Adaptive stabilization of a Kirchhoff’s non-linear beam with output disturbances

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Summary

This paper is concerned with adaptive stabilization of an undamped non-linear beam with disturbed outputs by boundary feedback control. The adaptive controller is constructed by the concept of high-gain adaptive feedback and the estimation mechanism for the unknown parameters of the measurement noise. The global existence and uniqueness of the solution of the closed-loop system is justified. The stability of the closed-loop system is proved such that the convergence of the system state to zero and the convergence of the estimated parameter to the unknown parameter are guaranteed for the smooth and small initial data.

Keywords: Kirchhoff’s non-linear beam, disturbed outputs, adaptive stabilization, boundary control, nonlinear observability.
1 Introduction

A non-linear model of vibrating strings was originally proposed by Kirchhoff in 1876. Kirchhoff type beams were considered in [24]. The solvability of Kirchhoff’s beam equations with dissipative terms is rather well known.

The boundary stabilization of non-linear beams was investigated in [19], where non-linear boundary velocity feedback is used.

The advantage of the adaptive control is that good control performance can be automatically achieved even in the presence of various types of uncertainties. Some attempts have been made to generalize traditional adaptive control algorithms to classes of distributed parameter systems [5]. Non-identifier-based adaptive stabilization has been also investigated for a class of distributed parameter systems [4, 6, 7, 8, 9, 11, 15, 16, 17]. The boundary adaptive stabilization of non-linear beams was investigated in [3, 2, 18].

In this paper we consider adaptive stabilization of an undamped Kirchhoff’s non-linear beam with disturbed outputs under boundary feedback control. The linearized system may have an infinite number of poles and zeros on the imaginary axis. The adaptive boundary feedback controller is constructed by the concept of high-gain adaptive feedback and the estimation mechanism for the unknown parameters of the disturbances. The global existence and uniqueness of the solution of the closed-loop system is justified. The stability of the closed-loop system is proved such that the convergence of the system state to zero and the convergence of the estimated parameter to the unknown parameter are guaranteed for the smooth and small initial data.

2 System description

In this paper we consider an elastic beam of unit length and unit mass density represented by the following non-linear partial differential equation [24]

$$z_{tt}(x,t) + z_{xxxx}(x,t) = M \left( \int_0^1 z_x^2(x,t)dx \right) z_{xx}(x,t)$$

$$= \left( a + b \int_0^1 z_x^2(x,t)dx \right) z_{xx}(x,t), \ x \in (0,1), \ t > 0, \ a > 0, \ b \geq 0$$

(1)

with non-linear boundary conditions

$$z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \ t \geq 0, \ t \geq 0,$$
\[ z_{xxx}(1, t) - M \left( \int_0^1 z_x^2(x,t) \, dx \right) z_x(1,t) = u(t), \quad t \geq 0, \] 

and initial conditions
\[ z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x), \quad x \in (0, 1), \] 

where \( z \) denotes the transversal displacement of the beam, \( z^0, z^1 \) are the initial data, \( u \) is a boundary control input and \( M \in C^1[0, \infty) \) is assumed to be a non-negative function. The observed output of the system is given by
\[ y_d(t) = y(t) + \tilde{\theta}^T \eta(t) = z_t(1, t) + \tilde{\theta}^T \eta(t), \] 

where \( \tilde{\theta}^T \eta(t) \) is measurement noise.

We assume that the measurement noise vector function \( \eta(t) \) is known and continuous with its bounded derivative \( \dot{\eta}(t) \), but \( \tilde{\theta} \) is the \( l \)-dimensional unknown constant vector. For example, we can consider \( \eta(t) \) such that
\[ \eta(t) = \begin{bmatrix} \sin t \\ \sin 5t \\ \cos 7t \end{bmatrix}, \]

Here it should be noted that the elements of \( \eta(t) \) are not necessarily assumed to satisfy a linear, time-invariant, finite-dimensional differential equation. We can consider signals such as a periodic rectangular pulse.

In this paper we shall prove that the adaptive boundary controller
\[
\begin{align*}
    u(t) &= k(t)[y_d(t) - \theta(t)^T \eta(t)] = k(t)[z_t(1, t) - (\theta(t) - \tilde{\theta})^T \eta(t)], \\
    \dot{k}(t) &= \gamma[z_t(1, t) - (\theta(t) - \tilde{\theta})^T \eta(t)]^2, \quad \gamma > 0, \quad k(0) = k_0 > 0, \\
    \dot{\theta}(t) &= k(t)\eta(t)[z_t(1, t) - (\theta(t) - \tilde{\theta})^T \eta(t)], \quad \theta(0) = \theta_0
\end{align*}
\]

asymptotically stabilizes and regulates the system (1)-(3), where \( k(t) \) is an adaptive feedback gain, \( \theta(t) \) is an estimate of the unknown parameter vector \( \tilde{\theta} \) and initial values \( k_0, \theta_0 \) can be designed. The adaptive controller (5) is specialized, since \( y_{d}(t) - \theta(t)^T \eta(t) \) is used in place of unknown output \( z_t(1, t) \) which can be usually used for the systems without output disturbances. If
\[ e(t) = y_d(t) - \theta(t)^T \eta(t) = z_t(1, t) - (\theta(t) - \tilde{\theta})^T \eta(t) \]

converges to 0 and \( \theta(t) \) converges to \( \tilde{\theta} \), then \( z_t(1, t) \) converges to 0.
3 Existence and uniqueness of the solution

In this section we investigate existence and uniqueness of the solution of the following closed-loop system

\[
\begin{cases}
z_{tt}(x,t) + z_{xxxx}(x,t) = M \left( \int_0^1 z_x^2(x,t) dx \right) z_{xx}(x,t), & 0 < x < 1, \; t > 0, \\
z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, & t \geq 0, \\
z_{xxx}(1,t) - M \left( \int_0^1 z_x^2(x,t) dx \right) z_x(1,t) = k(t)\left[ z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t) \right], & t \geq 0, \\
z(x,0) = z^0(x), \; z_t(x,0) = z^1(x), \\
\dot{k}(t) = \gamma [z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)]^2, & \gamma > 0, \; k(0) = k_0 > 0, \\
\dot{\theta}(t) = k(t)\eta(t)[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)], & \theta(0) = \theta_0.
\end{cases}
\] (6)

In this paper we use only standard notations, as in References [13, 19, 21]. Sometimes, a function \( z = z(x,t) \) will simply be denoted by \( z(t) \), when the \( x \)-variable is not in consideration. Our analysis is based on the Sobolev spaces

\[ V = \left\{ z \in H^2(0,1) | z(0) = z_x(0) = 0 \right\} \]

equipped with the norm \( \| z \|_V = \| z_{xx} \| \), and

\[ W = \left\{ z \in H^4(0,1) \cap V | z_{xx}(1) = 0 \right\} \]

equipped with the norm \( \| z \|^2_W = \| z_{xx} \|^2 + \| z_{xxxx} \|^2 \), where \( \| \cdot \| \) denotes the \( L^2(0,1) \) norm and \( H \) denotes \( L^2(0,1) \). Using the Poincare inequality, we see that \( \| \cdot \|_V \) and \( \| \cdot \|_W \) are equivalent to the standard norms of \( H^2(0,1) \) and \( H^4(0,1) \), respectively.

At first we define an energy-like function \( E(t) \) of the system (6) by

\[
E(t) = \frac{1}{2} \| z_t(t) \|^2 + \frac{1}{2} \| z_{xx}(t) \|^2 + \frac{1}{2} \tilde{M}(\| z_x(t) \|^2) + \frac{1}{2\gamma} k^2(t) + \frac{1}{2} \| \theta(t) - \tilde{\theta} \|^2, \] (7)

where \( \tilde{M}(s) = \int_0^s M(z)dz \). We shall prove the following existence and uniqueness result.

**Theorem 1** For any initial data \( z^0 \in W \), \( z^1 \in W \), \( \theta_0 \) and a positive initial data \( k_0 \) satisfying the compatibility condition

\[
z_{xxx}(1) - M \left( \int_0^1 z_x^2(x) dx \right) z_x^0(1) = k_0[z^1(1) - (\theta_0 - \tilde{\theta})^T \eta(0)], \] (8)
there exists a unique solution of the system (6) such that

\[
\begin{align*}
\left\{ \begin{array}{l}
z \in L^\infty(0,T; W), \quad z_t \in L^\infty(0,T; V), \\
z_{tt} \in L^\infty(0,T; H), \quad k \in C^1[0,T], \quad \theta \in (C^1[0,T])^l, \\
z_{ttt}(x,t) + z_{xx}(x,t) = M \left( \int_0^t z_x^2(x,t) dt \right) z_{xx}(x,t) \text{ in } L^\infty(0,T; H), \\
z(0,t) = z_x(0,t) = z_{xx}(1,t) = 0, \\
z_{xx}(1,t) - M \left( \int_0^t z_x^2(x,t) dt \right) z_x(1,t) = k(t)[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)], \\
z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x), \\
\dot{k}(t) = \gamma[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)]^2, \quad \gamma > 0, \quad k(0) = k_0 > 0, \\
\dot{\theta}(t) = k(t)\eta(t)[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)], \quad \theta(0) = \theta_0
\end{array} \right.
\end{align*}
\]

for any $T > 0$.

[Proof] [Existence]

We solve the variational problem associated with (6). Find $z(t) \in W$ such that

\[
\int_0^1 z_{tt}(t) w dx + \int_0^1 z_{xx}(t) w_{xx} dx + M(\|z_x(t)\|^2) \int_0^1 z_x(t) w_x dx \\
+ k(t)[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)] w(1) = 0
\]

(10)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{k}(t) = \gamma[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)]^2, \quad \gamma > 0, \quad k(0) = k_0 > 0, \\
\dot{\theta}(t) = k(t)\eta(t)[z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t)], \quad \theta(0) = \theta_0
\end{array} \right.
\end{align*}
\]

(11)

for all $w \in W$.

We use the Faedo-Galerkin method [12]. Since $W$ is separable, there exists a complete orthogonal system $\{w_j\}$ for which

\[
\{z^0, z^1\} \in \text{Span}\{w_1, w_2\}.
\]

For each $m$ put $W_m = \text{Span}\{w_1, w_2, \ldots, w_m\}$. We define an approximate solution $z_m$ of (10) as follows:

\[
\begin{align*}
z_m(t) &= \sum_{j=1}^m g_{mj}(t) w_j \\
\int_0^1 z_{mtt}(t) w dx + \int_0^1 z_{mxx}(t) w_{xx} dx + M(\|z_{mx}(t)\|^2) \int_0^1 z_{mx}(t) w_x dx \\
+ k_m(t)[z_{mt}(1,t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)] w(1) &= 0
\end{align*}
\]

(13)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{k}_m(t) = \gamma[z_{mt}(1,t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)]^2, \quad \gamma > 0, \quad k_m(0) = k_0 > 0, \\
\dot{\theta}_m(t) = k_m(t)\eta(t)[z_{mt}(1,t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)], \quad \theta_m(0) = \theta_0
\end{array} \right.
\end{align*}
\]

(14)
with the initial conditions
\[
z_m(0) = z^0, \quad z_{mt}(0) = z^1,
\]
which are possible, since \(z^0, z^1 \in W_m\) for \(m \geq 2\). It should be noted that (13) and (14) are a system of ordinary differential equations for \(g_{mj}(t)\), which is known to have a local solution in an interval \([0, t_m]\). There is a local solution \(z_m(t), k_m(t), \theta_m(t)\) in an interval \([0, t_m]\). After the estimate below the approximate solution \(z_m(t), k_m(t), \theta_m(t)\) will be extended to the interval \([0, T]\) for any \(T > 0\).

In order to prove the existence of the solution, we will give several lemmas.

Lemma 3.1
\[
\begin{align*}
\text{l} \quad & \quad z_{mt}(1, \cdot) - (\theta_m - \tilde{\theta})^T \eta \in L^2(0, \infty), \\
\text{r} \quad & \quad \|z_{mt}(t)\|^2 + \|z_{mxx}(t)\|^2 + \bar{M} \|z_{mx}(t)\|^2 + \frac{1}{\gamma} k_m^2(t) + \|\theta_m(t) - \tilde{\theta}\|^2 \leq 2E(0)
\end{align*}
\]

[Proof]

Putting \(w = z_{mt}\) in (13), we obtain from (13) and (14)
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|z_{mt}(t)\|^2 + \|z_{mxx}(t)\|^2 + \bar{M} \|z_{mx}(t)\|^2 + \frac{1}{\gamma} k_m^2(t) + \|\theta_m(t) - \tilde{\theta}\|^2 \right\}
\]
\[
= \int_0^1 z_{mxt}(t)z_{mt}(t)dx + \int_0^1 z_{mxx}(t)z_{mxx}(t)dx + \bar{M} \|z_{mx}(t)\|^2 \int_0^1 z_{mx}(t)z_{mxt}dx
\]
\[
+ k_m(t)[z_{mt}(1, t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)]^2
\]
\[
-k_m(t)[\theta_m(t) - \tilde{\theta}]^T \eta(t)[z_{mt}(1, t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)]
\]
\[
= -\bar{M} \|z_{mx}(t)\|^2 \int_0^1 z_{mx}(t)z_{mxt}dx
\]
\[
-k_m(t)[z_{mt}(1, t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)]z_{mt}(1, t)
\]
\[
+ \bar{M} \|z_{mx}(t)\|^2 \int_0^1 z_{mx}(t)z_{mxt}dx
\]
\[
+ k_m(t)[z_{mt}(1, t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)]^2
\]
\[
+ k_m(t)[\theta_m(t) - \tilde{\theta}]^T \eta(t)[z_{mt}(1, t) - (\theta_m(t) - \tilde{\theta})^T \eta(t)] = 0.
\]

Integrating from 0 to \(t\), \(t < t_m\), we get
\[
\|z_{mt}(t)\|^2 + \|z_{mxx}(t)\|^2 + \bar{M} \|z_{mx}(t)\|^2 + \frac{1}{\gamma} k_m^2(t) + \|\theta_m(t) - \tilde{\theta}\|^2
\]
\[
= \|z_{mt}(0)\|^2 + \|z_{mxx}(0)\|^2 + \bar{M} \|z_{mx}(0)\|^2 + \frac{1}{\gamma} k_m^2(0) + \|\theta_m(0) - \tilde{\theta}\|^2
\]
independent of $m$ and $t$. Therefore the approximate solution $z_m(t), k_m(t), \theta_m(t)$ can be extended to the any interval $[0, T]$. Moreover $z_{mt}(1, \cdot) - (\theta_m - \hat{\theta})^T \eta \in L^2(0, \infty)$. Similarly we can obtain

$$\frac{1}{2} \frac{d}{dt} \left\{ \|z_{mt}(t)\|^2 + \|z_{mxx}(t)\|^2 + \hat{M}(\|z_{mx}(t)\|^2) + \|\theta_m(t) - \hat{\theta}\|^2 \right\}$$

$$= -k_m(t)[z_{mt}(1, t) - (\theta_m(t) - \hat{\theta})^T \eta(t)]^2.$$

**Lemma 3.2**

$$\|z_{mtt}(0)\| \leq M(\|z_0^2\|\|z_{xx}\|)$$  (18)

**[Proof]**

Taking $w = z_{mtt}(0)$ in (13) and $t = 0$, we get

$$\|z_{mtt}(0)\|^2 + \int_0^1 z_{xx}^0 z_{mttx}(0)dx + M(\|z_0^2\|) \int_0^1 z_x^0 z_{mttx}(0)dx$$

$$+ k_0[z^1(1) - (\theta_0 - \hat{\theta})^T \eta(0)] z_{mtt}(1, 0) = 0.$$

Since

$$\int_0^1 z_{xx}^0 z_{mttx}(0)dx + M(\|z_0^2\|) \int_0^1 z_x^0 z_{mttx}(0)dx$$

$$= -[z_{xxx}(0, 0) - M(\|z_x^2\| z_0^3(1))] z_{mtt}(1, 0)$$

$$+ \int_0^1 z_{xxx}^0 z_{mtt}(0)dx - M(\|z_x^2\|) \int_0^1 z_x^0 z_{mtt}(0)dx,$$

it follows from the compatibility condition (8)

$$\|z_{mtt}(0)\|^2 = \int_0^1 z_{xxx}^0 z_{mtt}(0)dx + M(\|z_0^2\|) \int_0^1 z_x^0 z_{mtt}(0)dx,$$

which implies the estimate (18).

**Lemma 3.3**

$$\|z_{mtt}(t)\|^2 + \|z_{mxt}(t)\|^2 + \frac{\gamma^2}{2} \hat{e}_m^4(t) + k_m^2(t) e_m^2(t) \|\eta(t)\|^2 \leq M_1,$$  (19)

where $e_m(t) = z_{mt}(1, t) - (\theta_m(t) - \hat{\theta})^T \eta(t)$ and $M_1$ is a constant depending only on $T$.

**[Proof]**
To avoid differentiating the equation (13), we employ a difference argument as done in Reference [19]. Let us fix $t, \xi > 0$ such that $\xi < T - t$. Taking the difference of (13) with $t = t + \xi$ and $t = t$, and replacing $w$ by $z_{mt}(t + \xi) - z_{mt}(t)$, we get
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|z_{mt}(t + \xi) - z_{mt}(t)\|^2 + \|z_{mxx}(t + \xi) - z_{mxx}(t)\|^2 \right) = I_1 + I_2,
\end{equation}
where
\begin{align*}
I_1 &= -M(\|z_{mx}(t + \xi)\|^2) \int_0^1 z_{mx}(t + \xi)[z_{mx}(t + \xi) - z_{mx}(t)]dx \\
& \quad + M(\|z_{mx}(t)\|^2) \int_0^1 z_{mx}(t)[z_{mx}(t + \xi) - z_{mx}(t)]dx,
I_2 &= -k_m(t + \xi)e_m(t + \xi)[z_{mt}(1, t + \xi) - z_{mt}(1, t)] \\
& \quad + k_m(t)e_m(t)[z_{mt}(1, t + \xi) - z_{mt}(1, t)].
\end{align*}
Let us estimate $I_1$ first. For any $z \in W$, since $z(0) = z_x(0) = z_{xx}(1) = 0$, it follows that
\begin{equation*}
\|z\|_\infty \leq \|z_x\|, \quad \|z_x\|_\infty \leq \|z_{xx}\|, \quad \|z_x\| \leq \|z_{xx}\|.
\end{equation*}
Hence $I_1$ becomes
\begin{align*}
I_1 &= -M(\|z_{mx}(t + \xi)\|^2) \int_0^1 [z_{mx}(t + \xi) - z_{mx}(t)][z_{mx}(t + \xi) - z_{mx}(t)]dx \\
& \quad - [M(\|z_{mx}(t + \xi)\|^2) - M(\|z_{mx}(t)\|^2)] \int_0^1 z_{mx}(t)[z_{mx}(t + \xi) - z_{mx}(t)]dx.
\end{align*}
By the mean value theorem and the estimate (17), there exists a constant $C_1 > 0$ such that
\begin{equation}
|M(\|z_{mx}(t + \xi)\|^2) - M(\|z_{mx}(t)\|^2)| \leq C_1 \|z_{mx}(t + \xi) - z_{mx}(t)\|.
\end{equation}
Integrating $I_1$ by parts, we get
\begin{align*}
I_1 &= -M(\|z_{mx}(t + \xi)\|^2)[z_{mx}(1, t + \xi) - z_{mx}(1, t)][z_{mt}(1, t + \xi) - z_{mt}(1, t)] \\
& \quad + M(\|z_{mx}(t + \xi)\|^2) \int_0^1 [z_{mx}(t + \xi) - z_{mx}(t)][z_{mt}(t + \xi) - z_{mt}(t)]dx \\
& \quad + \Delta M z_{mx}(1, t)[z_{mt}(1, t + \xi) - z_{mt}(1, t)] \\
& \quad - \Delta M \int_0^1 z_{mxx}(t)[z_{mt}(t + \xi) - z_{mt}(t)]dx,
\end{align*}
where
\[
\Delta M = M(\|z_{mx}(t + \xi)\|^2) - M(\|z_{mx}(t)\|^2)
\]

Using Lemma 3.1 and Lemma 3.2, there exists a constant \( C_2 > 0, \rho > 0 \) such that
\[
|I_1| \leq C_2[\|z_m(t + \xi) - z_m(t)\|^2 + \|z_{mx}(t + \xi) - z_{mx}(t)\|^2]
\]
\[
+ \frac{\rho}{2}|z_m(1, t + \xi) - z_m(t, t)|^2
\]
\[
\leq C_2[\|z_m(t + \xi) - z_m(t)\|^2 + \|z_{mx}(t + \xi) - z_{mx}(t)\|^2]
\]
\[
+ \rho[e_m(t + \xi) - e_m(t)]^2 + \rho\|\theta_m(t + \xi) - \theta_m(t)\|^2\|\eta(t)\|^2
\]  \( (22) \)

Next,
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|z_m(t + \xi) - z_m(t)\|^2 + \|z_{mx}(t + \xi) - z_{mx}(t)\|^2 \right\}
\]
\[
= I_1 + I_2 + \frac{1}{2}[k_m(t + \xi) - k_m(t)]e^2_m(t + \xi) - e^2_m(t)
\]
\[
+ [\theta_m(t + \xi) - \theta_m(t)]^T[k_m(t + \xi)\eta(t + \xi)e_m(t + \xi) - k_m(t)\eta(t)e_m(t)]
\]
\[
= I_1 - \frac{1}{2}[k_m(t + \xi) + k_m(t)]e_m(t + \xi) - e_m(t)]^2
\]
\[
+ [k_m(t + \xi)e_m(t + \xi) - k_m(t)e_m(t)][e_m(t + \xi) - e_m(t)]
\]
\[
- [k_m(t + \xi)e_m(t + \xi) - k_m(t)e_m(t)]
\]
\[
\times \{e_m(t + \xi) + [\theta_m(t + \xi) - \tilde{\theta}]^T\eta(t + \xi) - e_m(t) - [\theta_m(t) - \tilde{\theta}]^T\eta(t)\}
\]
\[
+ k_m(t + \xi)e_m(t + \xi)[\theta_m(t + \xi) - \theta_m(t)]^T\eta(t + \xi)
\]
\[
- k_m(t)e_m(t)[\theta_m(t + \xi) - \theta_m(t)]^T\eta(t)
\]
\[
= I_1 - \frac{1}{2}[k_m(t + \xi) + k_m(t)]e_m(t + \xi) - e_m(t)]^2
\]
\[
+ k_m(t + \xi)e_m(t + \xi)[\theta_m(t + \xi) - \theta_m(t)]^T[\eta(t + \xi) - \eta(t)]
\]
\[
- [k_m(t + \xi)e_m(t + \xi) - k_m(t)e_m(t + \xi) + k_m(t)e_m(t + \xi) - k_m(t)e_m(t)]
\]
\[
\times [\theta_m(t + \xi) - \tilde{\theta}]^T[\eta(t + \xi) - \eta(t)]
\]
(taking $k_0 > 2\rho$)

\[
\leq C_2[\|z_{mt}(t + \xi) - z_{mt}(t)\|^2 + \|z_{mxx}(t + \xi) - z_{mxx}(t)\|^2] \\
+C_3\|k_m(t + \xi) - k_m(t)\|^2 + C_4\|\theta_m(t + \xi) - \theta_m(t)\|^2 \\
+ C_5 k^2_m(t + \xi)\epsilon^2_m(t + \xi)\|\eta(t + \xi) - \eta(t)\|^2 \\
+ C_6 \epsilon^2_m(t + \xi)\|\theta(t + \xi) - \tilde{\theta}\|^2 \|\eta(t + \xi) - \eta(t)\|^2 \\
+ \frac{1}{2} k_m(t)\|\theta(t + \xi) - \tilde{\theta}\|^2 \|\eta(t + \xi) - \eta(t)\|^2,
\]

(23)

where $C_3, C_4, C_5, C_6$ are positive constants. Here put

\[
\Phi_m(t, \xi) = \|z_{mt}(t + \xi) - z_{mt}(t)\|^2 + \|z_{mxx}(t + \xi) - z_{mxx}(t)\|^2 \\
+ \frac{1}{2\gamma}[k_m(t + \xi) - k_m(t)]^2 + \|\theta_m(t + \xi) - \theta_m(t)\|^2.
\]

(24)

From (23) we obtain for a positive constant $C_\gamma > 0$

\[
\frac{d}{dt}\Phi_m(t, \xi) \leq C_\gamma \Phi_m(t, \xi) + F_m(t, \xi),
\]

(25)

where

\[
F_m(t, \xi) = C_5 k^2_m(t + \xi)\epsilon^2_m(t + \xi)\|\eta(t + \xi) - \eta(t)\|^2 \\
+ C_6 \epsilon^2_m(t + \xi)\|\theta(t + \xi) - \tilde{\theta}\|^2 \|\eta(t + \xi) - \eta(t)\|^2 \\
+ \frac{1}{2} k_m(t)\|\theta(t + \xi) - \tilde{\theta}\|^2 \|\eta(t + \xi) - \eta(t)\|^2.
\]

(26)

By Gronwall’s inequality

\[
\Phi_m(t, \xi) \leq e^{C_\gamma t}[\Phi_m(0, \xi) + \int_0^T F_m(s, \xi)ds]
\]

Dividing the above inequality by $\xi^2$ and letting $\xi \to 0$, we get

\[
\|z_{mtt}(t)\|^2 + \|z_{mxx}(t)\|^2 + \frac{\gamma}{2} \epsilon^4_m(t) + k^2_m(t)\epsilon^2_m(t)\|\eta(t)\|^2 .
\]

\[
\leq [\|z_{mtt}(0)\|^2 + \|z_{mxx}^1\|^2 + \frac{\gamma}{2} \epsilon^4(0) + k^2_0\epsilon^2(0)\|\eta(0)\|^2]e^{C_\gamma T} \\
+ \{\lim_{\xi \to 0} \frac{1}{\xi^2} \int_0^T F_m(s, \xi)ds\} e^{C_\gamma T}.
\]

Then the estimate (19) is obtained.

With the estimates (15) and (19) we can use Lions-Aubin lemma to get the necessary compactness in order to pass (13) to the limit. Then it is a matter of routine to conclude the existence of global solutions in $[0, T]$ [2, 19].
Next, for the proof of uniqueness, let $z, k, \theta$ and $\hat{z}, \hat{k}, \hat{\theta}$ be two solutions of (6) and put $p = z - \hat{z}, \xi = \theta - \hat{\theta}, e(t) = z_\xi(1, t) - (\theta(t) - \hat{\theta})^T \eta(t), \hat{e}(t) = \hat{z}_\xi(1, t) - (\hat{\theta}(t) - \hat{\theta})^T \eta(t)$. By (10) we get

$$
\int_0^1 p_{tt}(t)w(t)dt + \int_0^1 p_{xx}(t)w_{xt}dt + M(\|z_{x}(t)\|^2)\int_0^1 z_x(t)w_xdt
$$

$$
-M(\|\hat{z}_{x}(t)\|^2)\int_0^1 \hat{z}_x(t)w_xdt + k(t)e(t)w(1, t) - \hat{k}(t)\hat{e}(t)w(1, t) = 0, \quad (27)
$$

Taking $w = 2p_{x}(x, t)$ in (27), we get

$$
\frac{d}{dt} \left\{ \int_0^1 p^2(t)dt + \int_0^1 p_{x}^2(t)dt + \int_0^1 p_{xx}^2(t)dt \right\}
$$

$$
+ M(\|z_{x}(t)\|^2)\int_0^1 p_{x}(t)p_{x}(t)dt + 2M'(\|z_{x}(t)\|^2)\int_0^1 z_{x}(t)z_{x}(t)dt \int_0^1 p_{x}^2(t)dt
$$

$$
+ \|k(t) - \hat{k}(t)\|^2 \|e(t) - \hat{e}(t)\|^2 + 2[\theta(t) - \hat{\theta}(t)]^T [k(t)e(t) - \hat{k}(t)\hat{e}(t)]\eta(t)
$$

$$
= 2 \int_0^1 p(t)p_t(t)dt - 2 \int_0^1 p_{xx}(t)p_{xxt}(t)dt - 2M(\|z_{x}(t)\|^2)\int_0^1 z_x(t)p_{x}(t)dt
$$

$$
+ 2M(\|\hat{z}_{x}(t)\|^2)\int_0^1 \hat{z}_x(t)p_{x}(t)dt - 2k(t)e(t)p_t(1, t) + 2\hat{k}(t)\hat{e}(t)p_t(1, t)
$$

$$
+ 2 \int_0^1 p_{xx}(t)p_{x}(t)dt + 2M(\|z_{x}(t)\|^2)\int_0^1 p_{x}(t)p_{x}(t)dt
$$

$$
+ 2M'(\|z_{x}(t)\|^2)\int_0^1 z_{x}(t)z_{x}(t)dt \int_0^1 p_{x}^2(t)dt
$$

$$
+ \|k(t) - \hat{k}(t)\|^2 \|e(t) + \hat{e}(t)\|^2 |p_t(1, t) - \xi(T\eta(t))|
$$

$$
+ 2[\theta(t) - \hat{\theta}(t)]^T [k(t)e(t) - \hat{k}(t)\hat{e}(t)]\eta(t)
$$

$$
= 2 \int_0^1 p(t)p_t(t)dt - 2M(\|z_{x}(t)\|^2)\int_0^1 z_x(t)p_{x}(t)dt
$$

$$
+ 2M(\|\hat{z}_{x}(t)\|^2)\int_0^1 \hat{z}_x(t)p_{x}(t)dt + 2M(\|z_{x}(t)\|^2)\int_0^1 p_{x}(t)p_{x}(t)dt
$$

$$
+ 2M'(\|z_{x}(t)\|^2)\int_0^1 z_{x}(t)z_{x}(t)dt \int_0^1 p_{x}^2(t)dt
$$

$$
+ \|k(t) + \hat{k}(t)\|^2 \|e(t) - \hat{e}(t)\|^2 |\xi(T\eta(t) - p_t(1, t))|
$$

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We have proved the theorem. For a sufficiently small positive numbers

\[ \gamma \] and some positive numbers \( C_1, C_2, C_3, C_4, C_5 \).

Applying Gronwall's inequality, we obtain

\[ (z, z_t, z_{xx}, k, \theta) = (\ddot{z}, \ddot{z}_t, \ddot{z}_{xx}, \ddot{k}, \ddot{\theta}), \quad 0 \leq t \leq T_0. \]

We have proved the theorem.
4 Boundedness of the solution

We shall discuss the boundedness of the solution of the system (6).

First we can obtain the global estimate by taking inner product between (1) and $2z_t$ like (15)

$$\|z_t(t)\|^2 + \|z_{xx}(t)\|^2 + \hat{M}(\|z_x(t)\|^2) + \frac{1}{\gamma}k^2(t) + \|\theta(t) - \tilde{\theta}\|^2 \leq 2E(0).$$  \hspace{1cm} (28)

Since

$$\hat{M}(\|z_x(t)\|^2) = a\|z_x(t)\|^2 + \frac{b}{2}\|z_x(t)\|^4 \leq 2E(0),$$

$$\|z_x(t)\|^2 \leq \frac{1}{b}[\sqrt{a^2 + 4bE(0)} - a].$$

From this

$$M(t) = M(\|z_x(t)\|^2) = a + b\|z_x(t)\|^2 \leq \sqrt{a^2 + 4bE(0)}.$$  \hspace{1cm} (29)

Taking the inner product between (1) and $2z_{xxt}$, we have

$$2\int_0^1 z_{tt}z_{xxt}dx + 2\int_0^1 z_{xxxx}z_{xxt}dx - 2M(t)\int_0^1 z_{xx}z_{xxt}dx$$

$$= 2z_{tt}(1,t)z_{xxt}(1,t) - \frac{d}{dt}\|z_{xxt}\|^2 - \frac{d}{dt}\|z_{xxx}\|^2 - M(t)\frac{d}{dt}\|z_{xx}\|^2 = 0.$$

Thus for

$$E_1(t) = \|z_{xxt}(t)\|^2 + \|z_{xxx}(t)\|^2 + \hat{M}(\|z_{xx}(t)\|^2)$$

we have

$$\frac{d}{dt}E_1(t) = 2z_{tt}(1,t)z_{xxt}(1,t)$$

$$= \frac{2}{k(t)}z_{xxxt}(1,t)z_{xxt}(1,t) - \frac{2M(t)}{k(t)}z_{xxxt}(1,t) - \frac{2\hat{M}(t)}{k(t)}z_{xx}(t)z_{xxt}(1,t)$$

$$+ 2[\hat{\theta}(t)^T \eta(t) + (\theta(t) - \tilde{\theta})^T \dot{\eta}(t) - \frac{\dot{k}(t)}{k(t)}c(t)]z_{xxt}(1,t)$$

$$\leq \delta_1 \frac{1}{k(t)}z_{xxxt}(1,t) + \frac{1}{\delta_1 k(t)}z_{xxxt}(1,t)$$

$$+ \frac{2M(t)}{k(t)}z_{xxxt}(1,t) - \frac{2\hat{M}(t)}{k(t)}z_{xx}(t)z_{xxt}(1,t)$$

$$+ 2[\hat{\theta}(t)^T \eta(t) + (\theta(t) - \tilde{\theta})^T \dot{\eta}(t) - \frac{\dot{k}(t)}{k(t)}c(t)]z_{xxt}(1,t)$$

$$\leq \delta_1 \frac{1}{k(t)}z_{xxxt}(1,t) + \frac{1}{\delta_1 k(t)}z_{xxxt}(1,t)$$

$$+ \frac{2M(t)}{k(t)}z_{xxxt}(1,t) - \frac{2\hat{M}(t)}{k(t)}z_{xx}(t)z_{xxt}(1,t)$$

$$+ 2[\hat{\theta}(t)^T \eta(t) + (\theta(t) - \tilde{\theta})^T \dot{\eta}(t) - \frac{\dot{k}(t)}{k(t)}c(t)]z_{xxt}(1,t)$$
for a positive constant $\delta_1$. We obtain the following estimate

$$\frac{d}{dt} E_1(t) \leq \frac{\delta_1}{k(t)} z_{xxt}(1, t) + F_1(t),$$

(31)

where

$$F_1(t) = \frac{1}{\delta_1 k(t)} z_{xxt}(1, t) - \frac{2M(t)}{k(t)} z_{xx}(1, t) - \frac{2M(t)}{k(t)} z_x(1, t) z_{xt}(1, t)$$

$$+ 2[\theta(t)^T \eta(t) + (\theta(t) - \tilde{\theta})^T \eta(t) - \frac{\dot{k}(t)}{k(t)} e(t)] z_{xt}(1, t).$$

Next taking the inner product between (1) and $2xz_{xxx}$, we have

$$2 \int_0^1 z_{t} x z_{xxx} dx + 2 \int_0^1 z_{xxxx} x z_{xxx} dx - 2M(t) \int_0^1 z_{xx} x z_{xxx} dx$$

$$= 2 \frac{d}{dt} \int_0^1 z_{tx} x z_{xxx} dx - 2 \int_0^1 z_t x z_{xxx} dx$$

$$+ 2 \int_0^1 z_{xxxx} x z_{xxx} dx - 2M(t) \int_0^1 z_{xx} x z_{xxx} dx = 0.$$

Here

$$2 \int_0^1 z_{tx} x z_{xxx} dx = -z_{xt}^2(1, t) - 2z_{xt}(1, t) z_t(1, t) + 3 \| z_{xt}(t) \|^2,$$

$$2 \int_0^1 z_{xxxx} x z_{xxx} dx = z_{xxx}^2(1, t) - \| z_{xxx}(t) \|^2,$$

$$2M(t) \int_0^1 z_{xx} x z_{xxx} dx = M(t) z_{xx}^2(1, t) - M(t) \| z_{xx}(t) \|^2$$

Thus

$$2 \frac{d}{dt} \int_0^1 z_{tx} x z_{xxx} dx + z_{xt}^2(1, t) + 2z_{xt}(1, t) z_t(1, t) - 3 \| z_{xt}(t) \|^2$$

$$+ z_{xx}(1, t)^2 - \| z_{xxx}(t) \|^2 - M(t) z_{xx}^2(1, t) + M(t) \| z_{xx}(t) \|^2 = 0.$$

We obtain the estimate

$$-2 \frac{d}{dt} \int_0^1 z_{tx} x z_{xxx} dx + E_1(t)$$

$$= \| z_{xt}(t) \|^2 + \| z_{xxx}(t) \|^2 + \hat{M}(\| z_{xx}(t) \|^2)$$

$$+ z_{xt}^2(1, t) + 2z_{xt}(1, t) z_t(1, t) - 3 \| z_{xt}(t) \|^2 + z_{xxx}^2(1, t)$$

$$- \| z_{xxx}(t) \|^2 - M(t) z_{xx}^2(1, t) + M(t) \| z_{xx}(t) \|^2$$

$$\leq \hat{M}(\| z_{xx}(t) \|^2) + z_{xx}^2(1, t) + 2z_{xt}(1, t) z_t(1, t) + z_{xxx}^2(1, t) + M(t) \| z_{xx}(t) \|^2.$$
\[ F_2(t). \]  

From (31) and (32) we have for a positive constant \( \epsilon_1 \)
\[
\frac{d}{dt} \left\{ E_1(t) - 2\epsilon_1 \int_0^1 z_t x z_{xxx} \, dx \right\} + \epsilon_1 E_1(t) \\
\leq \frac{\delta_1}{k(t)} z_{xxx}(1, t) + F_1(t) + \epsilon_1 F_2(t) \\
= \frac{\delta_1}{k(t)} z_{xxx}(1, t) + F(t). \tag{33}
\]

Next consider more smooth initial data such that
\[ z^0 \in H^5(0, 1) \cap W; \ z^1 \in H^4(0, 1) \cap W \]
satisfying the compatibility condition (8) and
\[
z_{xx}(1) = -\frac{1}{k(0)} z_{xxx}(1) + \frac{M(0)}{k(0)} z_{xx}(1) + \frac{\dot{M}(0)}{k(0)} z_{x}(1) \\
+ \frac{\dot{k}(0)}{k(0)} z_{x}(1) + g(k(0), \theta(0), \eta(0)), \tag{34}
\]
where we put
\[
g(k, \theta, \eta) = \frac{\dot{k}(t)}{k(t)} e(t) + \dot{\theta}(t)^T \eta(t) + [\theta(t) - \bar{\theta}]^T \dot{\eta}(t).
\]

Putting \( w(x,t) = z_x(x,t) \), we see from the equations (1) and (2)
\[
w_{tt}(x,t) + w_{xxxx}(x,t) = M(t) w_{xx}(x,t), \ x \in (0,1), t \geq 0 \tag{35}
\]
with the boundary condition
\[
\begin{cases}
  w(0,t) = w_x(1,t) = 0, \\
  w_{xx}(1,t) - M(t) w(1,t) = k(t) z_t(1,t) - [\theta(t) - \bar{\theta}]^T \eta(t) \\
  w_{xxx}(1,t) = -\frac{1}{k(t)} w_{xxxx}(1,t) + \frac{M(t)}{k(t)} w_t(1,t) + \frac{M(t)}{k(t)} w(1,t) \\
  + \frac{\dot{k}(t)}{k(t)} z_t(1,t) + g(k, \theta, \eta).
\end{cases} \tag{36}
\]

Taking the inner product between (35) and \( 2w_{xx} \), we have
\[
2 \int_0^1 w_{tt} w_{xx} \, dx + 2 \int_0^1 w_{xxxx} w_{xx} \, dx - 2M(t) \int_0^1 w_{xx} w_{xx} \, dx \\
= 2w_{tt}(1,t) w_{xt}(1,t) - \frac{d}{dt} \|w_{xt}\|^2 + 2w_{xxxx}(1,t) w_{xx}(1,t) \\
- \frac{d}{dt} \|w_{xxx}\|^2 - M(t) \frac{d}{dt} \|w_{xx}\|^2 = 0.
\]
Thus for

\[ E_2(t) = \|z_{xx}(t)\|^2 + \|z_{xxx}(t)\|^2 + M(t)\|z_{xx}(t)\|^2 \]

\[ = \|w_{xx}(t)\|^2 + \|w_{xxx}(t)\|^2 + M(t)\|w_{xx}(t)\|^2 \]

we obtain the following estimate for \( \epsilon_2 > 0 \)

\[
\frac{d}{dt} E_2(t) - \dot{M}(t)\|w_{xx}\|^2 + 2 \frac{M(t)}{k(t)} w_{xxt}(1, t)
\]

\[ = \frac{2M(t)}{k(t)} w(t,1)w_{xxt}(1, t) + \frac{2\dot{M}(t)}{k(t)} w(1,t)w_{xxt}(1, t) \]

\[ + 2\frac{\dot{k}(t)}{k(t)} z_{tt}(1, t)w_{xxt}(1, t) + 2g(k, \theta, \eta) w_{xxt}(1, t) \]

\[ \leq \frac{\epsilon_2}{k(t)} w_{xxt}^2(1, t) + \frac{M^2(t)}{\epsilon_2 k(t)} w_t^2(1, t) + 3\epsilon_2 w_{xxt}^2(1, t) + \frac{1}{\epsilon_2} g^2(k, \theta, \eta) \]

\[ + \frac{1}{\epsilon_2 k^2(t)} [\dot{M}^2(t) w^2(1, t) + \dot{k}^2(t) z^2_t(1, t)] \]

\[ = \frac{\epsilon_2}{k(t)} w_{xxt}^2(1, t) + 3\epsilon_2 w_{xxt}^2(1, t) + G_1(t), \]  

(38)

where

\[ G_1(t) = \frac{M^2(t)}{\epsilon_2 k(t)} w_t^2(1, t) + \frac{1}{\epsilon_2} g^2(k, \theta, \eta) \]

\[ + \frac{1}{\epsilon_2 k^2(t)} [\dot{M}^2(t) w^2(1, t) + \dot{k}^2(t) w_t^2(1, t)]. \]

Next taking the inner product between (35) and \( 2xw_{xxx} \), we have

\[ 2 \int_0^1 w_{tt}xw_{xxx}dx + 2 \int_0^1 w_{xxxx}xw_{xxx}dx - 2M(t) \int_0^1 w_{xx}xw_{xxx}dx \]

\[ = 2 \frac{d}{dt} \int_0^1 w_{tt}xw_{xxx}dx - 2 \int_0^1 w_{tt}xw_{xxx}dx \]

\[ + 2 \int_0^1 w_{xxxx}xw_{xxx}dx - 2M(t) \int_0^1 w_{xx}xw_{xxx}dx = 0. \]

Here

\[ 2 \int_0^1 w_{tt}xw_{xxx}dx = 2w_t(1, t)w_{xxt}(1, t) + 3\|w_{xt}(t)\|^2, \]

\[ 2 \int_0^1 w_{xxxx}xw_{xxx}dx = w_{xxx}^2(1, t) - \|w_{xxx}(t)\|^2, \]

\[ 2M(t) \int_0^1 w_{xx}xw_{xxx}dx = M(t)w_{xxx}^2(1, t) - M(t)\|w_{xx}(t)\|^2 \]
Thus
\[ 2 \frac{d}{dt} \int_{0}^{1} w_{1x} w_{xxx} dx - 2 w_{1}(1, t) w_{xxx}(1, t) - 3 \| w_{xt}(t) \|^2 + w_{xxx}(1, t) - \| w_{xxx}(t) \|^2 - M(t) w_{xx}^2(1, t) + M(t) \| w_{xx}(t) \|^2 = 0. \]

We obtain the estimate
\[
-2 \frac{d}{dt} \int_{0}^{1} w_{1x} w_{xxx} dx + \frac{1}{2} \| w_{xx}(t) \|^2 + \frac{1}{2} \| w_{xt}(t) \|^2 + \frac{1}{2} \| w_{xxx}(t) \|^2 + \frac{1}{2} M(t) \| w_{xx}(t) \|^2 \\
-2 w_{1}(1, t) w_{xxx}(1, t) - 3 \| w_{xt}(t) \|^2 + w_{xxx}(1, t) \\
- \| w_{xxx}(t) \|^2 - M(t) w_{xx}^2(1, t) + M(t) \| w_{xx}(t) \|^2 \\
\leq -2 w_{1}(1, t) w_{xxx}(1, t) + w_{xxx}(1, t) + \frac{3}{2} M(t) \| w_{xx}(t) \|^2 \\
\leq w_{1}^2(1, t) + w_{xxx}^2(1, t) + w_{xxx}^2(1, t) + \frac{3}{2} M(t) \| w_{xx}(t) \|^2 \tag{39}
\]

From (38) and (39) we have
\[
\frac{d}{dt} \left\{ E_{2}(t) - 2 \epsilon_{2} \int_{0}^{1} w_{1x} w_{xxx} dx \right\} + \frac{\epsilon_{2}}{2} E_{2}(t) \\
+ \left\{ \frac{2}{k(t)} - \frac{\epsilon_{2}}{k(t)} - 4 \epsilon_{2} - \frac{2 \epsilon_{2}}{k^2(t)} \right\} w_{xxx}^2(1, t) + \left[ \frac{\epsilon_{2}}{2} - M(t) \right] \| w_{xx}(t) \|^2 \\
\leq G_{1}(t) + \epsilon_{2} w_{1}^2(1, t) + \frac{3 \epsilon_{2}}{2} M(t) \| w_{xx}(t) \|^2 \\
+ 2 \epsilon_{2} g^2(k, \theta, \eta) + \frac{2 \epsilon_{2}}{k^2(t)} \left[ M^2(t) w_{1}^2(1, t) + \dot{M}^2(t) w_{1}^2(1, t) + k^2(t) z_{1}^2(1, t) \right] \\
= G_{1}(t) + \frac{3 \epsilon_{2}}{2} M(t) \| w_{xx}(t) \|^2 + G_{2}(t), \tag{40}
\]

where
\[
G_{2}(t) = \epsilon_{2} w_{1}^2(1, t) + 2 \epsilon_{2} g^2(k, \theta, \eta) \\
+ \frac{2 \epsilon_{2}}{k^2(t)} \left[ M^2(t) w_{1}^2(1, t) + \dot{M}^2(t) w_{1}^2(1, t) + k^2(t) z_{1}^2(1, t) \right].
\]

Putting
\[
E_{1}^{*}(t) = E_{1}(t) - 2 \epsilon_{1} \int_{0}^{1} z_{1} x z_{xxx} dx, \\
E_{2}^{*}(t) = E_{2}(t) - 2 \epsilon_{2} \int_{0}^{1} w_{1x} w_{xxx} dx,
\]

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we obtain from (33) and (40)

\[
\frac{d}{dt} E_1^*(t) + \epsilon_1 E_1(t) + \delta_1 \left\{ \frac{d}{dt} E_2^*(t) + \frac{\epsilon_2}{2} E_2(t) \right\} \\
+ \delta_1 \left\{ \frac{1}{k(t)} - \frac{\epsilon_2}{k(t)} - 4\epsilon_2 - \frac{2\epsilon_2}{k^2(t)} \right\} w_{xx}^2(1, t) \\
+ \delta_1 [\frac{\epsilon_2}{2} - \dot{M}(t)] \|w_{xx}(t)\|^2 - \frac{3\delta_1 \epsilon_2}{2} M(t) \|w_{xx}(t)\|^2 \\
\leq F(t) + \delta_1 G_1(t) + \delta_1 G_2(t) \equiv G(t). \quad (41)
\]

If

\[
0 < \epsilon_2 < \min \left\{ \frac{k_0}{8\gamma E(0) + \sqrt{2}\gamma E(0) + 2}, \frac{\epsilon_1}{3\delta_1 \sqrt{a^2 + 4bE(0)}} \right\} \equiv \epsilon_2^*, \quad (42)
\]

using the relations

\[
k_0 \leq k(t) \leq \sqrt{2\gamma E(0)}, \quad a \leq M(t) \leq \sqrt{a^2 + 4bE(0)},
\]

we have from (41)

\[
\frac{d}{dt} E_1^*(t) + \frac{\epsilon_1}{2} E_1(t) + \delta_1 \left\{ \frac{d}{dt} E_2^*(t) + \frac{\epsilon_2}{2} E_2(t) \right\} \\
+ \delta_1 [\frac{\epsilon_2}{2} - \dot{M}(t)] \|w_{xx}(t)\|^2 \leq G(t)
\]

Since

\[
- \int_0^1 z_{ttx} z_{xxx} dx \leq \frac{1}{2} (\|z_t(t)\|^2 + \|z_{xxx}(t)\|^2) \leq \frac{1}{2} E_1(t),
\]

we see that if \( \epsilon_1 \leq 1/2, \)

\[
\frac{1}{2} E_1(t) \leq E_1^*(t) \leq \frac{3}{2} E_1(t). \quad (43)
\]

Similarly if \( \epsilon_2 \leq 1/2, \)

\[
\frac{1}{2} E_2(t) \leq E_2^*(t) \leq \frac{3}{2} E_2(t).
\]

Thus we have for \( 0 < \epsilon_1 < 1/2 \) and \( 0 < \epsilon_2 < \min\{1/2, \epsilon_2^*\} \)

\[
\frac{d}{dt} E_1^*(t) + \frac{\epsilon_1}{4} E_1^*(t) + \delta_1 \left\{ \frac{d}{dt} E_2^*(t) + \frac{\epsilon_2}{4} E_2^*(t) \right\} \\
+ \delta_1 [\frac{\epsilon_2}{2} - \dot{M}(t)] \|w_{xx}(t)\|^2 \leq G(t). \quad (44)
\]
Now we shall show that
\[
\frac{\epsilon_2}{2} - \dot{M}(t) \geq 0 \quad \text{for } t \geq 0, \tag{45}
\]
where
\[
\dot{M}(t) = 2b \int_0^1 z_x(t)z_{xt}(t)dx \leq 2b\|z_x(t)\|\|w_t(t)\|.
\]
Since if \(\epsilon_2 > 4\sqrt{2a^{-1/2}b}\|z_x\|\sqrt{E(0)}\) we see that
\[
\frac{\epsilon_2}{2} - \dot{M}(0) > 0,
\]
where \(B_1\) is a positive constant such that \(E_1(t) \leq B_1^2\) for \(t \geq 0\). There exists \(T^* > 0\) such that
\[
\frac{\epsilon}{2} - \dot{M}(t) \geq 0, \quad 0 \leq t < T^*.
\]
Then we have from (44)
\[
\frac{d}{dt} E_1^*(t) + \frac{\epsilon_1}{4} E_1^*(t) + \delta_1 \left\{ \frac{d}{dt} E_2^*(t) + \frac{\epsilon_2}{4} E_2^*(t) \right\} \leq G(t)
\]
for \(0 \leq t < T^*\). From this, putting \(\epsilon = \min\{\epsilon_1, \epsilon_2\}\), we have the estimate for \(0 \leq t < T^*\)
\[
E_1^*(t) + \delta_1 E_2^*(t) \leq \exp(-\frac{\epsilon}{4} t)\{E_1^*(0) + \delta_1 E_2^*(0)\}
\]
\[
+ \int_0^t \exp\{-\frac{\epsilon}{4} (t - \tau)\} G(\tau) d\tau. \tag{46}
\]
For \(E_1(t)\) and \(E_2(t)\) it holds that
\[
E_1(t) + \delta_1 E_2(t) \leq 3\exp(-\frac{\epsilon}{4} t)\{E_1(0) + \delta_1 E_2(0)\}
\]
\[
+ 2\int_0^t \exp\{-\frac{\epsilon}{4} (t - \tau)\} G(\tau) d\tau. \tag{47}
\]
Thus we obtain the existence of constants \(B_1 > 0\) and \(B_2 > 0\) such that
\[
E_1(t) = \|z_{xt}(t)\|^2 + \|z_{xxx}(t)\|^2 + \dot{M}(\|z_{xx}(t)\|^2) \leq B_1^2, \tag{48}
\]
\[
E_2(t) = \|z_{xxt}(t)\|^2 + \|z_{xxxx}(t)\|^2 + M(t)\|z_{xxx}(t)\|^2 \leq B_2^2 \tag{49}
\]
for \(0 \leq t < T^*\). From this and (28)
\[
\dot{M}(t) = 2b \int_0^1 z_x(t)w_t(t)dx \leq 2b\|z_x\|\|w_t\| \leq 2bB_1\sqrt{\frac{2}{a}} E(0)
\]
Then for $0 \leq t < T^*$
\[
\frac{\varepsilon_2}{2} - \dot{M}(t) \geq \frac{\varepsilon_2}{2} - 2bB_1 \sqrt{\frac{2}{a}} E(0) > 0,
\]
if the initial data $z^0, z^1, k_0, \theta_0$ satisfy the smallness condition
\[
\varepsilon_2 > 4\sqrt{2a^{-1/2}}bB_1 \sqrt{E(0)} \quad (50)
\]
for
\[
0 < \epsilon_1 < 1/2, \quad 0 < \epsilon_2 < \min\{1/2, \epsilon_2^*\}. \quad (51)
\]
Therefore we deduce that $T^* = \infty$ [20, 22]. That is, (45) and hence (48) and (49) hold for $t \geq 0$. These are the desired boundedness estimates under the assumption of the smoothness and smallness of the initial data.

5 Asymptotic stability of the closed-loop system

For asymptotic stability of the closed-loop system we can show the following theorem.

**Theorem 2** Suppose that the initial data $z^0 \in H^5(0,1) \cap W$, $z^1 \in H^4(0,1) \cap W$, $\theta_0$ and a positive initial data $k_0$ satisfy the compatibility conditions (8), (34) and the smallness conditions (50) and (51). Moreover for the system
\[
\begin{aligned}
z(x,t) + z_{xxx}(x,t) &= M \left( \int_0^1 z_x^2(x,t) dx \right) zxx(x,t), \quad 0 < x < 1, \quad t > 0, \\
z(0,t) &= z_x(0,t) = z_{xx}(1,t) = 0, \quad t \geq 0, \\
z_{xxx}(1,t) - M \left( \int_0^1 z_x^2(x,t) dx \right) z_x(1,t) &= 0, \quad t \geq 0, \\
z_t(1,t) &= \theta^T \eta(t), \quad t \geq 0.
\end{aligned}
\] (52)

and a signal $\eta(t)$, suppose that the measurement data
\[
y_d(t) = z_t(1,t) + \theta^T \eta(t) = 0, \quad t \geq 0
\]
implies $\theta^* = 0$, where $\theta^*$ is an $l$-dimensional constant vector. Then the closed-loop system (6) is asymptotically stable such that
\[
\begin{aligned}
\lim_{t \to \infty} \left\{ \|z_t(t)\|^2 + \|z_{xx}(t)\|^2 \right\} &= 0, \\
\sup_{t \geq 0} k(t) &< \infty, \\
\lim_{t \to \infty} \theta(t) &= \bar{\theta}.
\end{aligned}
\] (53)

[Proof] First we introduce another energy-like function
\[
E_k(t) = \frac{1}{2} \|z_t(t)\|^2 + \frac{1}{2} \|z_{xx}(t)\|^2 + \frac{1}{2} \dot{M}(\|z_x(t)\|^2) + \frac{1}{2\gamma} [k(t) - \bar{k}]^2 + \frac{1}{2} \|\theta(t) - \bar{\theta}\|^2,
\]

\[\text{for} \quad 0 < \epsilon_1 < 1/2, \quad 0 < \epsilon_2 < \min\{1/2, \epsilon_2^*\}. \quad (51)\]
where \( \tilde{k} \geq k_0 \). Along the solution of the system (6) we obtain

\[
\dot{E}_k(t) = \int_0^1 \dot{z}_u(t)z_t(t)dx + \int_0^1 z_{xxt}(t)z_{xx}(t)dx + M(\|z_x(t)\|^2) \int_0^1 z_x(t)z_{xt}(t)dx \\
+ [k(t) - \tilde{k}]e^2(t) + k(t)[\theta(t) - \tilde{\theta}]^T \eta(t)e(t) \\
= -M(\|z_x(t)\|^2) \int_0^1 z_x(t)z_{xt}(t)dx \\
- k(t)e(t)z_t(1,t) - M(\|z_x(t)\|^2) \int_0^1 z_x(t)z_{xt}(t)dx \\
+ [k(t) - \tilde{k}]e^2(t) + k(t)[\theta(t) - \tilde{\theta}]^T \eta(t)e(t) = -\tilde{k}e^2(t).
\]

We have a dynamical system on \( V \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}^l \) with all orbits bounded. The injection of \( W \times V \times \mathbb{R} \times \mathbb{R}^l \) from \( V \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}^l \) is compact and the bounded set of \( W \times V \times \mathbb{R} \times \mathbb{R}^l \) is precompact in \( V \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}^l \) [18, 23]. Since (48) and (49) hold for any \( t \geq 0 \), it follows that each orbit is bounded in \( W \times V \times \mathbb{R} \times \mathbb{R}^l \). So each orbit is precompact in \( V \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}^l \).

According to LaSalle’s invariance principle [18, 23], all solutions of (6) asymptotically tend to the maximal invariant set of the following set

\[
S = \{(z, \dot{z}, k, \theta) | \dot{E}_k = 0\},
\]

if the solution trajectories for \( t \geq 0 \) are precompact in \( V \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}^l \). The proof will be accomplished if it is shown that \( S \) contains only single point \((0,0,\tilde{k},\tilde{\theta})\).

From \( \dot{E}_k = 0 \) it results in \( z_t(1,t) - (\theta(t) - \tilde{\theta})^T \eta(t) = 0 \) which implies that \( \dot{k}(t) = 0, \dot{\theta}(t) = 0 \).

The system (6) reduces to

\[
\begin{align*}
z_{tt}(x,t) + z_{xxxx}(x,t) &= M \left( \int_0^1 z_x^2(x,t)dx \right) z_{xx}(x,t) \\
z(0,t) &= z_x(0,t) = z_{xx}(1,t) = 0, \\
z_{xxx}(1,t) - M \left( \int_0^1 z_x^2(x,t)dx \right) z_x(1,t) &= 0 \\
z_t(1,t) &= (\theta - \tilde{\theta})^T \eta(t).
\end{align*}
\]

The assumption of the theorem implies that \( \theta = \tilde{\theta} \). Consequently the system (6) reduces to

\[
\begin{align*}
z_{tt}(x,t) + z_{xxxx}(x,t) &= M \left( \int_0^1 z_x^2(x,t)dx \right) z_{xx}(x,t) \\
z(0,t) &= z_x(0,t) = z_{xx}(1,t) = 0, \\
z_{xxx}(1,t) - M \left( \int_0^1 z_x^2(x,t)dx \right) z_x(1,t) &= 0 \\
z_t(1,t) &= 0.
\end{align*}
\]

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To prove that the system (6) is asymptotically stable, it is sufficient to show that the system (56) has only the zero solution. This is the observability problem for the controlled system (1)-(4).

Here introduce another energy-like functions such that

\[
\Phi(t) = \frac{1}{2} \| z_t(t) \|^2 + \frac{1}{2} \| z_{xx}(t) \|^2 + \frac{1}{2} \hat{M}(\| z_x(t) \|)^2),
\]

(56)

\[
\Psi(t) = \Phi(t) + \epsilon g(t), \quad \epsilon > 0,
\]

(57)

where \( g(t) \) is a multiplier function given by

\[
g(t) = \int_0^1 x z_t(x, t) z_x(x, t) dx.
\]

(58)

Firstly, from \( \dot{\Phi}(t) = 0 \) it follows that \( \Phi(t) \equiv \text{const.} \). Let us estimate \( g(t) \). Since

\[
\int_0^1 x z_t z_x dx
\]

\[
\leq \int_0^1 | z_t | | z_x | dx \leq \frac{1}{2} \int_0^1 z_t^2 dx + \frac{1}{2} \int_0^1 z_x^2 dx
\]

\[
\leq \frac{1}{2} \int_0^1 z_t^2 dx + \frac{1}{2} \int_0^1 z_x^2 dx
\]

there exists a constant \( c_g > 0 \) such that

\[
| g(t) | \leq c_g \Phi(t) \text{ for any } t \geq 0,
\]

(59)

where \( c_g = \max(1/2, 1/2a) \).

Thus for \( 0 < \epsilon < 1/c_g \) the function \( \Psi(t) \) satisfies

\[
0 \leq (1 - \epsilon c_g) \Phi(t) \leq \Psi(t) \leq (1 + \epsilon c_g) \Phi(t) \text{ for all } t \geq 0,
\]

(60)

which implies that \( \Psi(0) \geq 0 \) if \( 0 < \epsilon < 1/c_g \).

On the other hand

\[
\dot{g}(t) = \int_0^1 x z_{tt} z_x dx + \int_0^1 z_t z_{xt} dx.
\]

For each terms on the right hand side

\[
\int_0^1 x z_{tt} z_x dx = \int_0^1 x z_x (-z_{xxxx}) dx + \int_0^1 x z_x M(\| z_x(t) \|)^2 z_{xx} dx
\]

\[
= -z_{xxx}(1, t) z_x(1, t) + \int_0^1 (z_x + x z_{xx}) z_{xx} dx
\]

\[22\]
\[\begin{align*}
& + \frac{1}{2} M(\|z_x(t)\|^2) z_x^2(1, t) - \frac{1}{2} M(\|z_x(t)\|^2) \int_0^1 z_x^2 \, dx \\
& = - \frac{1}{2} M(\|z_x(t)\|^2) z_x^2(1, t) - \frac{3}{2} \int_0^1 z_{xx}^2 \, dx - \frac{1}{2} M(\|z_x(t)\|^2) \int_0^1 z_x^2 \, dx,
\end{align*}\]

\[\int_0^1 x t z_{xt} \, dx = \frac{1}{2} \int_0^1 (xz_t^2) \, dx - \frac{1}{2} \int_0^1 z_t^2 \, dx \]

\[= - \frac{1}{2} \int_0^1 z_t^2 \, dx.\]

Using these relations we obtain

\[\dot{g}(t) \leq -\frac{3}{2} \int_0^1 z_{xx}^2 \, dx - \frac{1}{2} M(\|z_x(t)\|^2) \int_0^1 z_x^2 \, dx - \frac{1}{2} \int_0^1 z_t^2 \, dx \]

\[\leq -\delta \Phi(t).\]

for some \(\delta > 0\). Since from (60) and (61)

\[\dot{\Psi}(t) = \dot{\Phi}(t) + \epsilon \dot{g}(t)\]

\[\leq -\epsilon \delta \Phi(t)\]

\[\leq -\frac{\epsilon \delta}{1 + \epsilon c_g} \Psi(t)\]

\[= -K_\epsilon \Psi(t),\]  

(61)

we have

\[0 \leq \Psi(t) \leq e^{-K_\epsilon t} \Psi(0) \text{ for all } t > 0,\]

\[0 \leq \Phi(t) \leq \frac{1}{1 - \epsilon c_g} e^{-K_\epsilon t} \Psi(0) \text{ for all } t > 0.\]  

(62)

This implies that \(\Phi(t) \equiv const. = 0\), from which it follows that \(z_t(x, t) \equiv 0, z_x(x, t) \equiv 0, z_{xx}(x, t) \equiv 0\). Moreover since \(z(0, t) = 0\) for all \(t \geq 0, |z(x, t)| \leq \|z_x(t)\|\). we conclude that \(z(x, t) \equiv 0\). We have proved the theorem.

6 Conclusion

In this paper we have considered adaptive regulator design for an undamped non-linear beam with disturbed outputs by boundary feedback control. The adaptive controller has been constructed by the concept of high-gain adaptive feedback and the estimation mechanism for the unknown parameters of the disturbances. The global existence and
uniqueness of the solution of the closed-loop system has been justified. The stability of the closed-loop system has been proved such that the convergence of the system state to zero and the convergence of the estimated parameter to the unknown parameter are guaranteed for the smooth and small initial data.

References


