'}) h^{q+1} \qquad \text{(w.p.1).} \qquad (14)$$

for all $n = 0, \ldots, M-1$ *and* $(p_1, \ldots, p_L) \in \{1, \ldots, d\}^L$ $(1 \leq L \leq 2q+1)$. *Here,* $(\boldsymbol{z})_{p_j}$ *and* $\mathcal{F}_n$ *denote, respectively, the* p_j*-th component of* $\boldsymbol{z}$ *and a non-anticipating sub-*σ*-algebra generated by the discretized Wiener process* $\boldsymbol{W}(\tau_i)$ $(i = 0, \ldots, n)$. *In addition,* $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$ *denotes the Itô-Taylor expansion of* $\boldsymbol{y}(\tau_{n+1}) - \boldsymbol{y}_n$ *centered at* $\boldsymbol{y}_n$ *and truncated up to the term* x *satisfying* $\lambda(x) \leq q$, *where* $\boldsymbol{y}(\tau_{n+1})$ *satisfies the following Itô's stochastic integral equation*

$$\boldsymbol{y}(\tau_{n+1}) \\ = \boldsymbol{y}_n + \int_{\tau_n}^{\tau_{n+1}} \left[\boldsymbol{g}_0(\boldsymbol{y}(s)) + \frac{1}{2} \sum_{j=1}^{m} \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}} \boldsymbol{g}_j(\boldsymbol{y}(s)) \right] \mathrm{d}s + \sum_{j=1}^{m} \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_j(\boldsymbol{y}(s))\, \mathrm{d}W_j(s)$$

with probability 1. *Then, the time discrete approximation* $\boldsymbol{y}_M$ *converges to the* $\boldsymbol{y}(\tau_M)$ *weakly with order* q *as* $h \to 0$.

Conditions (12) and (13) require the regularity of the time discretization itself. The most important assumption is (14) which provides a rule on how to construct a series of approximations $\{\boldsymbol{y}_n\}_{1 \leq n \leq M}$. Whereas (14) is expressed by the Itô-Taylor expansion, however, the results in Section 3 are expressed by the Stratonovich-Taylor expansion. For this, let us rewrite (14) in terms of the Stratonovich-Taylor expansion.

For $s > \tau_n$ and any natural number k, define the multiple stochastic integrals $J_{j_1 \cdots j_k}(s)$ and $I_{j_1 \cdots j_k}(s)$ by

$$J_{j_1 \cdots j_k}(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} \cdots \int_{\tau_n}^{s_2} \circ\, \mathrm{d}W_{j_1}(s_1) \cdots \circ\, \mathrm{d}W_{j_k}(s_k)$$

and

$$I_{j_1 \cdots j_k}(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} \cdots \int_{\tau_n}^{s_2} \mathrm{d}W_{j_1}(s_1) \cdots \mathrm{d}W_{j_k}(s_k),$$

respectively. The following relation holds between $J_{j_1 \cdots j_k}(s)$ and the Itô integral [10, p. 173]:

$$J_{j_1\cdots j_k}(s) = \int_{\tau_n}^{s} J_{j_1\cdots j_{k-1}}(s_k)\,\mathrm{d}W_{j_k}(s_k) + \frac{1}{2}\delta_{\{j_{k-1}=j_k\neq 0\}} \int_{\tau_n}^{s} J_{j_1\cdots j_{k-2}}(s_{k-1})\,\mathrm{d}s_{k-1}, \qquad (15)$$

where $\delta_{\{j_{k-1}=j_k\neq 0\}} = 1$ if $j_{k-1} = j_k \neq 0$, or 0 otherwise. From the repeated application of this equation and the observation that $\lambda(x)+\sigma(x)$ is invariant for any term x in the both hand side, we can see that $J_{j_1\cdots j_k}(s)$ is expressed by the sum of $I_{j_1\cdots j_{k'}}(s)$ ($k' \leq k$) satisfying $k + \sigma(J_{j_1\cdots j_k}(s)) = k' + \sigma(I_{j_1\cdots j_{k'}}(s))$. This means that $J_{j_1\cdots j_k}(s)$ satisfying $k + \sigma(J_{j_1\cdots j_k}(s)) > 2q$ can be expressed by the sum of $I_{j_1\cdots j_{k'}}(s)$ with $k' > q$. From this, we can see that $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n$ includes all the term in $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$. Hence, we can replace $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$ in (14) with $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n$ by noting that any term in $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n - \boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$ does not prevent the inequality from holding.

Next, let us consider the discrete approximate solution part in the inequality. For any $p_j, p_{j'} \in \{1, \ldots, d\}$,

$$\left| E[(\boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1,2q})_{p_j} (\boldsymbol{y}_{n+1,2q} - \boldsymbol{y}_n)_{p_{j'}} | \mathcal{F}_n] \right| \leq K_1 h^{q+1}$$

holds with probability 1, where K_1 is a constant. Hence, we can replace $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ in (14) with $\boldsymbol{y}_{n+1,2q} - \boldsymbol{y}_n$ when $L \geq 2$. From the point mentioned above and the results in Sections 3 and 4, for $L \geq 2$ we can rewrite the expression in the left-hand side of (14) as follows:

$$
\begin{aligned}
&E\left[\prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t\in\mathcal{T}_{A(0)}}} \nu(\hat{t}) \bar{\boldsymbol{\Psi}}_{(m+1)s+1}(t) \Big)_{p_j} - \prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(t)+r(t)=i \\ t\in T}} \nu(t) \boldsymbol{\Psi}(t) \Big)_{p_j} \,\middle|\, \mathcal{F}_n \right] \\
&= E\Bigg[\prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t\in\mathcal{T}_{A(0)}}} \nu(\hat{t}) \bar{\boldsymbol{\Psi}}_{(m+1)s+1}(t) \Big)_{p_j} \\
&\qquad\qquad - \prod_{j=1}^{L} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t\in\mathcal{T}_{A(0)}}} \nu(\hat{t}) \boldsymbol{\Psi}(\hat{t}) \Big)_{p_j} \,\Bigg|\, \mathcal{F}_n \Bigg] \qquad \text{(w.p.1)} \\
&= \sum_{i_1=1}^{2q} \sum_{\substack{\rho(\hat{t}_1)+r(\hat{t}_1)=i_1 \\ t_1\in\mathcal{T}_{A(0)}}} \cdots \sum_{i_L=1}^{2q} \sum_{\substack{\rho(\hat{t}_L)+r(\hat{t}_L)=i_L \\ t_L\in\mathcal{T}_{A(0)}}} \prod_{j=1}^{L} (\nu(\hat{t}_j)\beta(\hat{t}_j)(\boldsymbol{F}(\hat{t}_j))_{p_j}) \\
&\qquad\qquad \times E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j) - \prod_{j=1}^{L} \Phi(\hat{t}_j) \,\middle|\, \mathcal{F}_n \right] \qquad \text{(w.p.1).}
\end{aligned}
$$

On the other hand, for any $p_1 \in \{1, \dots, d\}$

$$\left| E[(\boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1,2q+1})_{p_1} | \mathcal{F}_n] \right| \leq K_2 h^{q+1}$$

holds with probability 1, where K_2 is a constant. Hence, when $L = 1$ we can replace $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ in (14) with $\boldsymbol{y}_{n+1,2q+1} - \boldsymbol{y}_n$, and this yields

$$\begin{aligned}
&E\left[\left(\sum_{i=1}^{2q+1} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_{A(0)}}} \nu(\hat{t}) \bar{\boldsymbol{\Psi}}_{(m+1)s+1}(t))_{p_1} - (\sum_{i=1}^{2q} \sum_{\substack{\rho(t)+r(t)=i \\ t \in T}} \nu(t) \boldsymbol{\Psi}(t))_{p_1} \right| \mathcal{F}_n \right] \\
&= \sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_{A(0)}}} \nu(\hat{t}) \beta(\hat{t}) (\boldsymbol{F}(\hat{t}))_{p_1} E\left[\bar{\Phi}_{(m+1)s+1}(t) - \Phi(\hat{t}) \middle| \mathcal{F}_n \right] \\
&\quad + \sum_{\substack{\rho(\hat{t})+r(\hat{t})=2q+1 \\ t \in \mathcal{T}_{A(0)}}} \nu(\hat{t}) \beta(\hat{t}) (\boldsymbol{F}(\hat{t}))_{p_1} E\left[\bar{\Phi}_{(m+1)s+1}(t) \middle| \mathcal{F}_n \right] \qquad \text{(w.p.1).}
\end{aligned}$$

Consequently, the inequality (14) holds if

$$E\left[\prod_{j=1}^{L} \bar{\Phi}_{(m+1)s+1}(t_j) \right] = E\left[\prod_{j=1}^{L} \Phi(\hat{t}_j) \right] \tag{16}$$

for any $t_1, \dots, t_L \in \mathcal{T}_{A^{(0)}}$ $(1 \leq L \leq 2q)$ satisfying $\sum_{j=1}^{L} (\rho(\hat{t}_j) + r(\hat{t}_j)) \leq 2q$ and

$$E\left[\bar{\Phi}_{(m+1)s+1}(t) \right] = 0 \tag{17}$$

for any $t \in \mathcal{T}_{A^{(0)}}$ satisfying $\rho(\hat{t}) + r(\hat{t}) = 2q+1$ since $\tilde{\eta}_{\cdot}^{(\cdot,\cdot)}$ is independent of $\boldsymbol{y}_n$.

5.2 Calculation rules for elementary weights or elementary numerical weights

In this subsection we will give a way of seeking the expectations appearing in weak order conditions with the help of MRTs. As preliminaries, we first introduce some rules to calculate elementary weights easily.

By means of the chain rule

$$J_{j_1\cdots j_k}(\tau_{n+1})J_{j_1\cdots j_{k'}}(\tau_{n+1}) = \int_{\tau_n}^{\tau_{n+1}} J_{j_1\cdots j_{k-1}}(s)J_{j_1\cdots j_{k'}}(s) \circ \mathrm{d}W_{j_k}(s) + \int_{\tau_n}^{\tau_{n+1}} J_{j_1\cdots j_k}(s)J_{j_1\cdots j_{k'-1}}(s) \circ \mathrm{d}W_{j_{k'}}(s),$$

we can express the product of elementary weights of some MRTs by the sum of elementary weights of other MRTs as the following example:

$$\Phi\left(\textcircled{0}\right)\Phi\begin{pmatrix} \textcircled{l} \\ \textcircled{j} \end{pmatrix} = \Phi\begin{pmatrix} \textcircled{l} \\ \textcircled{j} \\ \textcircled{0} \end{pmatrix} + \Phi\begin{pmatrix} \textcircled{0} & \textcircled{l} \\ \multicolumn{2}{c}{\textcircled{j}} \end{pmatrix} = \Phi\begin{pmatrix} \textcircled{l} \\ \textcircled{j} \\ \textcircled{0} \end{pmatrix} + \Phi\begin{pmatrix} \textcircled{l} \\ \textcircled{0} \\ \textcircled{j} \end{pmatrix} + \Phi\begin{pmatrix} \textcircled{0} \\ \textcircled{l} \\ \textcircled{j} \end{pmatrix}.$$

From the observation of the example we can obtain the following statement in order to rewrite the product of elementary weights or the composition of subtrees in an elementary weight:

- The product of elementary weights of two MRTs t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to the root of t_2 and an MRT generated by grafting t_2 to the root of t_1.
- The elementary weight of an MRT having subtrees t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to t_2's own root and an MRT generated by grafting t_2 to t_1's own root.

Hence, the product of elementary weights or the elementary weight of an MRT having a vertex with multi-branches can be expressed by the sum of MRTs whose each vertex has no more than one branch. Thus, next let us consider this type of MRT only.

For a multiple Stratonovich integral expressed as the elementary weight of the type of MRT, (15) holds. The recursive application gives a representation of the multiple Stratonovich integral in terms of the sum of multiple Itô integrals. In this application, only the second term in the right-hand side can yield a term having only integrals with respect to time. The condition for it is that the even number of indices j_i's satisfy $j_i \neq 0$ and $j_i = j_{i+1}$ or $j_i = j_{i-1}$ holds for each index j_i. Finally, by noting that the expectation of any multiple Itô integral is 0 if it includes an integral with respect to a Wiener process, we can obtain the following rules for any MRT whose each vertex has no more than one branch:

- The expectation of an elementary weight vanishes unless the even number of vertices are of colors different from 0 and each of these vertices has a parent or child vertex with the same color.
- Trace vertices in the direction from the root to upper vertices. Then, the expectation of an elementary weight of an MRT in which a vertex colored by $j \neq 0$ has a child vertex with the same color is equal to a half of that of

another MRT given by replacing the two vertices with one vertex colored by 0 ([10], p. 175). For example,

$$E\left[\Phi\left(\begin{smallmatrix}\textcircled{\scriptsize j}\\ | \\ \textcircled{\scriptsize j}\\ | \\ \textcircled{\scriptsize 0}\end{smallmatrix}\right)\right]=\frac{1}{2}E\left[\Phi\left(\begin{smallmatrix}\textcircled{\scriptsize 0}\\ | \\ \textcircled{\scriptsize 0}\end{smallmatrix}\right)\right]=\frac{1}{2}\Phi\left(\begin{smallmatrix}\textcircled{\scriptsize 0}\\ | \\ \textcircled{\scriptsize 0}\end{smallmatrix}\right).$$

Note that there is no longer need of the expectation in the right-hand side. By utilizing these rules, we give an example of the expectations of elementary weights or the products of them in Appendix A.

Next, we give a way of seeking the expectations of the $((m+1)s+1)$-st elements of elementary numerical weights or the products of them. From the observation of calculations for some elementary numerical weights according to Definition 17, we can see that the $((m+1)s+1)$-st element of an elementary numerical weight can be obtained directly from a diagram for an MRTL by the following procedure.

- Trace vertices in the direction from the root to upper vertices.
- For the root vertex, prepare indices i_1 and j'_1 and write down $c_{i_1}^{(j'_1,j)}\tilde{\eta}_{i_1}^{(j'_1,j)}$ if the color is j.
- For each vertex except the root, prepare new indices i_{k+1} and j'_{k+1} and write down $\alpha_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)}\tilde{\eta}_{i_{k+1}}^{(j'_{k+1},l)}$ if the label is $A^{(j)}$ and the color is l, or $\tilde{\alpha}_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)}\tilde{\eta}_{i_{k+1}}^{(j'_{k+1},l)}$ if the label is $\tilde{A}^{(j)}$ and the color is l, where i_k and j'_k mean the indices for the parent vertex and $\tilde{\alpha}_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)}\stackrel{\text{def}}{=}\alpha_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)}+\tilde{\gamma}_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)}$.
- Finally, take a summation possible values of $i_.$ and $j'_.$.

For example,

$$\bar{\Phi}_{(m+1)s+1}\left(\begin{smallmatrix}\textcircled{\scriptsize 0}\,\tilde{A}^{(j)}\\ | \\ \textcircled{\scriptsize j}\,A^{(0)}\end{smallmatrix}\right)=\sum_{i_1,i_2=1}^{s}\sum_{j'_1,j'_2=0}^{m}c_{i_1}^{(j'_1,j)}\tilde{\eta}_{i_1}^{(j'_1,j)}\tilde{\alpha}_{i_1 i_2}^{(j'_1,j,j'_2,0)}\tilde{\eta}_{i_2}^{(j'_2,0)}.$$

Now, let us assume the following.

Assumption 20 *The expectation of the $((m+1)s+1)$-st element of an elementary numerical weight or the product of those is equal to 0 if the odd number of vertices are of the same color $j(\neq 0)$.*

As we have seen above, the expectation of an elementary weight or the product of those vanishes if the odd number of vertices are of the same color $j(\neq 0)$. Assumption 20 ensures that (16) holds for such MRTL's and (17) holds. Hence, it is helpful to decrease the number of order conditions to be dealt with.

Then, we can obtain the expectations of the $((m+1)s+1)$-st elements of elementary numerical weights or the products of them for weak order 2 as in

Appendix B.

6 Summary and remarks

First, with MRTL's, we have obtained the expression of the weak order conditions of the stochastic Runge-Kutta family in the multi-dimensional Wiener process case. In the expressions there appear the expectations of elementary weights and elementary numerical weights. Second, we have shown the way of seeking the expectations directly from diagrams for MRTs or MRTL's. As a result, we can obtain weak order conditions for the stochastic Runge-Kutta family in quite a transparent way.

Because the Runge-Kutta family is sufficiently general, it is expected that high weak order schemes with good properties are derived from the family. For example, it includes a counterpart of a scheme proposed by Platen for the multi-dimensional Wiener process case (Appendix C). Our next work is, thus, to find the way of solving the order conditions in the multi-dimensional Wiener process case, and to construct schemes having good performance. As one of such works, we will consider the Runge-Kutta family for commutative SDEs as in [17,18] in the near future [11].

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Appendix

A Expectations of elementary weights

We show the expectations of elementary weights or the products of them for weak order 2 by utilizing the rules in Subsection 5.2. In what follows, we suppose $j, l \neq 0$ and $j \neq l$ for ease of notation. Only the expectations that do not vanish are shown here. Although only the results of calculations are shown here, note that they are obtained easily in a diagram-based manner.

Table A.1
Expectations of elementary weights or the products of them

(a) For $t \in T$ such that $\rho(t) + r(t) = 4$

t	$E[\Phi(t)]$	t	$E[\Phi(t)]$	t	$E[\Phi(t)]$
$[\tau^{(0)}]^{(0)}$	$\frac{1}{2}h^2$	$[[[\tau^{(j)}]^{(j)}]^{(j)}]^{(j)}$	$\frac{1}{8}h^2$	$[\tau^{(j)}, [\tau^{(l)}]^{(l)}]^{(j)}$	$\frac{1}{8}h^2$
$[[\tau^{(0)}]^{(j)}]^{(j)}$	$\frac{1}{4}h^2$	$[[[\tau^{(l)}]^{(l)}]^{(j)}]^{(j)}$	$\frac{1}{8}h^2$	$[\tau^{(l)}, [\tau^{(l)}]^{(j)}]^{(j)}$	$\frac{1}{4}h^2$
$[[\tau^{(j)}]^{(j)}]^{(0)}$	$\frac{1}{4}h^2$	$[[\tau^{(j)}, \tau^{(j)}]^{(j)}]^{(j)}$	$\frac{1}{4}h^2$	$[\tau^{(j)}, \tau^{(j)}, \tau^{(j)}]^{(j)}$	$\frac{3}{4}h^2$
$[\tau^{(j)}, \tau^{(j)}]^{(0)}$	$\frac{1}{2}h^2$	$[[\tau^{(l)}, \tau^{(l)}]^{(j)}]^{(j)}$	$\frac{1}{4}h^2$	$[\tau^{(j)}, \tau^{(l)}, \tau^{(l)}]^{(j)}$	$\frac{1}{4}h^2$
$[\tau^{(0)}, \tau^{(j)}]^{(j)}$	$\frac{1}{4}h^2$	$[\tau^{(j)}, [\tau^{(j)}]^{(j)}]^{(j)}$	$\frac{3}{8}h^2$		

(b) For $t \in T$ such that $\rho(t) + r(t) = 3$

t	t_1	$E[\Phi(t)\Phi(t_1)]$	t	t_1	$E[\Phi(t)\Phi(t_1)]$
$[\tau^{(0)}]^{(j)}$	$\tau^{(j)}$	$\frac{1}{2}h^2$	$[[\tau^{(l)}]^{(l)}]^{(j)}$	$\tau^{(j)}$	$\frac{1}{4}h^2$
$[\tau^{(j)}]^{(0)}$	$\tau^{(j)}$	$\frac{1}{2}h^2$	$[\tau^{(j)}, \tau^{(j)}]^{(j)}$	$\tau^{(j)}$	h^2
$[[\tau^{(j)}]^{(j)}]^{(j)}$	$\tau^{(j)}$	$\frac{1}{2}h^2$	$[\tau^{(j)}, \tau^{(l)}]^{(j)}$	$\tau^{(l)}$	$\frac{1}{4}h^2$
$[[\tau^{(l)}]^{(j)}]^{(j)}$	$\tau^{(l)}$	$\frac{1}{4}h^2$	$[\tau^{(l)}, \tau^{(l)}]^{(j)}$	$\tau^{(j)}$	$\frac{1}{2}h^2$

(c) For $t \in T$ such that $\rho(t) + r(t) = 2$

t	t_1	$E[\Phi(t)\Phi(t_1)]$	t	t_1	t_2	$E[\Phi(t)\Phi(t_1)\Phi(t_2)]$
$[\tau^{(j)}]^{(j)}$	$\tau^{(0)}$	$\frac{1}{2}h^2$	$[\tau^{(j)}]^{(j)}$	$\tau^{(j)}$	$\tau^{(j)}$	$\frac{3}{2}h^2$
$[\tau^{(j)}]^{(j)}$	$[\tau^{(j)}]^{(j)}$	$\frac{3}{4}h^2$	$[\tau^{(j)}]^{(j)}$	$\tau^{(l)}$	$\tau^{(l)}$	$\frac{1}{2}h^2$
$[\tau^{(j)}]^{(j)}$	$[\tau^{(l)}]^{(l)}$	$\frac{1}{4}h^2$				
$[\tau^{(l)}]^{(j)}$	$[\tau^{(l)}]^{(j)}$	$\frac{1}{2}h^2$	$[\tau^{(l)}]^{(j)}$	$\tau^{(j)}$	$\tau^{(l)}$	$\frac{1}{2}h^2$
$\tau^{(0)}$	$\tau^{(0)}$	h^2	$\tau^{(0)}$	$\tau^{(j)}$	$\tau^{(j)}$	h^2

t	$E[\Phi(t)]$	t	$E[\Phi(t)]$
$[\tau^{(j)}]^{(j)}$	$\frac{1}{2}h$	$\tau^{(0)}$	h

(d) For $t \in T$ such that $\rho(t) + r(t) = 1$

t	t_1	$E\left[\{\Phi(t)\}^2\right]$	$E\left[\{\Phi(t)\}^4\right]$	$E\left[\{\Phi(t)\}^2\{\Phi(t_1)\}^2\right]$
$\tau^{(j)}$	$\tau^{(l)}$	h	$3h^2$	h^2

B Expectations of elementary numerical weights

To save space, we omit expectations for elementary numerical weights corresponding to elementary weights in Section A. Except them, we show only the expectations that do not vanish, of elementary numerical weights or the products of them for weak order 2. In addition, for ease of notation we omit all indices and the range of values of all indices in all summations.

Table B.1
Expectations of elementary numerical weights or the products of them

(a) For $t \in \mathcal{T}_{A(0)}$ such that $\rho(\hat{t}) + r(\hat{t}) = 4$

t	$E[\bar{\Phi}_{(m+1)s+1}(t)]$
$[[\tau^{(j)}_{\tilde{A}(0)}]^{(0)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,0)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,0,j'_3,j)}_{i_2 i_3} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,0)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3}\right]$
$[[[\tau^{(l)}_{\tilde{A}(j)}]^{(j)}_{\tilde{A}(l)}]^{(l)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,l,j'_3,j)}_{i_2 i_3} \tilde{\alpha}^{(j'_3,j,j'_4,l)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[[[\tau^{(j)}_{\tilde{A}(l)}]^{(l)}_{\tilde{A}(l)}]^{(l)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,l,j'_3,l)}_{i_2 i_3} \tilde{\alpha}^{(j'_3,l,j'_4,j)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[[\tau^{(l)}_{A(l)}, \tau^{(j)}_{A(l)}]^{(l)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \alpha^{(j'_2,l,j'_3,l)}_{i_2 i_3} \alpha^{(j'_2,l,j'_4,j)}_{i_2 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$
$[\tau^{(l)}_{A(j)}, [\tau^{(j)}_{\tilde{A}(l)}]^{(l)}_{A(j)}]^{(j)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \alpha^{(j'_1,j,j'_2,l)}_{i_1 i_2} \alpha^{(j'_1,j,j'_3,l)}_{i_1 i_3} \tilde{\alpha}^{(j'_3,l,j'_4,j)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$

(b) For $t \in \mathcal{T}_{A(0)}$ such that $\rho(\hat{t}) + r(\hat{t}) = 3, 2$

t	t_1	$E[\bar{\Phi}_{(m+1)s+1}(t)\bar{\Phi}_{(m+1)s+1}(t_1)]$
$[[\tau^{(j)}_{\tilde{A}(l)}]^{(l)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$\tau^{(l)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} \tilde{\alpha}^{(j'_2,l,j'_3,j)}_{i_2 i_3} c^{(j'_4,l)}_{i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,j)}_{i_3} \tilde{\eta}^{(j'_4,l)}_{i_4}\right]$
$[\tau^{(l)}_{\tilde{A}(j)}]^{(j)}_{A(0)}$	$[\tau^{(j)}_{\tilde{A}(l)}]^{(l)}_{A(0)}$	$\sum c^{(j'_1,j)}_{i_1} \tilde{\alpha}^{(j'_1,j,j'_2,l)}_{i_1 i_2} c^{(j'_3,l)}_{i_3} \tilde{\alpha}^{(j'_3,l,j'_4,j)}_{i_3 i_4} E\left[\tilde{\eta}^{(j'_1,j)}_{i_1} \tilde{\eta}^{(j'_2,l)}_{i_2} \tilde{\eta}^{(j'_3,l)}_{i_3} \tilde{\eta}^{(j'_4,j)}_{i_4}\right]$

The above expectations are expressed with the notation in (7). By substituting $\tilde{\eta}^{(\cdot,j_b)}_{i_a} = \eta^{(\cdot,j_b)}_{i_a} b^{(\cdot,j_b)}_{i_a}$ and $\tilde{\gamma}^{(\cdot,j_b,*,j_d)}_{i_a i_b} = \gamma^{(\cdot,j_b,*,j_d)}_{i_a i_b} / b^{(\cdot,j_b)}_{i_a}$ for $j_b, j_d \in \{0, j, l\}$, however, we can readily obtain the expressions of the expectations for (6).

C An example of stochastic Runge-Kutta schemes in the multi-dimensional Wiener process case

We show an example of schemes belonging to the stochastic Runge-Kutta family (6) in the multi-dimensional Wiener process case. This scheme is of weak order 2 for Stratonovich SDEs with a multi-dimensional Wiener process. In fact, the scheme is the counterpart of a weak second order explicit scheme for Itô SDEs, proposed by Platen ([10], p. 486).

When $s = 7$ in (6), set random variables and parameters as in Tables C.1, C.2 and C.3. Suppose that $j, j' \neq 0$ in all tables and $j \neq j'$ only in the second line for $i = 3, 4$. The other random variables and parameters vanish. Then, we have

$$
\begin{aligned}
\boldsymbol{y}_{n+1} = \quad & \boldsymbol{y}_n + \frac{2-m}{2} \sum_{j=1}^{m} \triangle W_j \boldsymbol{g}_j(\boldsymbol{y}_n) + \frac{h}{2} \boldsymbol{g}_0(\boldsymbol{y}_n) + \frac{h}{4} \sum_{j=1}^{m} \boldsymbol{g}^{(1)}_j(\boldsymbol{y}_n) \boldsymbol{g}_j(\boldsymbol{y}_n) \\
& + \frac{1}{4} \sum_{j=1}^{m} \left\{ \triangle W_j + ((\triangle W_j)^2 - h) \big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\bar{\boldsymbol{y}}^j_+ \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{j=1}^{m}\left\{\triangle W_j-((\triangle W_j)^2-h)\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}_-^j\right)\\
&+\frac{1}{4}\sum_{j=1}^{m}\sum_{\substack{j'=1\\ j'\neq j}}^{m}\left\{\triangle W_j+(\triangle W_j\triangle W_{j'}+V_{j',j})\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\tilde{\boldsymbol{y}}_+^{j'}\right)\\
&+\frac{1}{4}\sum_{j=1}^{m}\sum_{\substack{j'=1\\ j'\neq j}}^{m}\left\{\triangle W_j-(\triangle W_j\triangle W_{j'}+V_{j',j})\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\tilde{\boldsymbol{y}}_-^{j'}\right)\\
&+\frac{h}{2}\boldsymbol{g}_0(\bar{\boldsymbol{y}})+\frac{\sqrt{h}}{24}\sum_{j=1}^{m}\left[8\left\{\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}+\frac{\sqrt{h}}{2}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)-\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}-\frac{\sqrt{h}}{2}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)\right\}\right.\\
&\qquad\left.-\left\{\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}+\sqrt{h}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)-\boldsymbol{g}\left(\bar{\boldsymbol{y}}-\sqrt{h}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)\right\}\right] \qquad \text{(C.1)}
\end{aligned}
$$

($m \geq 2$) with the intermediate variables

$$
\begin{aligned}
\bar{\boldsymbol{y}} &= \boldsymbol{y}_n + h\Big(\boldsymbol{g}_0(\boldsymbol{y}_n)+\frac{1}{2}\sum_{j=1}^{m}\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)\boldsymbol{g}_j(\boldsymbol{y}_n)\Big)+\sum_{j=1}^{m}\triangle W_j\boldsymbol{g}_j(\boldsymbol{y}_n),\\
\bar{\boldsymbol{y}}_{\pm}^j &= \boldsymbol{y}_n + h\Big(\boldsymbol{g}_0(\boldsymbol{y}_n)+\frac{1}{2}\sum_{l=1}^{m}\boldsymbol{g}_l^{(1)}(\boldsymbol{y}_n)\boldsymbol{g}_l(\boldsymbol{y}_n)\Big)\pm\sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n),\\
\tilde{\boldsymbol{y}}_{\pm}^j &= \boldsymbol{y}_n \pm \sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n),
\end{aligned}
$$

where $\triangle W_j$'s and $V_{j',j}$'s ($j' \neq j$) are mutually independent random variables satisfying

$$
\begin{aligned}
&E\left[(\triangle W_j)^k\right] = \begin{cases} 0 & (k=1,3,5),\\ (k-1)h^{k/2} & (k=2,4),\\ O(h^3) & (k\geq 6),\end{cases}\\
&E\left[V_{j',j}\right]=0,\quad E\left[(V_{j',j})^2\right]=h^2,\quad E\left[V_{j',j}V_{j,j'}\right]=-h^2,
\end{aligned}
$$

and they are also independent of $\boldsymbol{y}_n$.

Because the last term in (C.1) can be rewritten by $(h/4)\sum_{j=1}^{m}\boldsymbol{g}_j^{(1)}(\bar{\boldsymbol{y}})\boldsymbol{g}_j(\bar{\boldsymbol{y}}) + O(h^3)$, (C.1) is equivalent to Platen's scheme for Itô SDEs except the terms being of $O(h^3)$ at least. Consequently, (C.1) is of weak order 2. (It should be noted that there exists a typographical error in the scheme in [10].)

Table C.1
A set of random variables

i	Random variables		i	Random variables	
1	$\eta_i^{(0,0)}$	h	5	$\eta_i^{(0,0)}$	h
	$\eta_i^{(0,j)}$	$\sqrt{h}$		$\eta_i^{(j,j)}$	$\sqrt{h}$
	$\eta_i^{(j,j)}$	$\triangle W_j$	6	$\eta_i^{(0,j)}, \eta_i^{(1,j)}$	$\sqrt{h}$
2	$\eta_i^{(j,j)}$	$\sqrt{h}$	7	$\eta_i^{(0,j)}, \eta_i^{(1,j)}$	$\sqrt{h}$
3	$\eta_i^{(j,j)}$	$\triangle W_j + ((\triangle W_j)^2 - h) / \sqrt{h}$			
	$\eta_i^{(j,j')}$	$\triangle W_{j'} + (\triangle W_j \triangle W_{j'} + V_{j,j'}) / \sqrt{h}$			
4	$\eta_i^{(j,j)}$	$\triangle W_j - ((\triangle W_j)^2 - h) / \sqrt{h}$			
	$\eta_i^{(j,j')}$	$\triangle W_{j'} - (\triangle W_j \triangle W_{j'} + V_{j,j'}) / \sqrt{h}$			

Table C.2
A set of parameter values for $b_{i_a}^{(j_a,j_b)}$ and $c_i^{(j_a,j_b)}$

i	Parameters				i	Parameters				i	Parameters			
1	$b_i^{(0,0)}$	1	$c_i^{(0,0)}$	$\frac{1}{2}$		$b_i^{(j,j')}$	1	$c_i^{(j,j')}$	$\frac{1}{4}$	6	$b_i^{(0,j)}$	1	$c_i^{(0,j)}$	$-\frac{1}{24}$
	$b_i^{(0,j)}$	1			4	$b_i^{(j,j)}$	1	$c_i^{(j,j)}$	$\frac{1}{4}$		$b_i^{(1,j)}$	1	$c_i^{(1,j)}$	$\frac{1}{24}$
	$b_i^{(j,j)}$	1	$c_i^{(j,j)}$	$\frac{2-m}{2}$		$b_i^{(j,j')}$	1	$c_i^{(j,j')}$	$\frac{1}{4}$	7	$b_i^{(0,j)}$	1	$c_i^{(0,j)}$	$\frac{1}{3}$
2			$c_i^{(j,j)}$	$\frac{1}{2}$	5	$b_i^{(0,0)}$	1	$c_i^{(0,0)}$	$\frac{1}{2}$		$b_i^{(1,j)}$	1	$c_i^{(1,j)}$	$-\frac{1}{3}$
3	$b_i^{(j,j)}$	1	$c_i^{(j,j)}$	$\frac{1}{4}$		$b_i^{(j,j)}$	1							

Table C.3
A set of parameter values for $\alpha_{i_a,i_b}^{(j_a,j_b,j_c,j_d)}$ and $\gamma_{i_a,i_b}^{(j_a,j_b,j_c,j_d)}$

i	Parameters		i	Parameters	
2	$\gamma_{i1}^{(j,j,0,j)}$	$\frac{1}{2}$	6	$\alpha_{i1}^{(0,j,0,0)}, \alpha_{i1}^{(0,j,j',j')}, \alpha_{i2}^{(0,j,j',j')}$	1
3	$\alpha_{i1}^{(j,j,0,0)}, \alpha_{i1}^{(j,j,0,j)}, \alpha_{i2}^{(j,j,j',j')}$	1		$\alpha_{i1}^{(1,j,0,0)}, \alpha_{i1}^{(1,j,j',j')}, \alpha_{i2}^{(1,j,j',j')}$	1
	$\alpha_{i1}^{(j,j',0,j)}$	1		$\alpha_{i5}^{(0,j,j,j)}, -\alpha_{i5}^{(1,j,j,j)}$	1
4	$\alpha_{i1}^{(j,j,0,0)}, -\alpha_{i1}^{(j,j,0,j)}, \alpha_{i2}^{(j,j,j',j')}$	1	7	$\alpha_{i1}^{(0,j,0,0)}, \alpha_{i1}^{(0,j,j',j')}, \alpha_{i2}^{(0,j,,j',j')}$	1
	$\alpha_{i1}^{(j,j',0,j)}$	-1		$\alpha_{i1}^{(1,j,0,0)}, \alpha_{i1}^{(1,j,j',j')}, \alpha_{i2}^{(1,j,j',j')}$	1
5	$\alpha_{i1}^{(0,0,0,0)}, \alpha_{i1}^{(0,0,j,j)}, \alpha_{i2}^{(0,0,j,j)}$	1		$\alpha_{i5}^{(0,j,j,j)}, -\alpha_{i5}^{(1,j,j,j)}$	$\frac{1}{2}$
	$\alpha_{i1}^{(j,j,0,0)}, \alpha_{i1}^{(j,j,j',j')}, \alpha_{i2}^{(j,j,j',j')}$	1			

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