PAPER

Dihedral Butterfly Digraph and Its Cayley Graph Representation

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SUMMARY In this paper, we present a new extension of the butterfly digraph, which is known as one of the topologies used for interconnection networks. The butterfly digraph was previously generalized from binary to d-ary. We define a new digraph by adding a signed label to each vertex of the d-ary butterfly digraph. We call this digraph the dihedral butterfly digraph and study its properties. Furthermore, we show that this digraph can be represented as a Cayley graph. It is well known that a butterfly digraph can be represented as a Cayley graph on the wreath product of two cyclic groups [1]. We prove that a dihedral butterfly digraph can be represented as a Cayley graph in two ways.
key words: *butterfly digraph, dihedral butterfly digraph, Cayley graph, wreath product*

1. Introduction

The butterfly digraph is an important class of graphs not only for the FFT algorithm, but also as one of the topologies of interconnection networks for parallel computers. Other such useful classes include the hypercube, the de Bruijn digraph, the Kautz digraph, and the CCC. Many extensions of the binary butterfly digraph have been proposed including the d-ary butterfly digraph and a butterfly digraph for multiple-dimensional signal processing [5]. The necessity for various topologies for parallel signal processing will increase gradually when we take into account the requirements of current computers. We think that some of the properties required for these topologies will be capability for multiple signal processing and an algebraically symmetric structure. In this paper, we propose a graph class with such properties.

We discuss a new extension of the d-ary butterfly digraph and call it the *dihedral butterfly digraph*. The vertices of the butterfly digraph are defined by the pair of a string and a number (the number is called the level). We also define each vertex of the dihedral butterfly digraph as the pair of a string and the level. Our extension is to append a sign to each letter of the strings, and the adjacencies are also regarded as a condition with a sign. The extension from d-ary letters to signed d-ary letters is one possible extension of the butterfly digraph. This extension may provide new viewpoints, allowing us to reconsider the butterfly digraph and a fundamental result for Cayley graphs.

We describe the fundamental properties of the dihedral butterfly digraph. Appending signs increases the number of representable strings; thus, it increases the number of vertices of the dihedral butterfly digraph. It is known that the order and size of the d-ary n-dimensional butterfly digraph $BF(d,n)$ are nd^n and nd^{n+1}, respectively. The d-ary n-dimensional dihedral butterfly digraph $DBF(d,n)$ has $n(2d)^n$ vertices and $n(d+1)(2d)^n$ arcs. This digraph has similar properties to the butterfly digraph. Relations between the butterfly digraph and the dihedral butterfly digraph are, for instance, $DBF(d,n)$ includes 2^n (resp. d^n) $BF(d,n)$'s (resp. $BF(2,n)$'s) and $DBF(d,n)$ is included in $BF(2d,n)$ as a subgraph. We prove that the diameter of $DBF(d,n)$ is $3n-1$, is $(d+1)$-strongly connected and is Hamiltonian.

It is interesting that the dihedral butterfly digraph is a Cayley graph. It is known that a butterfly digraph can be represented as a Cayley graph on the wreath product of two cyclic groups. A dihedral butterfly digraph can also be represented as a Cayley graph in two ways.

2. Definitions and Preliminary Results

A *digraph* D is defined as a finite nonempty set of *vertices* and a set of *arcs* which are ordered pairs of two vertices. We denote the vertex set of D by $V(D)$, the arc set of D by $A(D)$, and an arc from u to v by (u,v). If $(u,v) \in A(D)$, u is *adjacent to* v, and v is *adjacent from* u. Moreover, (u,v) is *incident from* u and is *incident to* v. The *order* of D is $|V(D)|$, and the *size* of D is $|A(D)|$.

The *indegree* of a vertex v is the number of arcs that are incident to v, and the *outdegree* of v is the number of arcs that are incident from v. A digraph D is called *d-regular* when the indegree and outdegree of every vertex of D are equal to d.

A digraph H is called a *subgraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. We denote this by $H \subset D$. Furthermore, a subgraph H of a digraph D is a *factor* of D if $V(H) = V(D)$. A subgraph H of a digraph D is an *induced subgraph* if for any vertices $u,v \in V(H)$, $(u,v) \in A(H)$ whenever $(u,v) \in A(D)$.

A digraph D is *isomorphic* to a digraph H if there exists a one-to-one mapping ϕ from $V(D)$ onto $V(H)$ such that $(u,v) \in A(D)$ if and only if $(\phi u, \phi v) \in A(H)$. We denote this by $D \cong H$. An isomorphism is called an *automorphism* if the mapping is from $V(D)$ onto $V(D)$. In addition, a digraph

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D is *vertex-transitive* if for any two vertices $u, v \in V(D)$, there exists an automorphism ϕ of D such that $\phi u = v$.

A *(u, v)-path* of D is a vertex sequence of $V(D)$ beginning with u and ending with v such that there is no repeated vertex and that when we write a sequence as $u = u_0, u_1, \ldots, u_{n-1} = v$, there exists $(u_k, u_{k+1}) \in A(D)$ for every $0 \le k \le n-2$. The *length* of a path including n vertices is $n-1$. The *distance* between u and v is defined as the minimum length of paths from u to v (if there is no path from u to v, the distance is infinite). The *terminal vertex* of a (u, v)-path is v. An *internal vertex* of a u-v path P is any vertex of P different from u or v. A collection $\{P_1, \ldots, P_k\}$ of paths is called *internally disjoint* if no internal vertex of P_i $(1 \le i \le k)$ lies on path P_j $(j \ne i)$.

A digraph D is (*weakly*) *connected* if there is a path between any two vertices in D. Moreover, for any vertices u, v in D, if there are paths from u to v and from v to u, then D is *strongly connected*. When a digraph D is strongly connected, the *diameter* of D can be defined as the longest distance between any two vertices in D. A *component* of D is a subgraph of D that is a maximal subgraph with respect to the property of being connected. For nonadjacent vertices u, v of a digraph D, if every u-v path has a vertex in a set $S \subseteq V(D)$, S *separates* u and v. If the minimum cardinality of such a set of D is k, D is *k-strongly connected*.

A *cycle* of a digraph is a nontrivial path such that the beginning and ending vertices are the same. If a cycle of a digraph includes all vertices of the digraph, the cycle is called a *Hamiltonian cycle*.

We denote nD for n distinct copies of a digraph D.

Let Γ be a group generated by a set Δ. A *Cayley graph* $\mathrm{Cay}(\Gamma, \Delta)$ is defined by the vertex set Γ, and its adjacencies are characterized with Δ. There is an arc from a vertex u to a vertex v in $\mathrm{Cay}(\Gamma, \Delta)$ if and only if there exists $\alpha \in \Delta$ such that $u\alpha = v$.

The set $\mathbb{Z}_n = \{0, \ldots, (n-1)\}$ forms a group with respect to addition modulo n. This group is called a *cyclic group*, and we write this group $\mathbb{Z}_n$.

A group is called a *dihedral group* $\mathbb{D}_n = \{\sigma^a\tau^b \mid 0 \le a < n,\ b = \{0, 1\}\}$ if it is represented by the following relations:

$$\sigma^n = e, \qquad \tau^2 = e, \qquad \tau\sigma\tau = \sigma^{-1},$$

where e is the identity element of $\mathbb{D}_n$. Unless otherwise specified, we use σ as the generator of order n and τ as the generator of order 2. Note that $\mathbb{D}_n$ is not commutative.

We denote $\oplus_n$ for addition modulo n and $\ominus_n$ for subtraction modulo n.

For two cyclic groups $\mathbb{Z}_d$ and $\mathbb{Z}_n$, their *direct product* is denoted as $\mathbb{Z}_d \times \mathbb{Z}_n$. Then, $\mathbb{Z}_d \times \mathbb{Z}_n = \{(a, b) \mid a \in \mathbb{Z}_d \text{ and } b \in \mathbb{Z}_n\}$, and for (a, b), $(a', b') \in \mathbb{Z}_d \times \mathbb{Z}_n$, $(a, b)(a', b') = (a \oplus_d a', b \oplus_n b')$. Thus, this group is commutative.

Let A, B be groups. In this paper, we assume that $B = \mathbb{Z}_n$ (for the general definition, see [4]). The *wreath product of groups* $A \wr B$ is a set consisting of all elements π represented by

$$\pi = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\,;\, \beta), \qquad \alpha_k \in A\ (0 \le k < n),\ \beta \in B.$$

$A \wr B$ forms a group under the following binary operation for $\rho = (\delta_0, \ldots, \delta_{n-1}\,;\, \gamma) \in A \wr B$:

$$\pi\rho = (\alpha_0\delta_{0\ominus_n\beta}, \ldots, \alpha_k\delta_{k\ominus_n\beta}, \ldots, \alpha_{n-1}\delta_{(n-1)\ominus_n\beta}\,;\, \beta \oplus_n \gamma).$$

The order of $A \wr B$ is $|A|^n|B|$.

We define a set $\Psi_d = \{\pm 0, \ldots, \pm(d-1)\}$. Each element of Ψ_d is a letter with a sign, in particular, 0 is also signed; $+0$ and -0 are distinct elements. For $x \in \Psi_d$, $|x|$ is the letter without its sign, $\mathrm{sgn}(x)$ is the sign of x, and $-x$ is the sign inversion of x; $-(\pm|x|) = \mp|x|$ (the signs are in the same order). In addition, we denote $(-)^k x$ for k operations of $-$ to x; if k is even, $\mathrm{sgn}((-)^k x) = \mathrm{sgn}(x)$, and if k is odd, $\mathrm{sgn}((-)^k x) = \mathrm{sgn}(-x)$.

Definition 1: The d-ary n-dimensional *butterfly digraph* $BF(d, n)$ is defined as follows: for integers $d \ge 2, n \ge 1$,

$$V(BF(d,n)) = \left\{ (x_{n-1} \cdots x_0; i) \;\middle|\; \begin{array}{l} x_k \in \mathbb{Z}_d\ (0 \le k < n), \\ i \in \mathbb{Z}_n \end{array} \right\},$$

$$A(BF(d,n)) = \left\{ \begin{array}{l} ((x_{n-1} \cdots x_i \cdots x_0; i), \\ \quad (x_{n-1} \cdots x_{i+1}\, x'_i\, x_{i-1} \cdots x_0; i \oplus_n 1)) \end{array} \;\middle|\; x'_i \in \mathbb{Z}_d \right\}.$$

There is another definition of the butterfly digraph in which the vertices are written as $(x_0 \cdots x_{n-1}; i)$. It is well known that Definition 1 is equivalent to the other definition. Definition 1 is useful when we use its recursive structure. Algebraic representations of several digraphs including de Bruijn digraphs, butterfly digraphs, and CCC are studied by Annexstein et al. [1], and the following theorem is the starting point for our research.

Theorem 1 (Annexstein et al. [1]): $BF(d, n)$ can be represented as a Cayley graph $\mathrm{Cay}(\Gamma, \Delta)$, where

$$\Gamma = \mathbb{Z}_d \wr \mathbb{Z}_n,$$
$$\Delta = \{(k, 0, \ldots, 0; 1) \mid k \in \mathbb{Z}_d\}.$$

3. Dihedral Butterfly Digraph

In this section, we define and discuss the properties of the dihedral butterfly digraph.

3.1 Definition

The dihedral butterfly digraph is defined in a similar way to the butterfly digraph so that each vertex is defined by the pair of a string and a level. The difference from the butterfly digraph is that each letter in the string is signed.

Definition 2: The d-ary n-dimensional dihedral butterfly digraph $DBF(d, n)$ is defined as follows: for integers $d, n \ge 1$, where $\Psi_d = \{\pm 0, \ldots, \pm(d-1)\}$,

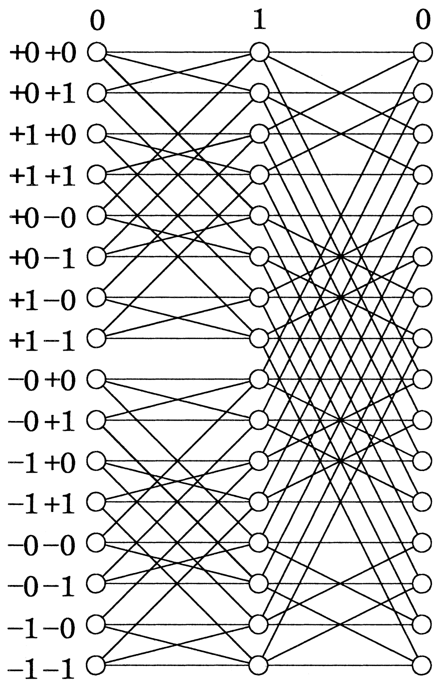

Fig. 1 $DBF(2,2)$.

$$V(DBF(d,n)) = \left\{ (x_{n-1}\cdots x_0; i) \;\middle|\; \begin{array}{l} x_k \in \Psi_d \ (0 \le k < n), \\ i \in \mathbb{Z}_n \end{array} \right\},$$

$$A(DBF(d,n)) = \left\{ \begin{array}{l} ((x_{n-1}\cdots x_i \cdots x_0; i), \\ \quad (x_{n-1}\cdots x_{i+1}\, x'_i\, x_{i-1}\cdots x_0; i \oplus_n 1)) \end{array} \;\middle|\; \begin{array}{l} x'_i = -x_i \text{ or} \\ x'_i = \mathrm{sgn}(x_i)|x|, \\ x \in \Psi_d \end{array} \right\}.$$

In the similar way to the butterfly digraph, another definition can be given, in which the vertices of the dihedral butterfly digraph are written as $(x_0 \cdots x_{n-1}; i)$. This definition is equivalent to Definition 2, and we use it in Sect. 4.

From the definition, $DBF(d,n)$ is $(d+1)$-regular, the order is $n(2d)^n$, and the size is $n(d+1)(2d)^n$. In Fig. 1, $DBF(2,2)$ is illustrated. Every arc in Fig. 1 is directed from left to right. The number written above each column is the level of the vertices of the column, and each string written on the left side is the string of the vertices of the row. Usually, we draw figures of the dihedral butterfly and butterfly digraphs so that the left terminal column is the same as the right terminal column.

We define the *sign sequence* and *absolute sequence* as follows. When $x = (x_{n-1}\cdots x_0; i)$ is a vertex of $DBF(d,n)$, the sign sequence of x is the sign sequence of the string of x, namely, $(\mathrm{sgn}(x_{n-1})\cdots \mathrm{sgn}(x_0))$. The absolute sequence of x is the sequence of letters without signs of the string of x, namely, $(|x_{n-1}|\cdots|x_0|)$.

3.2 Properties

3.2.1 Inclusion Relations

The dihedral butterfly digraph has many relations with the butterfly digraph. From one of the inclusion relations between the butterfly and dihedral butterfly digraphs, we can determine that $DBF(d,n)$ includes $2^n BF(d,n)$ as a factor.

Theorem 2: $DBF(d,n)$ includes $2^n BF(d,n)$ as a factor.

Proof. We consider an induced subgraph D of $DBF(d,n)$ such that the sign sequence of any vertex of D is the same. To show that D is isomorphic to $BF(d,n)$, we state a mapping f from $V(D)$ to $V(BF(d,n))$ as follows:

$$f((x_{n-1}\cdots x_0; i)) = (|x_{n-1}|\cdots|x_0|; i).$$

For some $x, x' \in V(D)$, we assume that $f(x) = f(x')$. This means that for letters $|x_k|$ of x and $|x'_k|$ of x' $(0 \le k < n)$, $|x_k| = |x'_k|$. Therefore, $x = x'$ because any vertex of D has the same sign sequence; thus, f is an injection. Since there are d letters for each sign, D has d^n distinct strings; thus, D has nd^n vertices. It follows that $|V(D)| = |V(BF(d,n))|$; hence, f is a bijection.

Next for the adjacencies, a vertex $x = (x_{n-1}\cdots x_i \cdots x_0; i)$ of D is adjacent to each vertex $(x_{n-1}\cdots x_{i+1}\, x'_i\, x_{i-1}\cdots x_0; i \oplus_n 1)$, where $x'_i = \mathrm{sgn}(x_i)|a|$ for any $a \in \Psi_d$. It follows that $f(x) = (|x_{n-1}|\cdots|x_i|\cdots|x_0|; i)$ is adjacent to the vertex $(|x_{n-1}|\cdots|x_{i+1}|\,|x'_i|\,|x_{i-1}|\cdots|x_0|; i\oplus_n 1)$, where $|x'_i| \in \mathbb{Z}_d$, because $|x'_i| = |\mathrm{sgn}(x_i)|a|| = |a|$, and a is any element of Ψ_d. This is the same as the definition of $BF(d,n)$; therefore, D is isomorphic to $BF(d,n)$.

There are 2^n such distinct induced subgraphs in $DBF(d,n)$ because the number of sign sequences with length n is 2^n. Hence, there exists $2^n BF(d,n)$ in $DBF(d,n)$, as required. □

Furthermore, we prove that $DBF(d,n)$ includes $d^n BF(2,n)$ as a factor.

Theorem 3: $DBF(d,n)$ includes $d^n BF(2,n)$ as a factor.

Proof. We consider an induced subgraph D of $DBF(d,n)$ such that the absolute sequence of any vertex of D is the same. To show that D is isomorphic to $BF(2,n)$, we state a mapping f from $V(D)$ to $V(BF(2,n))$ as follows:

$$f((x_{n-1}\cdots x_0; i)) = (y_{n-1}\cdots y_0; i),$$

where for all k $(0 \le k < n)$, if $\mathrm{sgn}(x_k)$ is $+$ then $y_k = 1$, else if $\mathrm{sgn}(x_k)$ is $-$ then $y_k = 0$. For some $x, x' \in V(D)$, we assume that $f(x) = f(x')$. This means that $\mathrm{sgn}(x_k) = \mathrm{sgn}(x'_k)$ for all letters in the strings of x, x'. Since any vertex of D has the same absolute sequence, we have $x = x'$; thus, f is an injection. There are 2^n distinct sign sequences for the same absolute sequence. It follows that $|V(D)| = |V(BF(2,n))|$; hence, f is a bijection.

Regarding the adjacencies of D, a vertex $x = (x_{n-1}$

$\cdots x_i \cdots x_0; i)$ is adjacent to $(x_{n-1} \cdots x_{i+1}\, x'_i\, x_{i-1} \cdots x_0;\ i \oplus_n 1)$ when $x'_i = x_i$ or $-x_i$. Thus, $f(x)$ is adjacent to $(y_{n-1} \cdots y_{i+1}\, y'_i\, y_{i-1} \cdots y_0; i \oplus_n 1)$ when $y'_i = 0$ or 1. Since this relation is the same as that in $BF(2, n)$, D is isomorphic to $BF(2, n)$.

There are d^n such disjoint induced subgraphs in $DBF(d, n)$. Hence, there exist $d^n BF(2, n)$ in $DBF(d, n)$, as required. □

Finally, we show that the dihedral butterfly digraph is included in the butterfly digraph. This is an opposite relation to Theorems 2 and 3.

Theorem 4: $DBF(d, n)$ is a factor of $BF(2d, n)$.

Proof. We state a mapping f from $V(DBF(d, n))$ to $V(BF(2d, n))$ as follows:

$$f((x_{n-1} \cdots x_0; i)) = (y_{n-1} \cdots y_0; i),$$

where for all k ($0 \le k < n$), if $\mathrm{sgn}(x_k)$ is + then $y_k = |x_k|$, else if $\mathrm{sgn}(x_k)$ is − then $y_k = |x_k| + d$. The inverse mapping can be written as follows:

$$f^{-1}((y_{n-1} \cdots y_0; i)) = (x_{n-1} \cdots x_0; i),$$

where if $y_k < d$ then $x_k = +|y_k|$ and if $y_k \ge d$ then $x_k = -|y_k - d|$ (for $a \in \mathbb{Z}_n$, $\pm|a|$ is the sign appended to a). Thus, f is a bijection.

On the arc from $x = (x_{n-1} \cdots x_i \cdots x_0; i)$ to $x' = (x_{n-1} \cdots x_{i+1}\, x'_i\, x_{i-1} \cdots x_0; i \oplus_n 1)$, if $\mathrm{sgn}(x_i)$ is +, x is adjacent to each vertex x' when $x'_i = +|a|$ for any $a \in \Psi_d$ or $x'_i = -x_i$. The image of x' is $(y_{n-1} \cdots y_{i+1}\, y'_i\, y_{i-1} \cdots y_0, i \oplus_n 1)$ such that if $\mathrm{sgn}(x'_i)$ is + then $y'_i = |x'_i| \in \mathbb{Z}_d \subset \mathbb{Z}_{2d}$, else if $\mathrm{sgn}(x'_i)$ is − then $y'_i = |x_i| + d \in \mathbb{Z}_{2d}$. Thus, these arcs are included in $BF(2d, n)$. For the other case when $\mathrm{sgn}(x_i)$ is −, we can prove that these arcs are included in $BF(2d, n)$ similarly to above. □

The dihedral butterfly digraph can be represented without its signs by using the mapping in Theorem 4.

Corollary 1: $DBF(d, n)$ can be represented as the following digraph D:

$$V(D) = V(BF(2d, n)),$$

$$A(D) = \left\{ \begin{aligned} &((x_{n-1} \cdots x_i \cdots x_0; i), \\ &\quad (x_{n-1} \cdots x_{i+1}\, x'_i\, x_{i-1} \cdots x_0; i \oplus_n 1)) \\ &\quad \left| \begin{array}{l} 0 \le x'_i < d \text{ or } x'_i = x_i + d, \text{ if } x_i < d. \\ d \le x'_i < 2d \text{ or } x'_i = x_i - d, \text{ if } x_i \ge d. \end{array} \right. \end{aligned} \right\}.$$

The following corollary can be derived from Theorems 2, 3, and 4.

Corollary 2:

$$2^n BF(d, n) \subset DBF(d, n) \subset BF(2d, n).$$
$$d^n BF(2, n) \subset DBF(d, n) \subset BF(2d, n).$$
$$1^n BF(2, n) \cong DBF(1, n) \cong BF(2, n).$$

3.2.2 Strong Connectedness

We can show the strong connectedness of the dihedral butterfly digraph using the connectedness of the butterfly digraph and the two theorems proved above. Let u, v be the vertices of $DBF(d, n)$ and w be a vertex of $DBF(d, n)$ such that its sign sequence is equal to that of u and its absolute sequence is equal to that of v. As discussed in Theorem 2, the induced subgraph of $DBF(d, n)$ with the same sign sequence of vertices is isomorphic to $BF(d, n)$. Thus, there is a subgraph $BF(d, n)$ including u and w in $DBF(d, n)$. Since the butterfly digraph is strongly connected, there is a path from u to w. Similarly, a path from w to v can also be found by Theorem 3. Hence, a path from u to v can be found, and we may also find a path from v to u by the same discussion. Thus, the following theorem holds.

Theorem 5: $DBF(d, n)$ is strongly connected.

3.2.3 Diameter

We can define the diameter of $DBF(d, n)$ since it is strongly connected.

Theorem 6: The diameter of $DBF(d, n)$ is $3n - 1$.

Proof. Let $x = (x_{n-1} \cdots x_0; i)$ be a vertex of $DBF(d, n)$. We consider a vertex $y = (y_{n-1} \cdots y_0; i)$ of $DBF(d, n)$ such that $\mathrm{sgn}(y_k) \ne \mathrm{sgn}(x_k)$ and $|y_k| \ne |x_k|$ for all k ($0 \le k < n$). The string of a vertex that is adjacent from x is different from that of x by at most one letter. Furthermore, the difference is either one of the sign or of the absolute value. There are $2n$ letters that must be changed. Therefore, the length of a path from x to y is $2n$.

We consider a path from y to $y' = (y_{n-1} \cdots y_0; i \oplus_n (n-1))$. The length of this path is $n-1$; thus, the length of a path from x through y to y' is at most $3n - 1$.

Here, let the length of some path from a vertex of level i to a vertex of level $i \oplus_n (n-1)$ be L. Then,

$$L \equiv i + (n-1) - i = n - 1 \pmod{n}.$$

If we assume that the length of the path from x to y' is less than $3n - 1$, it is less than or equal to $2n - 1$ by the above congruence expression. However, this is impossible because the length of such a path is at least $2n$, as discussed in the first paragraph. Therefore, the distance between x and y' is $3n - 1$. The strings of any two vertices in $DBF(d, n)$ are different at most n letters, and their levels are different at most $n-1$ letters. Thus the distance between any two vertices is at most $3n - 1$; hence, the diameter of $DBF(d, n)$ is $3n - 1$. □

The diameter of $BF(d, n)$ is $2n - 1$, while that of $DBF(d, n)$ is $3n - 1$. We might say that the latter value is small, since the order of $BF(d, n)$ is nd^n, while that of $DBF(d, n)$ is $n(2d)^n$. On the other hand, the order of $BF(2d, n)$ is equal to that of $DBF(d, n)$; however, the size of $DBF(d, n)$ is about half that of $BF(2d, n)$.

3.2.4 Connectivity

We discuss the connectivity of $DBF(d,n)$. There is an important theorem called Menger's theorem concerned with the connectivity.

Theorem 7 (McCuaig [6]): If no set of fewer than k vertices separates nonadjacent vertices u and v in a digraph D, then there are k internally disjoint u-v paths.

In other words, if a digraph D is k-strongly connected, there are k internally disjoint paths between any two vertices of D. From this theorem, the following corollary can be deduced (see [3]).

Corollary 3: If a digraph D is k-strongly connected, there are k disjoint paths from any k vertices to any k vertices of D.

It is known that the butterfly digraph $BF(d,n)$ is d-strongly connected. Thus, $BF(d,n)$ has d disjoint paths from any d vertices to any d vertices of $BF(d,n)$ by Menger's theorem.

We prove the following theorem by using these results.

Theorem 8: The dihedral butterfly digraph $DBF(d,n)$ is $(d+1)$-strongly connected.

Proof. We prove that there are $d+1$ internally disjoint paths from vertex $x = (x_{n-1}\cdots x_0; i)$ to $y = (y_{n-1}\cdots y_0; j \oplus_n 1)$ of $DBF(d,n)$. x is adjacent to $d+1$ vertices. As shown in Fig. 2, we write the vertex $(x_{n-1}\cdots x_{i+1}\ {-x_i}\ x_{i-1}\cdots x_0; i\oplus_n 1)$ as x^- and the other vertices $(x_{n-1}\cdots x_{i+1}\ x_i'\ x_{i-1}\cdots x_0; i\oplus_n 1)$ $(x_i' = \mathrm{sgn}(x_i)|a|$ for all $a \in \Psi_d)$ as $x^{x_i'}$. Similarly, for the $d+1$ vertices adjacent to y, we write the vertex $(y_{n-1}\cdots y_{j+1}\ {-y_j}\ y_{j-1}\cdots y_0; j)$ as y^- and the other vertices $(y_{n-1}\cdots y_{j+1}\ y_j'\ y_{j-1}\cdots y_0; j)$ $(y_j' = \mathrm{sgn}(y_j)|a|$ for all $a \in \Psi_d)$ as $y^{y_j'}$. Then, $d+1$ internally disjoint paths from x to y are deemed as $d+1$ disjoint paths from x^- and $x^{x_i'}$ to y^- and $y^{y_j'}$. In the figures in this section, paths are drawn as thick lines. We write some of the 2^n induced subgraphs, $BF(d,n)$, discussed in Theorem 2 as BF_x, BF_x^-, BF_y, and BF_y^-, which include x, x^-, y, and y^-, respectively. We draw these $BF(d,n)$'s as boxes in the figures in this section.

We consider two paths from x to BF_y and BF_y^- on vertices with the same absolute sequence. x^- and x^{x_i} are the second vertices of such two paths, respectively, as shown in Fig. 3. We consider two conditions of the two paths as follows: 1. they are internally disjoint and 2. a path to BF_y (resp. BF_y^-) does not pass through BF_y^- (resp. BF_y). First, we assume that $BF_x \neq BF_y^-$.

Since the ith sign of either x^{x_i} or x^- is equal to $\mathrm{sgn}(y_i)$, we consider a path from such a vertex to a vertex of BF_y (however, if $BF_x^- = BF_y^-$, we deem x^- as a trivial path and consider a path from x^{x_i} to BF_y). Assume, without loss of generality, that the vertex whose ith sign is equal to $\mathrm{sgn}(y_i)$ is x^{x_i}. x^{x_i} is adjacent to vertex $x_1^{x_i}$ so that its $(i+1)$th sign is equal to $\mathrm{sgn}(y_{i+1})$. $x_2^{x_i}$ is adjacent to vertex $x_2^{x_i}$ so that its $(i+2)$th sign is equal to $\mathrm{sgn}(y_{i+2})$. In this way, a path follows

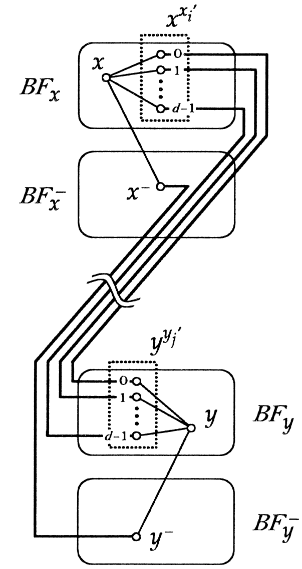

Fig. 2 $d+1$ internally disjoint paths from x to y on $DBF(d,n)$.

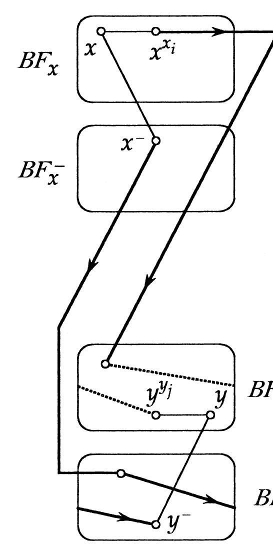

Fig. 3 Two paths from x to BF_y and BF_y^-.

vertex $x_k^{x_i}$ so that its $(i+k)$th sign is equal to $\mathrm{sgn}(y_{i+k})$. This method constructs a path from x^{x_i} to a vertex of BF_y using the same absolute sequences. The length of such a path is at most $n-1$ (since the ith $= (i+n)$th sign is equal to $\mathrm{sgn}(y_i)$). In a similar way, we can find a path from x^- to a vertex of BF_y^- using the same absolute sequences (the length of this path is at most n). These two paths P_1 and P_2 are internally disjoint because the ith sign of any internal vertex of P_1 is different from that of P_2. Regarding the second condition, if a path from x^{x_i} passes through BF_y^-, there is the smallest integer k such that $x_k^{x_i}$ is a vertex of BF_y^-, as shown in Fig. 4. Then, the vertex $x_{k-1}^{x_i}$ is not included in BF_y^-. Therefore, we proceed a path from $x_{k-1}^{x_i}$ to ${x'}_k^{x_i}$ such that its $(i+k)$th sign is not equal to $\mathrm{sgn}(y_{i+k})$. As a result, the path does not pass through BF_y^- until the $(i+k)$th sign changes. The remaining process is similar to that mentioned above; the path proceeds from ${x'}_k^{x_i}$ to $x_{k+1}^{x_i}$ so that the $(i+k+1)$th sign is equal to $\mathrm{sgn}(y_{i+k+1})$.

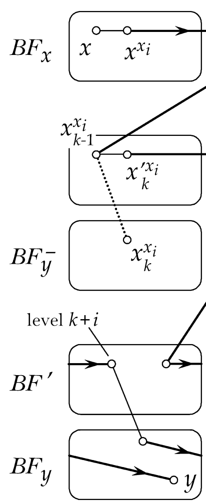

Fig. 4 A path from x^{x_i} to BF_y avoiding BF_y^-.

Then, the process constructs a path from x^{x_i} to a vertex $x^{x_i}_{n-1}$ of BF', which is one of the 2^n induced subgraphs, $BF(d, n)$, such that only the $(i + k)$th sign is different from that of y. In BF', a vertex that is adjacent to a vertex of BF_y has level $i + k$. Since BF' is strongly connected, there is a path from $x^{x_i}_{n-1}$ to any vertex of level $i+k$, as shown in Fig. 4. Hence, we have found a path from x^{x_i} to a vertex of BF_y satisfying the two conditions. We apply a similar process to a path from x^- to BF_y^- satisfying the second condition. This path might pass through an induced subgraph $BF(d, n) = BF$ such that only the $(i + k)$th and ith signs are the same as that of x^- and that only these signs are different from those of the vertices in BF_y^-. A vertex u that is adjacent from any vertex of level $i + k$ in BF is not included in BF_y, because the ith sign of u is still different from those of vertices in BF_y. Therefore, we can proceed to u while satisfying the second condition. Moreover, since BF is strongly connected, we can find a path from u to a vertex adjacent to a vertex in BF_y^-. Hence, we have found two paths satisfying the two conditions.

Now, we consider the other $d - 1$ paths from $x^{x'_i}$ ($x'_i = \mathrm{sgn}(x_i)|a|$ for all $a \in \Psi_d$ and $x'_i \neq x_i$). There are two cases depending on the connection of the two paths.

We assume that a path from x^{x_i} arrives at BF_y (a path from x^- arrives at BF_y^-). The $d - 1$ paths from $x^{x'_i}$ ($x'_i \neq x_i$) follow a path from x^{x_i}. That is, all absolute sequences in the paths are the same, and when we write a path from $x^{x'_i}$ as $x^{x'_i}$, $x_1^{x'_i}$, $x_2^{x'_i}, \ldots$, the sign sequence of $x_k^{x'_i}$ is equal to that of $x_k^{x_i}$. These paths are disjoint because their absolute sequences are different at the ith letter. d paths from $x^{x'_i}$ end with d vertices in BF_y. In BF_y, there are d disjoint paths from the d terminal vertices to $y^{y'_j}$ ($y'_j = \mathrm{sgn}(y_j)|a|$ for all $a \in \Psi_d$) by Menger's theorem. y is not included in these d paths, because if a path includes y, then the path must pass through some $y^{y'_j}$. The other path from x^- ends with a vertex in BF_y^-. Since BF_y^- is strongly connected, there is a path from the terminal vertex to y^-. Therefore, there are $d + 1$ internally disjoint paths from x to y.

Conversely, we assume that a path from x^- arrives at BF_y. As shown in Fig. 5, each path from $x^{x'_i}$ ($x'_i \neq x_i$) traces each vertex with the same string to a vertex with level i. Then, each path can proceed to a vertex in BF_x^- and to BF_y along a path from x^-, similarly to in the previous paragraph. These paths are disjoint. Now, there are d terminal vertices of the paths from x^- and $x^{x'_i}$ ($x'_i \neq x_i$) in BF_y. By Menger's theorem, we can find d disjoint paths from these d terminal vertices to $y^{y'_j}$ ($y'_j = \mathrm{sgn}(y_j)|a|$ for all $a \in \Psi_d$) in BF_y. The other path from x^{x_i} ends with a vertex in BF_y^-, and we can find a path to y^-, since BF_y^- is strongly connected.

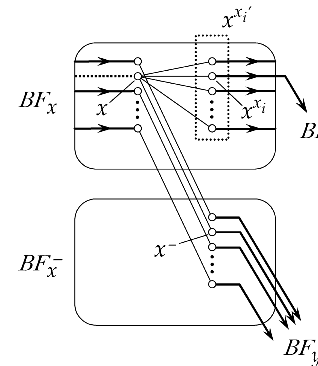

Fig. 5 Route when a path from x^- arrives at BF_y.

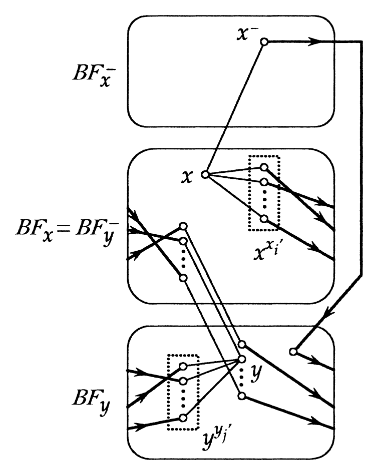

Fig. 6 Route when $BF_x = BF_y^-$.

We consider the exceptional case of $BF_x = BF_y^-$. As shown in Fig. 6, there are d disjoint paths in BF_x from $x^{x'_i}$ ($x'_i = \mathrm{sgn}(x_i)|a|$ for all $a \in \Psi_d$) to vertices written as $(y_{n-1} \cdots y_{j+1}\, y'_j\, y_{j-1} \cdots y_0; j)$ ($y'_j = -\mathrm{sgn}(y_j)|a|$ for all $a \in \Psi_d$) by Menger's theorem. We assume that none of the terminal vertices of these paths are x. There are d vertices in BF_y that are adjacent from these terminal vertices, one of which is y. Thus, we have found a path from x to y. On the other hand, we can find another path from x^- to BF_y, as shown in Fig. 6, by a similar route as the first two paths. Now, there are d terminal vertices in BF_y, one of which is the terminal vertex of the path from x^-, and the others are the terminal vertices except y of the paths from $x^{x'_i}$. By Menger's theorem, there are d disjoint paths from these d terminal vertices to $y^{y'_j}$ in BF_y.

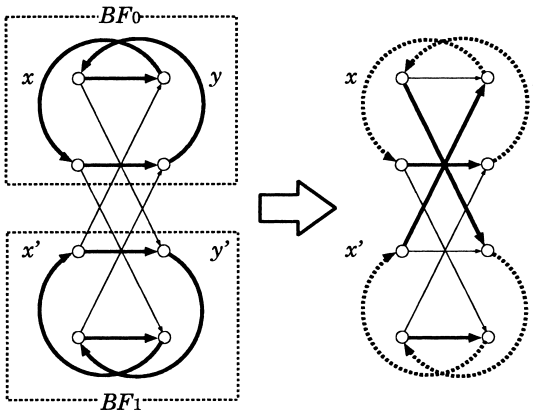

Fig. 7 Exchange of arcs when $|x_i'| = |x_i|$.

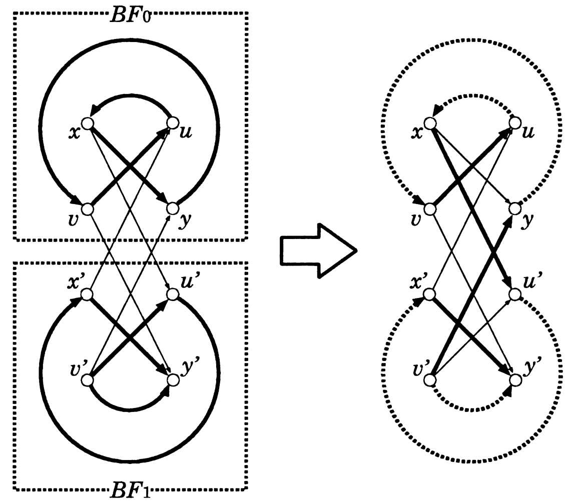

Fig. 8 Exchange of arcs when $|x_i'| \neq |x_i|$.

Finally, we assume that one of the terminal vertices of the d paths in BF_x considered above is x. Then $BF_x^- = BF_y$, and y is one of the vertices in BF_y that are adjacent from these terminal vertices. Thus, there are a path from x to y and $d-1$ internally disjoint paths from x ending with vertices in BF_y. Let the starting vertex of the path ending with x be x'. We can find another path from x' to a vertex in BF_y by the same route as the first two paths. Now, there are d internally disjoint paths ending with vertices except y in BF_y. By Menger's theorem, we can find d disjoint paths in BF_y from these terminal vertices to $y^{y_j'}$. □

3.2.5 Hamiltonicity

We investigate the Hamiltonicity of $DBF(d,n)$ using the well-known fact that the butterfly digraph is Hamiltonian.

Theorem 9: The butterfly digraph is Hamiltonian.

(Refer to [2] for details of the Hamiltonicity of the butterfly digraph.)

Theorem 10: The dihedral butterfly digraph is Hamiltonian.

Proof. Since $BF(2,n)$ has a Hamiltonian cycle from Theorem 9, we first construct a Hamiltonian cycle of $DBF(2,n)$ from 2^n cycles, each of which is a Hamiltonian cycle of $BF(2,n)$ by Theorem 2. Then, we construct a Hamiltonian cycle of $DBF(d,n)$ by induction on d.

Let BF_0 be a subgraph of $DBF(2,n)$, which is one of the $BF(2,n)$ components of the factor $2^n BF(2,n)$ of $DBF(2,n)$ mentioned in Theorem 2, and let $x = (x_{n-1} \cdots x_i \cdots x_0; i)$ be a vertex of BF_0. x is adjacent to a vertex $y = (x_{n-1} \cdots x_{i+1}\, x_i'\, x_{i-1} \cdots x_0; i \oplus_n 1)$ in a Hamiltonian cycle of BF_0, where $\mathrm{sgn}(x_i') = \mathrm{sgn}(x_i)$ and $|x_i'| = 0$ or 1. Moreover, we consider another component BF_1 such that the sign sequence of any vertex of BF_1 is written as $(\mathrm{sgn}(x_{n-1}) \cdots \mathrm{sgn}(x_{i+1})\mathrm{sgn}(-x_i)\ \mathrm{sgn}(x_{i-1}) \cdots \mathrm{sgn}(x_0))$.

If $|x_i'| = |x_i|$, the situation is shown in Fig. 7, which depicts the construction of a cycle including all vertices of BF_0 and BF_1. The vertices x' and y' have the same absolute sequences as x and y, respectively. In Fig. 7, the solid arcs are actual arcs, and the dashed arcs are paths in a Hamiltonian cycle of each component. We can construct a cycle including all vertices of BF_0 and BF_1 by deleting arcs (x,y) and (x',y') and adding arcs (x,y') and (x',y).

If $|x_i'| \neq |x_i|$, a cycle including all vertices of BF_0 and BF_1 can be constructed, as shown in Fig. 8. Similarly to that mentioned above, we construct a cycle by deleting arcs (x,y) and (x',y') and adding arcs (x,u') and (v',y).

For two induced subgraphs, $BF(2,n)$, of $DBF(2,n)$, let one $BF(2,n)$ be different from the other $BF(2,n)$ at the ith sign if the string of every vertex of one $BF(2,n)$ is different from the string of every vertex of the other $BF(2,n)$ at the ith sign. It can be shown that for two induced subgraphs, $BF(2,n)$, of $DBF(2,n)$ that are different at the ith sign, a cycle can be constructed. In the construction, there is no change in any arc other than arcs between levels i and $i+1$. Therefore, for another induced subgraph $BF(2,n)$ that is different from BF_0 or BF_1 at the kth sign, the method for level k can be used to construct a cycle joining the third $BF(2,n)$ (since $k \neq i$). Specifically, first, we construct 2^{n-1} cycles from each pair of induced subgraphs differing at the 0th sign. Let one of these cycles be C. The two $BF(2,n)$'s in C are different only at the 0th sign. We consider another cycle C' such that the $2 \sim (n-1)$th signs of the vertices of C' are the same as those of C. Then, there is some $BF(2,n)$ in C' that is different from one of the $BF(2,n)$'s in C at the 1st sign. Thus, a cycle can be constructed from C and C' by using the method at the 1st sign for these two $BF(2,n)$'s differing at the 1st sign. Therefore, we can construct 2^{n-2} cycles from these 2^{n-1} cycles. Similarly, we can construct a Hamiltonian cycle of $DBF(2,n)$ by repeating this process up to the $(n-1)$th sign.

We show that there exists a Hamiltonian cycle of $DBF(d,n)$ by induction on d. The case of $d = 2$ is described above, and we show that $DBF(d+1,n)$ is Hamiltonian, assuming the case holds for some d. It is clear that the induced subgraph with the vertices excluding letters $+d$ and $-d$ is isomorphic to $DBF(d,n)$. By the induction hypothesis, we can construct a cycle C including all vertices of $DBF(d, n)$ included in $DBF(d + 1, n)$. The remaining vertices can be *joined* as follows.

The remaining vertices not included in C can be written as $(x_{n-1}, \ldots, x_{t+1}, \pm d, y_{t-1}, \ldots, y_0; i)$ for $0 \leq t < n$, where $x_j \in \Psi_d$ $(t < j < n)$, $y_k \in \Psi_{d+1}$ $(0 \leq k < t)$. In $DBF(d+1, n)$, a subgraph induced by vertices with the same sign sequence is isomorphic to $BF(d + 1, n)$, as discussed in Theorem 2. We focus on one such induced subgraph. We write it as BF and omit the sign sequences of the vertices in BF because they are the same. We classify the vertices $(x_{n-1} \cdots x_{t+1}\, d\, y_{t-1} \cdots y_0; i)$ (each $x_j \in \mathbb{Z}_d$ and $y_k \in \mathbb{Z}_{d+1}$) not included in C by the strings $x_{n-1} \cdots x_{t+1}$; strings $x_{n-1} \cdots x_{t+1}$ of all vertices in the set are the same. For a given t, there are d^{n-t-1} such sets because the number of strings $x_{n-1} \cdots x_{t+1}$ is d^{n-t-1}. Thus, for all t $(0 \leq t < n)$, there are

$$d^{n-1} + d^{n-1-1} + \cdots + d^0 = \frac{d^n - 1}{d - 1}$$

sets. Let H be a subgraph induced by one of these sets. A subgraph F of H induced by the vertices between levels 0 and t is isomorphic to $BF(d + 1, t)$, if we identify each vertex $(X; 0)$ with the vertex $(X; t)$ (X is the string of any vertex of F). To show the isomorphism, we omit the strings $x_{n-1} \cdots x_{t+1}$ and the tth letters of the vertices of H, since these are the same in H. Then, the vertex set of F can be written as

$$V(F) = \{(y_{t-1} \cdots y_0; i) \mid y_k \in Z_{d+1}\ (0 \leq k < t),\ i \in Z_t)\},$$

and the vertex $(y_{t-1} \cdots y_i \cdots y_0; i)$ of F is adjacent to each vertex $(y_{t-1} \cdots y_{i+1}\, y'_i\, y_{i-1} \cdots y_0; i \oplus_t 1)$ for all $y' \in Z_{d+1}$. These adjacencies are the same as the definition of $BF(d + 1, t)$; thus, $F \cong BF(d + 1, t)$. By Theorem 5, F has a Hamiltonian cycle. We can construct a cycle including all vertices of H by adding the remaining vertices between levels t and $n - 1$ as paths to form a Hamiltonian cycle of F as follows. In H, we array the vertices between levels 0 and t in the same order as a Hamiltonian cycle of F. Then, the vertex sequence can be written as

$$\begin{gathered}(X_0; 0),\ (X_1; 1), \ldots, (X_{t-1}; t-1),\ (X_t; t),\\ (X_t; 0), \ldots, (X_{2t-1}; t-1),\ (X_{2t}; t),\ (X_{2t}; 0), \ldots, (X_0; 0),\end{gathered}$$

where X_k (k is any integer) is some string. If there are disjoint paths from $(X_{mt}; t)$ to $(X_{mt}; 0)$ (m is any integer), inserting these paths into the sequence makes a cycle. Since there is an arc from level i to level $i + 1$ for any two vertices with the same strings, we can construct each path from level t to level 0 including the vertices with the same strings. Hence, a cycle C_H including all vertices of H can be constructed.

We construct a cycle including all vertices of C and C_H by joining C_H to C as follows. We erase arcs $((x_{n-1} \cdots x_t \cdots x_0; t),\ (x_{n-1} \cdots x_{t+1}\, x'_t\, x_{t-1} \cdots x_0; t \oplus_n 1))$ of C and $((x_{n-1} \cdots x_{t+1}\, d\, y_{t-1} \cdots y_0; t),\ (x_{n-1} \cdots x_{t+1}\, d\, y_{t-1} \cdots y_0; t \oplus_n 1))$ of C_H, and then we add arcs $((x_{n-1} \cdots x_{t+1}\, d\, y_{t-1} \cdots y_0; t),\ (x_{n-1} \cdots x_{t+1}\, x'_t\, x_{t-1} \cdots x_0; t \oplus_n 1))$ and $((x_{n-1} \cdots x_t \cdots x_0; t),\ (x_{n-1} \cdots x_{t+1}\, d\, y_{t-1} \cdots y_0; t \oplus_n 1))$. Here, we assume that $y_k = x_k$ $(0 \leq k < t)$; this assumption is possible since $x_k \in \mathbb{Z}_d$ and $y_k \in \mathbb{Z}_{d+1}$.

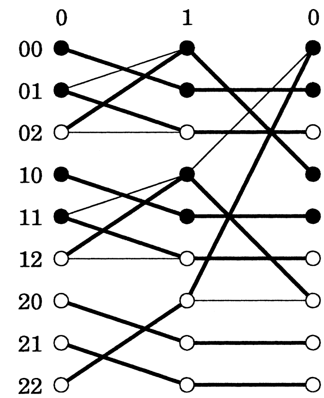

Fig. 9 Construction of a Hamiltonian cycle by adding vertices to $BF(2, 2)$.

We construct each induced subgraph as a cycle C'_H similarly to the method for H, and join C'_H to C (C already includes H). We need other arcs from the construction above when C'_H is joined to C. However, such arcs exist since the strings $x_{n-1} \cdots x_{t+1}$ of each induced subgraph are distinct. Therefore, all remaining vertices of BF can be included in a cycle constructed from C. It follows that we can construct a cycle including all vertices of $DBF(d, n)$ and BF in $DBF(d + 1, n)$.

Figure 9 shows an example of the construction of a Hamiltonian cycle for a butterfly digraph (this is almost the same as the construction for a dihedral butterfly digraph). We constructed a Hamiltonian cycle of $BF(3, 2)$ by adding vertices to a cycle of $BF(2, 2)$. In Fig. 9, the filled circles are the vertices of $BF(2, 2)$, and the open circles are the remaining vertices. We can construct a Hamiltonian cycle from the bold lines.

As described above, we can construct a Hamiltonian cycle of $DBF(d + 1, n)$ by applying the construction to all $2^n BF(d + 1, n)$'s included in $DBF(d + 1, n)$. □

4. Cayley Graph Representation

It is known that the butterfly digraph is a Cayley graph on the wreath product of two cyclic groups [1]. Similarly, we show that the dihedral butterfly digraph can also be represented as a Cayley graph.

In this section, we use the other definition of the dihedral butterfly digraph for brevity. To avoid confusion, we write down the definition.

Definition 3: The dihedral butterfly digraph $DBF(d, n)$ is defined as follows for integers $d, n \geq 1$, where $\Psi_d = \{\pm 0, \ldots, \pm(d - 1)\}$:

$$V(DBF(d, n)) = \left\{ (x_0 \cdots x_{n-1}; i) \;\middle|\; \begin{array}{l} x_k \in \Psi_d\ (0 \leq k < n), \\ i \in \mathbb{Z}_n \end{array} \right\},$$

$A(DBF(d, n))$

$$= \left\{ \begin{array}{l} ((x_0 \cdots x_i \cdots x_{n-1}; i), \\ \quad (x_0 \cdots x_{i-1}\, x'_i\, x_{i+1} \cdots x_{n-1}; i \oplus_n 1)) \end{array} \middle| \begin{array}{l} x'_i = -x_i \text{ or} \\ x'_i = \mathrm{sgn}(x_i)|x|, \\ x \in \Psi_d \end{array} \right\}.$$

Then, we discuss the main result in this section. We show the result in two ways, i.e., we obtain two different Cayley graph representations of the dihedral butterfly digraph.

Theorem 11: Let

$$\Gamma = \mathbb{D}_d \wr \mathbb{Z}_n,$$
$$\Delta = \left\{ (\omega, e, \ldots, e\,;\, 1) \,\middle|\, \begin{array}{l} e \text{ is the identity of } \mathbb{D}_d, \\ \omega = \sigma^k \ (0 \le k < d) \text{ or } \tau \end{array} \right\}.$$

Then $DBF(d,n) \cong \mathrm{Cay}(\Gamma, \Delta)$.

Proof. We write $\pi \in \Gamma$ as

$$\pi = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}; \beta),$$

where $\alpha_k \in \mathbb{D}_d$ $(0 \le k < n)$, $\beta \in \mathbb{Z}_n$. (Then, the order of Γ is $|\mathbb{D}_d|^n|\mathbb{Z}_n| = (2d)^n n$.) We state a mapping f from Γ to $V(DBF(d,n))$ such that $f(\pi) = x = (x_0 \cdots x_{n-1}; i)$ if we write $\alpha_k = \sigma^a\tau^b$ $(0 \le k < n)$, $x_k = (-)^b a$, and $i = \beta$ $(0 \le a < d,\ b = \{0,1\})$. The inverse can be written as $f^{-1}(x) = \pi$, where $\beta = i$, $\alpha_k = \sigma^a\tau^b$ $(0 \le k < n)$, and $a = |x_k|$. If $\mathrm{sgn}(x_k)$ is $+$, then $b = 0$, and if $\mathrm{sgn}(x_k)$ is $-$, then $b = 1$. Thus, f is a bijection.

Let $\rho \in \Delta$. A vertex π of $\mathrm{Cay}(\Gamma, \Delta)$ is adjacent to the vertex

$$\begin{aligned} \pi\rho &= (\alpha_0, \ldots, \alpha_{n-1}\,;\, \beta)(\omega, e, \ldots, e; 1) \\ &= (\alpha_0, \ldots, \alpha_{\beta-1}, \alpha_\beta\omega, \alpha_{\beta+1}, \ldots, \alpha_{n-1}; \beta \oplus_n 1). \end{aligned}$$

We write $f(\pi)$ as $(x_0 \cdots x_\beta \cdots x_{n-1}; \beta)$, $f(\pi\rho)$ as $(x_0 \cdots x_{\beta-1}\, x'_\beta\, x_{\beta+1} \cdots x_{n-1}; \beta \oplus_n 1)$, and $x_\beta = (-)^b a$, where $\alpha_\beta = \sigma^a\tau^b$. If $\omega = \tau$, $\alpha_\beta\omega = \sigma^a\tau^{b\oplus_2 1}$; thus, $x'_\beta = (-)^{b\oplus_2 1}a = -(-)^b a = -x_\beta$. This is equivalent to the inverse-sign condition of the adjacencies of $DBF(d,n)$. If $\omega = \sigma^k$ $(0 \le k < d)$, $\alpha_\beta\omega = \sigma^a\tau^b\sigma^k = \sigma^{a\ominus_d k}\tau^b$; thus, $x'_\beta = (-)^b(a \ominus_d k)$. It follows that the sign of x'_β is equal to that of x_β and is an element of Ψ_d. Therefore, these adjacencies are the same as remaining adjacencies of $DBF(d,n)$. Hence, f preserves the adjacencies, as required. □

The second representation is shown as follows.

Theorem 12: The dihedral butterfly digraph $DBF(d,n)$ can be represented as a Cayley graph on the following group Γ and the generating set Δ:

$$\Gamma = (\mathbb{Z}_2 \times \mathbb{Z}_d) \wr \mathbb{Z}_n,$$
$$\Delta = \left\{ (\omega, e, \ldots, e; 1) \,\middle|\, \begin{array}{l} \omega = (1,0) \text{ or } (0,b) \quad ({}^\forall b \in \mathbb{Z}_d), \\ e = (0,0) \end{array} \right\}.$$

Proof. Let a vertex of $DBF(d,n)$ be denoted $(x_0 \cdots x_{n-1}\,;\, i)$ and an element of group Γ be denoted $(\alpha_0 \cdots \alpha_{n-1}\,;\, \beta)$. We consider a mapping from Γ to $V(DBF(d,n))$ such that for $\alpha_k = (a,b)$, if $a = 0$, then $x_k = +|b|$, and if $a = 1$, then $x_k = -|b|$ $(0 \le k < n)$, and $i = \beta$. Here, each letter x_k $(0 \le k < n)$ of any vertex of $DBF(d,n)$ has a sign $+$ or $-$, and its absolute value is in $\mathbb{Z}_d$. Thus, it follows that there is a corresponding letter α_k that is an element of Γ. That is, if $\mathrm{sgn}(x_k)$ is $+$, then $\alpha_k = (0, |x_k|)$, and if $\mathrm{sgn}(x_k)$ is $-$, then $\alpha_k = (1, |x_k|)$. It follows that the strings $x_0 \cdots x_{n-1}$ correspond to some $\alpha_0 \cdots \alpha_{n-1}$. Since the level is $i \in \mathbb{Z}_n$ and β of an element of Γ is in $\mathbb{Z}_n$, there is at least one element of Γ corresponding to a vertex of $DBF(d,n)$. Hence, this mapping is a surjection, and since the cardinalities of the two sets are equal, this mapping is a bijection.

We consider the adjacencies of $\mathrm{Cay}(\Gamma, \Delta)$. Action of an element of Δ to an element of Γ gives

$$\begin{aligned} &(\alpha_0, \ldots, \alpha_{n-1}; \beta)(\omega,\ e, \ldots, e; 1) \\ &\quad = (\alpha_0, \ldots, \alpha_{\beta-1}, \alpha_\beta\omega,\ \alpha_{\beta+1}, \ldots, \alpha_{n-1}; \beta \oplus_n 1). \end{aligned}$$

This element is mapped to $(x_0, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_{n-1}; i \oplus_n 1)$. We now discuss $\alpha_\beta\omega$ and x'_i. We write $\alpha_\beta = (a,b)$ and $\omega = (a', b')$. Then $\alpha_\beta\omega = (a,b)(a',b') = (a \oplus_2 a', b \oplus_d b')$. If $\omega = (0, b')$, $b \oplus_d b'$ can take any value in $\mathbb{Z}_d$ because b' can take any value in $\mathbb{Z}_d$. Hence, $\alpha_\beta\omega = (a,c)$ $({}^\forall c \in \mathbb{Z}_d)$. This is equivalent to the condition that x'_i is any element of Ψ_d with $\mathrm{sgn}(x'_i) = \mathrm{sgn}(x_i)$. On the other hand, if $\omega = (1,0)$, $\alpha_\beta\omega = (a \oplus_2 1, b)$. This is equivalent to the inverse-sign condition. For the reason described above, the adjacencies are preserved; thus, $DBF(d,n) \cong \mathrm{Cay}(\Gamma, \Delta)$. □

Note that $\mathbb{D}_d$ and $\mathbb{Z}_2 \times \mathbb{Z}_d$ are not isomorphic, since $\mathbb{D}_d$ is not commutative but $\mathbb{Z}_2 \times \mathbb{Z}_d$ is commutative.

Corollary 4: $DBF(d,n)$ is vertex-transitive.

5. Conclusion

In this paper, we define a new class of graphs called dihedral butterfly digraphs, and we derive some of their fundamental properties, such as the diameter, connectivity, and Hamiltonicity. The dihedral butterfly digraph has many analogies to the butterfly digraph since the definition of the dihedral butterfly digraph is given as an extension of the butterfly digraph.

It is known that the butterfly digraph can be represented as a Cayley graph on the wreath product of two cyclic groups. Similarly, we show that the dihedral butterfly digraph can also be represented as a Cayley graph in two ways.

Our future subjects of research will include Hamiltonian cycle decomposition and applications to signal processing.

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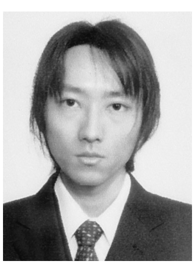

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