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Blowup in infinite time of radial solutions for a parabolic-elliptic system in high dimensional Euclidean spaces

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Abstract

We consider radial solutions blowing up in infinite time to a parabolic-elliptic system in $N$-dimensional Euclidean space. The system was introduced to describe the gravitational interaction of particles. In the case where $N \geq 2$, we can find positive and radial solutions blowing up in finite time. In the present paper, in the case where $N \geq 11$, we find positive and radial solutions blowing up in infinite time and investigate those blowup speeds, by using so called asymptotic matched expansion techniques and parabolic regularity.

Keywords: parabolic-elliptic system; Keller-Segel system; blowup in infinite time; growup

1 Introduction

In this paper, we consider the system

\[
\begin{aligned}
U_t &= \nabla \cdot (\nabla U - U \nabla V) \quad \text{in } \mathbb{R}^N \times (0, T), \\
0 &= \Delta V + U \quad \text{in } \mathbb{R}^N \times (0, T), \\
U(\cdot, 0) &= U_0 \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]  

(1)

Here, $U_0$ is radial, nonnegative and continuous in $\mathbb{R}^N$ with $N \geq 1$, and satisfies $\sup_{x \in \mathbb{R}^N} |x|^2 U_0(x) < \infty$. Then, we consider classical solutions $(U, V)$ to (1) such that $U(\cdot, t)$ is radial, nonnegative and bounded in $\mathbb{R}^N$ for each $t \in (0, \infty)$.

The system (1) was introduced to describe the gravitational interaction of particles (see [2]).

In this paper, we consider blowup in infinite time of solutions to (1). We say that a solution $(U, V)$ to (1) blows up at $x = q$ and $t = T$, if there exists a sequence $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathbb{R}^N \times (0, T)$ such that $\lim_{n \to \infty} (x_n, t_n) = (q, T)$ and that $\lim_{n \to \infty} |U(x_n, t_n)| = \infty$. Then, we say that the point $q$ and the

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time $T$ are a blowup point and a blowup time, respectively. Furthermore, we say that a solution blows up in finite time and infinite time, if the blowup time $T$ is finite and infinite, respectively.

In the case where $N \geq 2$, the system (1) has solutions blowing up in finite time (see [5, 14, 15, 11]). In particular, Mizoguchi and the author [11] constructed radial and positive solutions blowing up in finite time, by using so called asymptotic matched expansion techniques and parabolic regularity.

In this paper, we use an argument similar to the one in [11].

Concerning blowup in infinite time, some related results are treated in [3, 4, 12]. In [3], radial and positive solutions to (1) globally exist in time, if $|U_0|_1 < 8\pi$. The solutions satisfy that $|U(\cdot, t)|_1 = |U_0|_1$ for $t > 0$. In particular, those solutions are bounded, if $|U_0|_1 < 8\pi$. Concerning non radial case, Dolbeault and Perthame [4] showed that positive solutions to (1) globally exist in time, if $|U_0|_1 < 8\pi$.

Here and henceforth, for $p \in [1, \infty]$, $\cdot |_p$ denotes the standard $L^p(\mathbb{R}^N)$ norm.

In [12], radial solutions $(U, V)$ to the system

$$\begin{cases}
U_t = \nabla \cdot (\nabla U - U \nabla V) & \text{in } B_R \times (0, \infty), \\
0 = \Delta V + U & \text{in } B_R \times (0, \infty), \\
\partial U/\partial \nu - U \partial V/\partial \nu = 0 & \text{on } \partial B_R \times (0, \infty), \\
V = 0 & \text{on } \partial B_R \times (0, \infty), \\
U(\cdot, 0) = U_0 \geq 0 & \text{in } B_R
\end{cases}$$

blow up in infinite time, if $|U_0|_1 = 8\pi$, where $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$ and $R \in (0, \infty)$.

However, in the case of $N \geq 3$, there is few related results.

In this paper, we construct radial solutions to (1) with $N \geq 11$ which blow up in infinite time, and we investigate those blowup speeds. For $N \geq 11$ and $J \geq 0$, let us put

$$\nu = -\frac{(N + 2) + \sqrt{(N - 10)(N - 2)}}{4} \quad (2)$$

and

$$a_J = \frac{N + 2J + 6\nu + 4}{-2\nu - 2}.$$  

**Theorem 1** For each nonnegative even integer $J$, there exist a classical solution $(U, V)$ to (1) which satisfies that

$$\sup_{(x, t) \in \mathbb{R}^N \times [0, \infty)} |x|^2 U(x, t) < \infty$$

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and that

\[ ct^{a_J} \leq |U(\cdot, t)|_\infty \leq Ct^{a_J} \]

for any sufficiently large \( t \)

with some positive constants \( c \) and \( C \).

**Theorem 2** Let \( U_\infty(x) = 2(N - 2)/|x|^2 \), and let \( U_0 \) be a radial, nonnegative and continuous function such that \( U_0 \leq U_\infty \) in \( \mathbb{R}^N \). Then, the solution \((U, V)\) to (1) exists globally in time, and there exists a positive constant \( C \) such that

\[ |U(\cdot, t)|_\infty \leq Ct^{a_0} \]

for any sufficiently large \( t \).

We have \( a_0 = 0 \) in the case of \( N = 11 \). Then, the solutions \((U, V)\) in Theorem 2 satisfy that \( \sup_{(x,t) \in \mathbb{R}^{11} \times (0, \infty)} U(x, t) < \infty \), in the case of \( N = 11 \).

In order to find those solutions, we use an argument similar to the one in [10]. Mizoguchi [10] showed the existence of radial solutions \( u \) to

\[ u_t = \Delta u + u^p \quad \text{in} \quad \mathbb{R}^N \times (0, T) \quad (3) \]

blowing up in infinite time for \( p > (N - 2\sqrt{N - 1})/(N - 4 - 2\sqrt{N - 1}) \) and \( N \geq 11 \). That is to say, the global strategy of [9] and the one of this paper are similar. Those strategies are as follows.

In the present paper, we first put

\[ m(y, s) = (t + 1)M(r, t) = \frac{(t + 1)}{\omega_N r^N} \int_{|x| < r} U(x, t) dx \]

with

\[ s = \log(t + 1) \quad \text{and} \quad y = r/\sqrt{t + 1} \]

for a radial solution \((U, V)\) to (1), where \( \omega_N \) is the area of a unit sphere in \( \mathbb{R}^N \). Then, \( m \) satisfies that

\[ L(m) = m_s - m_{yy} - \left( \frac{N + 1}{y} + \frac{y}{2} \right) m_y - m (ym_y + Nm) \quad \text{in} \quad \mathbb{R}^+ \times (0, \infty), \]

\[ m_y(0, \cdot) = 0 \quad \text{in} \quad (0, \infty) \quad (4) \]

and that

\[ m(\cdot, 0) = M(\cdot, 0) \quad \text{in} \quad \mathbb{R}^+, \]

where \( \mathbb{R}_+ = (0, \infty) \).

In order to find a solution to (1) blowing up in infinite time, we consider the solution \( m \) to (4) with (5) and (6) which satisfies that \( \lim_{s \to \infty} m(y, s) = \frac{3}{\omega_N} \).
\[ \frac{2}{y^2} \] for \( y > 0 \). Let us put \( m_\infty(y) = \frac{2}{y^2} \). The function \( m_\infty \) is a singular stationary solution to (4). For a solution \( m \) to (4) with (5) and (6), let us put \( \phi(y, s) = e^{y^2/4(m(y, s) - m_\infty(y))} \). Then, \( \phi \) satisfies that

\[
\begin{cases}
\phi_s = -\mathcal{A}\phi + F(\cdot, \phi) & \text{in } \mathbb{R}_+ \times (0, \infty), \\
\phi(y, 0) = \exp \left( \frac{y^2}{4} \right) \{ m(y, 0) - m_\infty(y) \} & \text{in } \mathbb{R}_+,
\end{cases}
\]

where

\[
\mathcal{A}\varphi = -\frac{d^2\varphi}{dy^2} - \frac{N + 3}{y} \frac{d\varphi}{dy} + \frac{y}{2} \frac{d\varphi}{dy} + \frac{N + 2}{2} \varphi - \frac{4(N - 1)}{y^2} \varphi,
\]

\[
F(y, \varphi) = e^{y^2/4} \left[ \frac{y}{2} \frac{d}{dy} \left\{ e^{-y^2/4} \varphi \right\}^2 + N \left\{ e^{-y^2/4} \varphi \right\}^2 \right].
\]

The operator \( \mathcal{A} \) is a self-adjoint operator in a suitable function space. Let \( \{ \lambda_j \}_{j=0}^\infty \) be eigenvalues of the operator, and let \( \{ \varphi_j \}_{j=0}^\infty \) be eigenfunctions corresponding to the eigenvalues. Then, the solution \( \phi \) to (7) has the representation

\[
\phi(y, s) = \sum_{j=0}^\infty a_j(s) \varphi_j(y).
\]

In order to show that \( \phi(\cdot, s) \) is in the suitable function space for \( s > 0 \), we need an estimate of \( \phi \) under some conditions (see Lemma 2.3). Furthermore, we see that \( \lambda_j > 0 \) for \( j \geq 0 \) (see Section 2). By using those, we can find a solution \( \phi \) satisfying

\[
\phi(\cdot, s) \to 0 \quad \text{as } s \to \infty \quad \text{in the region } y \geq Ke^{-\eta s}.
\]

Since \( m_\infty \) is not smooth at \( y = 0 \), then an argument similar to establish (9) does not work in the region \( y \leq Ke^{-\eta s} \). Then, we construct a solution \( \phi \) in the region \( y \leq Ke^{-\eta s} \), by using rescaled stationary solutions and the comparison theorem. Furthermore, we can connect the solution in the region \( y \geq Ke^{-\eta s} \) and the one in the region \( y \leq Ke^{-\eta s} \). Then, we can get a desired solution \( \phi \) in \( \mathbb{R}_+ \).

That is the global strategy of the present paper, which is similar to the one of [10].

In [10], Mizoguchi showed some estimates of \( w(y, s) = (t + 1)^{1/(p-1)}u(x, t) \) and \( q(y, s) = e^{y^2/4}(w(y, s) - w_\infty(y)) \), in order to get the results, where \( u \) is a radial solution to (3) and \( w_\infty \) is a singular stationary solution to (3). The function \( q \) satisfies that

\[
q_s + Bq - h(\cdot, q) = 0,
\]
where
\[ Bq = -\frac{d^2 q}{dy^2} - \left( \frac{N - 1}{r} - \frac{y}{2} \right) \frac{dq}{dy} + \left( \frac{N}{2} - \frac{1}{p - 1} \right) q - pw_{\infty}^{p-1}q \]
and
\[ h(y, q) = e^{y^2/4} \left[ (e^{-y^2/4}q + w_{\infty})^p - w_{\infty}^p \right] - pw_{\infty}^{p-1}q. \]

Then, those estimates follow from the representation
\[ q(\cdot, s) = e^{-sB}q(\cdot, 0) + \int_0^s e^{-(s-\tilde{s})B}h(\cdot, q(\cdot, \tilde{s}))d\tilde{s} \]
and some estimates of the kernel of \( e^{-sB} \). In this paper, some estimates of the solution \( \phi \) to (7) are necessary for the proof of our theorems, and those estimates follow from the representation
\[ \phi(\cdot, s) = e^{-sA}\phi(\cdot, 0) + \int_0^s e^{-(s-\tilde{s})A}F(\cdot, \phi(\cdot, \tilde{s}))d\tilde{s} \]
and some estimates of the kernel of \( e^{-sA} \) and its differentiation. Because \( F \) includes the differentiation of \( \phi \). Moreover, we must estimate the differentiation of \( \phi \) in order to show our theorems. Then, our proof is more complicated than the one in [10].

2 Proof of Main theorems

Here and henceforth, we treat only radial and classical solutions \((\overline{U}(r, t), \overline{V}(r, t)) = (U(x, t), V(x, t))\) to (1), where \( r = |x| \). We first transform the original equation (1). Putting
\[ M(r, t) = \frac{1}{r^N} \int_0^r \overline{U}(\rho, t)\rho^{N-1}d\rho, \]
\( M \) satisfies
\[ M_t = M_{rr} + \frac{N + 1}{r} M_r + M(rM_r + NM) \quad \text{in } \mathbb{R}_+ \times (0, \infty), \quad (11) \]
\[ M_r(0, \cdot) = 0 \quad \text{in } (0, \infty), \]
where \( \mathbb{R}_+ = (0, \infty) \). \( M_\infty(r) = 2/r^2 \) is a singular stationary solution to (11).

Putting
\[ m(y, s) = (t + 1)M(r, t) \quad (12) \]
with
\[ y = \frac{r}{\sqrt{t + 1}} \quad \text{and} \quad s = \log(t + 1) \]
for a solution $M$ to (11), $m$ satisfies (4). Since $M_\infty$ is a singular stationary solutions to (11),

$$m_\infty(y) = \frac{2}{y^2} = (t + 1)M_\infty(r)$$  \hspace{1cm} \text{(13)}$$

is a singular stationary solution to (4). Let

$$L_w^2 = \left\{ h \in L^2_{loc}(\mathbb{R}_+) \mid \int_0^\infty h(y)^2 y^{N+3} e^{-y^2/4} \, dy < \infty \right\}$$

and

$$H_w^1 = \left\{ h \in L^2_{loc}(\mathbb{R}_+) \mid h, h' \in L_w^2 \right\}.$$  

Denote by $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ the natural inner product and norm in $L_w^2$, respectively. The following facts were originally given in [11, 15] for readers’ convenience: if $N \geq 11$, then the spectrum of the operator $A$ in (7) consists of countable eigenvalues $\{\lambda_j\}_{j=0}^\infty$ so that

$$\lambda_j = j + \frac{N + 2}{2} + 2\nu \quad \text{for } j = 0, 1, 2, \ldots$$

with $\nu$ in (2) and

$$\varphi_j(y) = \sqrt{\frac{j!}{2^{N+4} \Gamma(j + \alpha + 1)}} \left( \frac{y^2}{4} \right)^\nu L_\alpha^\nu \left( \frac{y^2}{4} \right)$$

is its corresponding eigenfunction satisfying $\|\varphi_j\| = 1$. Here,

$$\alpha = 2\nu + \frac{N + 2}{2} = \frac{1}{2} \sqrt{N(N - 10)(N - 2)}$$

and $L_\alpha^\nu$ is the associated Laguerre polynomial satisfying $L_\alpha^\nu(0) > 0$ (see [1]). Then, it holds that

$$\varphi_j(y) = \gamma_j y^{2\nu} + o(y^{2\nu}) \quad \text{as } y \to 0$$  \hspace{1cm} \text{(14)}$$

and

$$\varphi_j(y) = \tilde{\gamma}_j y^{2\nu+2j} + o(y^{2\nu+2j}) \quad \text{as } y \to \infty$$

for some $\gamma_j > 0$ and $(-1)^j \tilde{\gamma}_j > 0$. Let $\nabla(\cdot ; \ell)$ be a unique solution of

$$-\nabla_{rr} - \frac{N - 1}{r} \nabla_r = e^\nu \quad \text{in } \mathbb{R}_+$$

with $\nabla(0) = \ell$ and $\nabla_r(0) = 0$.
for \( \ell > 0 \). It follows from [11, Lemma 2.5] that \( V \) is increasing with respect to \( \ell > 0 \),

\[
U(r; \ell) = \exp \left( \frac{\ell - 2}{\ell^2} - \gamma(\ell) (1 + o(1)) r^{-2\nu} \right) \quad \text{for } r \gg 1
\]

for some \( \gamma(\ell) > 0 \) and that for \( J \geq 2 \) and \( 0 < \varepsilon \ll 1 \) there exist a positive constants \( \ell_+ > 0 \) and \( \ell_+^* > 0 \) satisfying \( \gamma(\ell_+) = \gamma_J + 3\varepsilon \) and \( \gamma(\ell_+^*) = \gamma_J - 3\varepsilon \), where \( \nu \) is the constant in (2) and \( \gamma_J \) is the constant in (14) with \( j = J \).

Here and henceforth, for positive constants \( C \) and \( \tilde{C} \) we write \( C \lesssim \tilde{C} \), if \( \tilde{C} / C \) is sufficiently large.

Let \( \chi = \frac{(2\nu + 1)(2\nu + 2)}{2\nu + 1} \) \( \eta = \lambda_j / (-2\nu - 1) \), \( \tilde{U}_1(y) = U(y; \ell_+), \tilde{U}_2(y) = U(y; \ell_+^*) \), \( m_1(y, s) = (1 + \varepsilon K^{2\nu + 2} e^{-\eta s}) \frac{e^{2\chi_\eta s}}{(e^{\eta s} y)^N} \int_0^{e^{\eta s} y} \tilde{U}_1(\rho) \rho^{N-1} d\rho \)

and

\[
m_2(y, s) = (1 - \varepsilon K^{2\nu + 2} e^{-\eta s}) \frac{e^{2\chi_\eta s}}{(e^{\eta s} y)^N} \int_0^{e^{\eta s} y} \tilde{U}_2(\rho) \rho^{N-1} d\rho.
\]

Then, the following lemma is shown by an argument similar to the one in [11, Lemma 2.6].

**Lemma 2.1** For \( y \in (0, Ke^{-\eta s}] \) and \( s \geq s_0 \), it holds that

\[
0 < m_1(y, s) < m_2(y, s) < m_\infty(y)
\]

and that

\[
0 < m_2(y, s) - m_1(y, s) < C \varepsilon m_1(y, s),
\]

if \( 0 < \varepsilon \ll 1 \) and \( s_0 \gg 1 \).

For \( k \in [\tilde{K}, K] \), it holds that

\[
m_1(ke^{-\eta s}, s) \leq m_\infty(ke^{-\eta s}) - (\gamma_J + 2\varepsilon) e^{-\lambda_J s} k^{2\nu} e^{\eta s}
\]

and that

\[
m_2(ke^{-\eta s}, s) \geq m_\infty(ke^{-\eta s}) - (\gamma_J - 2\varepsilon) e^{-\lambda_J s} k^{2\nu} e^{\eta s},
\]

if \( 0 < \varepsilon \ll 1 \), \( K \gg \tilde{K} \gg 1 \) and \( s_0 \gg 1 \).

Here and henceforth, \( C \) represents a positive constant which is independent of \( \varepsilon, K, \tilde{K} \), any spatial and time variables and any solutions. Then, each \( C \) may be different from the other \( C \)‘s.

Theorem 1 is an immediate conclusion of the following lemma.
Lemma 2.2 For any nonnegative even integer $J$, there exist $K \gg 1$, $s_0 \geq 0$ and a radial solution $m$ to (4) in $R_+ \times (s_0, \infty)$ satisfying the following.

(i) For $s \geq s_0$, it holds that $Z(m(\cdot, s) - m_\infty) = J$, where $Z(h)$ denotes the supremum over all $j$ such that there exists $0 < y_1 < y_2 < \cdots < y_{j+1} < \infty$ with $h(y_i) - h(y_{i+1}) < 0$ for $i = 1, 2, \ldots, j$ and a continuous function $h$ on $R_+$.

(ii) It holds that
\[ m_1(y, s) \leq m(y, s) \leq m_2(y, s) \quad \text{for } y \in [0, K e^{-\eta s}] \] and $s \geq s_0$.

(iii) For any sufficiently small $\varepsilon > 0$, it holds that
\[ e^{y^2/4} \left\{ m(y, s) - m_\infty(y, s) \right\} + e^{-\lambda_j s} \varphi_J(y) \leq \varepsilon e^{-\lambda_j s} \left( y^{2\nu} + y^{2J+2\nu} \right) \]
for $y \in [K e^{-\eta s}, e^{\sigma s}]$ and $s \geq s_0$ with some $\sigma \in (0, 1/2)$.

Take $\sigma = 2 \lambda_j/(4 \lambda_j + 1)$. For $K \gg 1$, $0 < \varepsilon \ll 1$, $s_1 \geq s_0 \geq 0$ and $\theta \in [0, 1]$, let $A(s_0, s_1; \theta)$ be the class of functions $h \in L^\infty((s_0, s_1); L_{\text{loc}}^1(R_+))$ satisfying
\[ e^{y^2/4} \left\{ h(y, s) - m_\infty(y, s) \right\} + e^{-\lambda_j s} \varphi_J(y) \leq \theta e^{-\lambda_j s} \left( y^{2\nu} + y^{2J+2\nu} \right) \]
for $y \in [K e^{-\eta s}, e^{\sigma s}]$ and $s \in [s_0, s_1]$. Choose $\tilde{K} > 0$ and $\tilde{\sigma} > 0$ with $K \gg \tilde{K}$ and $\tilde{\sigma} = (\sigma + 1)/2$.

We first suppose that $J$ is a positive even integer and take
\[ \phi(y, s_0) = e^{y^2/4} \left\{ m(y, s_0) - m_\infty(y) \right\} = \sum_{j=0}^{J-1} d_j \varphi_j(y) - e^{-\lambda_j s_0} \tilde{\varphi}_J(y) \]
as follows:

(\phi1) $\sum_{j=0}^{J-1} |d_j| < \varepsilon \theta_1 e^{-\lambda_j s_0}$, where
\[ \theta_1 = \left\{ \sum_{j=0}^{J-1} \left( \sup_{0 < y < \infty} \frac{|\varphi_j(y)|}{y^{2\nu} + y^{2J+2\nu}} \right) + 1 \right\}^{-1}; \quad (18) \]

(\phi2) For $y \in [0, \tilde{K} e^{-\eta s_0}]$,
\[ \tilde{\varphi}_J(y) = e^{\lambda_j s_0} \left\{ e^{y^2/4} \left[ m_\infty(y) - m(y, s_0) \right] + \sum_{j=0}^{J-1} d_j \varphi_j(y) \right\}, \]

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where \(\overline{m} = \theta_1m_1 + (1 - \theta_1)m_2\) and \(\theta_1 \in [0,1]\) is a constant satisfying
\[
\varphi_J(Ke^{-\eta s_0}) = \tilde{\varphi}_J(Ke^{-\eta s_0});
\]
\((\phi 3)\) For \(y \in [Ke^{-\eta s_0}, e^{\sigma s_0}]\), \(\varphi_J(y) = \tilde{\varphi}_J(y)\);
\((\phi 4)\) For \(y \in [e^{\sigma s_0}, \infty)\),
\[
\phi(y, s) = e^{y^2/4} \min \left\{ m_E(s_0) - m_\infty(y), 0 \right\},
\]
where
\[
m_E(s_0) = \exp \left( -e^{2\sigma s_0}/4 \right) \left\{ \sum_{j=0}^{J-1} d_j \varphi_J(e^{\sigma s_0}) - e^{-\lambda_J s_0} \varphi_J(e^{\sigma s_0}) \right\} + m_\infty(e^{\sigma s_0});
\]
\((\phi 5)\) \(Z(\phi(\cdot, s_0)) = J\).

Since \(J\) is a positive even integer, we see that \(\varphi_J(y) > 0\) for \(y \gg 1\) and that
\(\phi(e^{\sigma s_0}, s_0) \leq 0\) for \(s_0 \gg 1\).

For \(d = (d_0, d_1, \ldots, d_{J-1}) \in \mathbb{R}^J\), denote by \(\phi(y, s_0; d)\) and \(m(y, s_0; d)\) the above \(\phi(y, s_0)\) and the solution to (4) with \(m(y, s_0; d) = m_\infty(y) + e^{-y^2/4} \phi(y, s_0; d)\), respectively, and write
\[
\phi(y, s_0; d) = e^{y^2/4} \left\{ m(y, s_0; d) - m_\infty(y) \right\}.
\]

Let \(U(s_0, s_1)\) be the set of \(d \in \mathbb{R}^J\) with \((\phi 1)\) such that \(m(\cdot, \cdot; d) \in \mathcal{A}(s_0, s_1; 1)\), where \(\phi(\cdot, s_0; d)\) is taken to satisfy \((\phi 2)\)\(-(\phi 5)\). If \(d \in U(s_0, s_1)\) and \(J\) is a positive even integer, then
\[
m(r, s; d) \leq m_\infty(y) \quad \text{for} \quad y \in [e^{\sigma s}, \infty) \quad \text{and} \quad s \in [s_0, s_1]. \quad (19)
\]

Actually, for any positive even integer \(J\), any sufficiently small \(\varepsilon > 0\) and any sufficiently large \(s_0 \geq 0\) it holds that \(m(e^{\sigma s_0}, s; d) \leq m_\infty(e^{\sigma s_0})\) in \([s_0, s_1]\) with
\(s_1 \geq s_0\), since \(m(\cdot, \cdot; d) \in \mathcal{A}(s_0, s_1^1)\). Combining this with \(m(\cdot, s_0; d) \leq m_\infty\)
in \([e^{\sigma s_0}, \infty)\) and the comparison theorem implies (19).

In order to prove Lemma 2.2, we need the following proposition.

**Proposition 2.1** Let \(s_0 \gg 1\), \(K \gg 1\) and \(s_1 \geq s_0\). If \(d \in \overline{U(s_0, s_1)}\) (the closure of \(U(s_0, s_1)\)), then
\[
m_1(y, s) < m(y, s; d) < m_2(y, s)
\]
for \(y \in [0, Ke^{-\eta s}]\) and \(s \in [s_0, s_1]\).

**Proof.** Since \(m\) satisfies (17) for \(y \in [Ke^{-\eta s}, e^{\sigma s}]\) and \(s \in [s_0, s_1]\), it follows from Lemma 2.1 that
\[
(1 + \varepsilon_1)m_1(y, s) < m(y, s; d) < (1 - \varepsilon_1)m_2(y, s)
\]
for \(y \in [0, Ke^{-\eta s}]\) and \(s \in [s_0, s_1]\).
for $y = Ke^{-\eta s}$ and $s \in [s_0, s_1]$ with some $\varepsilon_1 > 0$ if $K \gg 1$ and $0 < \varepsilon \ll 1$. From $(\phi 2)$ it holds that
\[ m_1(y, s_0) \leq m(y, s_0; d) \leq m_2(y, s_0) \]
for $y \in [0, Ke^{-\eta s_0}]$.
Then, we have that
\[ m_1(y, s) < m(y, s; d) < m_2(y, s) \quad \text{for} \quad y \in [0, e^{-\eta s}] \quad \text{and} \quad s \in (s_0, s_1) \]
by the comparison theorem and $m(\cdot, \cdot; d) \in \mathcal{A}(s_0, s_1; 1)$, if we obtain that
\[ \mathcal{L}(m_1) \leq 0 \quad \text{and} \quad \mathcal{L}(m_2) \geq 0 \quad \text{for} \quad y \in [0, Ke^{-\eta s}] \quad \text{and} \quad s \in [s_0, s_1], \quad (20) \]
where $\mathcal{L}$ is the operator in (4). We shall show (20).
Setting
\[ \tilde{m}(\zeta, \tau) = e^{-2\chi \eta \zeta} m(y, s) \quad \text{with} \quad \zeta = e^{\chi \eta s} y \quad \text{and} \quad \tau = e^{2\chi \eta s}/(2\chi \eta), \]
$\tilde{m}$ satisfies
\[
\hat{\mathcal{L}}(\tilde{m}) = \frac{1}{(2\chi \eta)^2} \mathcal{L}(m) = \tilde{m}_\zeta - \frac{N + 1}{\zeta} \tilde{m}_\zeta - \tilde{m}_\zeta = 0.
\]
By using an argument similar to the proof of [11, Proposition 4.1], we obtain that
\[
\hat{\mathcal{L}}(\tilde{m}_1) \leq (1 + \varepsilon K^{2\nu + 2} e^{-\eta s}) \left\{ -(N - 2)\varepsilon K^{2\nu + 2} e^{-\eta s} \frac{C}{\tau} \right\} \tilde{m}_1
\]
and that
\[
\hat{\mathcal{L}}(\tilde{m}_2) \geq (1 - \varepsilon K^{2\nu + 2} e^{-\eta s}) \left\{ (N - 2)\varepsilon K^{2\nu + 2} e^{-\eta s} \frac{C}{\tau} \right\} \tilde{m}_2
\]
with $\tilde{m}_i(y) = y^{-N} \int_0^y U_i(\rho) \rho^{N-1} d\rho \ (i = 1, 2)$.
By Lemma 2.1, we have that
\[ e^{-\eta s} \tilde{m}_i(\zeta) \geq \frac{1}{2K^2 e^{(2\chi - 1)\eta s}} \geq \frac{C}{\tau} \quad \text{for} \quad \zeta \in [0, Ke^{(\chi - 1)\eta s}] \quad \text{and} \quad i = 1, 2, \]
if $s \gg 1$. Then, we obtain that
\[ \hat{\mathcal{L}}(\tilde{m}_1) < 0 \quad \text{and} \quad \hat{\mathcal{L}}(\tilde{m}_2) > 0 \quad \text{in} \quad \mathcal{R}_I, \]
where
\[ R_t = \{ (\zeta, \tau) : 0 \leq \zeta \leq K(2\chi\eta \tau)^{(\chi-1)/(2\chi)}, \tau \geq \tau_0 \} \]
and \( \tau_0 = e^{2\chi s_0}/(2\chi\eta) \), if \( s_0 \gg 1 \). Then, we get (20). Thus, we have this proposition. \( \square \)

For \( s_1 \geq s_0 > 0 \), define an operator \( P(\cdot; s_0, s_1) \) from \( U(s_0, s_1) \) into \( \mathbb{R}^J \) by
\[
P(d; s_0, s_1) = (p_0, p_1, \cdots, p_{J-1}) \quad \text{with} \quad p_j = \langle \phi(\cdot, s_1; d), \varphi_j \rangle \quad \text{for} \quad j = 0, 1, \cdots, J - 1.
\]
Then, we see that \( P \) is continuous with respect to \( d, s_0 \) and \( s_1 \).

**Proposition 2.2** Let \( K \gg \hat{K} \gg 1, \ s_0 \gg 1 \) and \( s_1 > s_0 \). If there is a \( d \in U(s_0, s_1) \) such that \( P(d; s_0, s_1) = 0 \), then
\[
\sum_{j=0}^{J-1} |d_j| < \frac{\varepsilon}{2} e^{-\lambda_j s_0}
\]
and \( m(\cdot, \cdot; d) \in A(s_0, s_1; \theta) \) with some \( \theta \in (0, 1) \).

In order to prove Proposition 2.2, we need the following lemma.

**Lemma 2.3** Suppose that \( m(\cdot, \cdot; d) \in A(s_0, s_1; 1) \) with \( s_1 > s_0 \gg 1 \) and let \( M \) be the corresponding solution to (11) through (12). Then, there is a constant \( C_1 > 0 \) such that
\[
|M(r, t) - M_{\infty}(r)| < C_1 \exp \left( -\frac{\nu^2}{4(t+1)} \right) \quad \text{in} \quad \mathbb{R}_+ \times [t_0, t_1],
\]
where \( t_i = e^{s_i} - 1 \) \( (i = 0, 1) \).

**Proof.** Since \( m(\cdot, \cdot; d) \in A(s_0, s_1; 1) \), it holds that
\[
|m(y, s; d) - m_{\infty}(y)| \leq C e^{-\lambda_j y^2(1+y^2)}e^{-y^2/4} \quad \text{for} \quad y \in [Ke^{-\eta s}, e^{\sigma s}] \text{ and } s \in [s_0, s_1].
\]
Then, we obtain that
\[
|M(r, t) - M_{\infty}(r)| \leq \frac{C}{t+1} \frac{(t+1)^{-\lambda_j} y^{2\nu}(1+y^{2J})e^{-y^2/4}}{t+1} e^{-\nu^2/4(t+1)}
\]
\[
\leq C(t+1)^{-\lambda_j} \left( y^{2\nu+2} + y^{2J+2\nu+2} \right) \frac{1}{r^2} \exp \left( -\frac{r^2}{4(t+1)} \right)
\]
\[
\leq C(t+1)^{-\lambda_j} \left\{ (K(t+1)^{-\eta})^{2\nu+2} + (t+1)^{2\sigma(J+\nu+1)} \right\} \frac{1}{r^2} \exp \left( -\frac{r^2}{4(t+1)} \right)
\]

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for \( r \in [K(t+1)^{\frac{1}{2}-\eta}, (t+1)^{\frac{1}{2}+\sigma}] \) and \( t \in [t_0, t_1] \). Since it follows from \( N \geq 11 \) and \( J \geq 2 \) that \(-\lambda_J - \eta(2\nu + 2) < 0\) and that \(-\lambda_J + 2\sigma(J + \nu + 1) < 0\), for any \( \delta > 0 \) it hold that

\[
|M(r, t) - M_\infty(r)| \leq \frac{\delta}{r^2} \exp \left( -\frac{r^2}{4(t+1)} \right) \tag{23}
\]

for \( r \in [K(t+1)^{\frac{1}{2}-\eta}, (t+1)^{\frac{1}{2}+\sigma}] \) and \( t \in [t_0, t_1] \), if \( t_0 \gg 1 \).

Since it holds that

\[
0 \leq m(y, s; d) \leq m_\infty(y) = \frac{2}{y^2} \quad \text{for } y \in (0, Ke^{-\eta}] \text{ and } s \in [s_0, s_1],
\]

we have that

\[
0 \leq M(r, t) \leq M_\infty(r) \quad \text{for } r \in (0, K(t+1)^{\frac{1}{2}-\eta}] \text{ and } t \in [t_0, t_1].
\]

By this and \( \eta > 1/2 \), we have that

\[
|M(r, t) - M_\infty(r)| \leq \frac{3}{r^2} \exp \left( -\frac{r^2}{4(t+1)} \right)
\]

for \( r \in (0, K(t+1)^{\frac{1}{2}-\eta}] \) and \( t \in [t_0, t_1] \), if \( t_0 \gg 1 \). Combining this with (23) implies that

\[
|M(r, t) - M_\infty(r)| \leq \frac{3}{r^2} \exp \left( -\frac{r^2}{4(t+1)} \right) \tag{24}
\]

for \( r \in (0, (t+1)^{\frac{1}{2}+\sigma}] \) and \( t \in [t_0, t_1] \), if \( t_0 \gg 1 \).

We can take a positive constant \( C_0 \geq 4 \) satisfying

\[
|M(r, t_0) - M_\infty(r)| \leq \frac{C_0}{r^2} \exp \left( -\frac{r^2}{4(t_0+1)} \right) \quad \text{in } \mathbb{R}_+.
\]

Let us put \( C_1 = 2C_0 \). We shall assume that the assertion is not valid. Then, putting

\[
t^* = \sup \left\{ \tilde{t} \in (t_0, t_1) : (22) \text{ with } C_1 = 2C_0 \text{ is valid in } \mathbb{R}_+ \times [t_0, \tilde{t}] \right\}
\]

we get that \( t^* < t_1 \) and that

\[
|M(r^*, t^*) - M_\infty(r^*)| = \frac{C_1}{(r^*)^2} \exp \left( -\frac{(r^*)^2}{4(t^*+1)} \right)
\]

for some \( r^* \in \mathbb{R}_+ \). It follows from (24) that \( r^* > (t^*+1)^{\sigma+\frac{3}{2}} \). Take \( X^* \in \mathbb{R}^{N+2} \) satisfying \( |X^*| = r^* \) and put

\[
G(X, t) = \frac{1}{(4\pi t)^{(N+2)/2}} \exp \left( -\frac{|X|^2}{4t} \right) \quad \text{for } (X, t) \in \mathbb{R}^{N+2} \times (0, \infty).
\]
The solution $M$ to (11) is written as

$$M(|X|, t) = \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - t_0)M(|\tilde{X}|, t_0)d\tilde{X}$$

$$+ \int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \left\{ \tilde{X} \cdot \nabla_{\tilde{X}} M(|\tilde{X}|, \tilde{t}) + N M(|\tilde{X}|, \tilde{t}) \right\} M(|\tilde{X}|, \tilde{t})d\tilde{X}d\tilde{t}. \quad (25)$$

Since we have that

$$\int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \left\{ \frac{\tilde{X}}{2} \cdot \nabla_{\tilde{X}} M(|\tilde{X}|, \tilde{t}) \right\} d\tilde{X}d\tilde{t}$$

$$= \int_{t_0}^t \int_{\mathbb{R}^{N+2}} \left\{ - \frac{N + 2}{2} \left( \frac{\tilde{X} - X}{4(t - \tilde{t})} \right) \right\} G(X - \tilde{X}, t - \tilde{t})M(|\tilde{X}|, \tilde{t})^2d\tilde{X}d\tilde{t},$$

$M$ satisfies that

$$M(|X|, t) = \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - t_0)M(|\tilde{X}|, t_0)d\tilde{X}$$

$$+ \int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \frac{\tilde{X} - X}{4(t - \tilde{t})} M(|\tilde{X}|, \tilde{t})^2d\tilde{X}d\tilde{t}$$

$$+ \frac{N - 2}{2} \int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t})M(|\tilde{X}|, \tilde{t})^2d\tilde{X}d\tilde{t}. \quad (26)$$

Since $M_\infty$ satisfies (25) and (26), we have that

$$M(|X|, t) - M_\infty(|X|) = \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - t_0) \left( M(|\tilde{X}|, t_0) - M_\infty(|\tilde{X}|) \right) d\tilde{X}$$

$$+ \int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \frac{\tilde{X} - X}{4(t - \tilde{t})} \left\{ M(|\tilde{X}|, \tilde{t})^2 - M_\infty(|\tilde{X}|)^2 \right\} d\tilde{X}d\tilde{t}$$

$$+ \frac{N - 2}{2} \int_{t_0}^t \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \left\{ M(|\tilde{X}|, \tilde{t})^2 - M_\infty(|\tilde{X}|)^2 \right\} d\tilde{X}d\tilde{t}$$

$$= I(X, t) + II(X, t) + III(X, t). \quad (27)$$

We see that

$$|I(X, t)| \leq C_0 \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - t_0) \frac{1}{|X|^2} \exp \left( - \frac{|\tilde{X}|^2}{4(t_0 + 1)} \right) d\tilde{X}$$

$$= C_0 \tilde{I}(X, t) \quad \text{for} \quad (X, t) \in \mathbb{R}^{N+2} \times [t_0, \infty). \quad (28)$$
Putting

\[ \mathcal{M} = \frac{\partial}{\partial t} - \Delta \quad \text{in } \mathbb{R}^{N+2} \times (0, \infty) \]

and

\[ \overline{\mathcal{M}}(X, t; k) = \frac{1}{|X|^k} \exp \left( -\frac{|X|^2}{4(t + 1)} \right) , \]

it holds that

\[ \mathcal{M}(\overline{\mathcal{M}}(\cdot, \cdot; k)) = \exp \left( -\frac{|X|^2}{4(t + 1)} \right) \left\{ k(N - k) \frac{N - 2k + 2}{2(t + 1)|X|^k} \right\} . \quad (29) \]

Then, we have that

\[ \mathcal{M}(\overline{\mathcal{M}}(\cdot, \cdot; 2)) > 0 \quad \text{in } \mathbb{R}^{N+2} \times [t_0, \infty). \]

It holds that

\[ \mathcal{M}(\tilde{I}) = 0 \quad \text{in } \mathbb{R}^{N+2} \times (t_0, \infty) \]

and that

\[ \tilde{I}(X, t_0) = \frac{1}{|X|^2} \exp \left( -\frac{|X|^2}{4(t_0 + 1)} \right) = \overline{\mathcal{M}}(X, t_0; 2). \]

By those, the comparison theorem and (28), we get that

\[ |I(X, t)| \leq C_0 \frac{|X|^2}{|X|^2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right) \quad \text{in } \mathbb{R}^{N+2} \times [t_0, \infty). \quad (30) \]

Since it follows from (24) and \( m(\cdot, \cdot, d) \in \mathcal{A}(s_0, s_1; 1) \) that

\[ 0 \leq M(r, t) \leq M_\infty(r) \quad \text{for } r \in (0, \tilde{K}(t+1)^{\frac{1}{2}-\eta}] \cup [(t+1)^{\frac{1}{2}+\sigma}, \infty) \text{ and } t \in [t_0, t_1], \]

we have that

\[ |M(r, t)| + |M_\infty(r)| \leq \frac{7}{r^2} \quad \text{in } \mathbb{R}_+ \times [t_0, t_1]. \]

By this and the definition of \( t^* \), we see that

\[ |M(r, t)^2 - M_\infty(r)^2| \leq \frac{7C_1}{r^4} \exp \left( -\frac{r^2}{4(t + 1)} \right) \quad \text{for } (r, t) \in \mathbb{R}_+ \times [t_0, t^*] \]

and that

\[ |III(X^*, t^*)| \leq \frac{7(N - 2)C_1}{2} \int_{t_0}^{t^*} \int_{\mathbb{R}^{N+2}} \mathcal{G}(X^* - \tilde{X}, t^* - \tilde{t}) \overline{\mathcal{M}}(\tilde{X}, \tilde{t}; 4)d\tilde{X}d\tilde{t}. \]
By (29) with $k = 4$, $|X^*| = r^* > (t^* + 1)^{\frac{3}{2} + \sigma}$ and the comparison theorem, we have that

$$|IIII(X^*, t^*)| \leq \frac{7(N - 2)C_1}{2} \int_{t^*}^{t} \frac{1}{|X^*|^4} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right) d\tilde{t}$$

$$\leq \frac{7(N - 2)C_1}{2} \frac{t^* - t_0}{(t^* + 1)^{1+2\sigma}} \frac{1}{|X^*|^2} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right)$$

$$= \frac{7(N - 2)C_1}{2(t^* + 1)^{2\sigma}} \frac{1}{|X^*|^2} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right). \quad (31)$$

It follows from (31) that

$$|II(X,t)| \leq 7C_1 \int_{t_0}^{t} \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \frac{|X - \tilde{X}|}{4(t - \tilde{t})} M(\tilde{X}, \tilde{t}; 3) d\tilde{X} d\tilde{t}$$

$$\leq \frac{7}{4} C_1 \int_{t_0}^{t} \left\{ \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) M(\tilde{X}, \tilde{t}; 6) d\tilde{X} \right\}^{1/2}$$

$$\times \left\{ \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \frac{|X - \tilde{X}|^2}{(t - \tilde{t})^2} \exp \left( -\frac{|\tilde{X}|^2}{4(\tilde{t} + 1)} \right) d\tilde{X} \right\}^{1/2} d\tilde{t}$$

$$= \frac{7}{4} C_1 \int_{t_0}^{t} II_1(X, t, \tilde{t})^{1/2} II_2(X, t, \tilde{t})^{1/2} d\tilde{t} \quad \text{for } (X, t) \in \mathbb{R}^{N+2} \times [t_0, t^*]. \quad (32)$$

By using (29) with $k = 6$, $N \geq 11$ and the comparison theorem, we get that

$$II_1(X, t, \tilde{t}) \leq M(X, t; 6) \quad \text{for } (X, t) \in \mathbb{R}^{N+2} \times [t_0, \infty) \text{ and } \tilde{t} \in [t_0, t). \quad (33)$$

Putting

$$g_{II,2}(X,t,\tilde{t}) = \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) \exp \left( -\frac{|\tilde{X}|^2}{4(\tilde{t} + 1)} \right) d\tilde{X}, \quad (34)$$

we see that

$$g_{II,2}(X,t,\tilde{t}) = \left\{ 4\pi(\tilde{t} + 1) \right\}^{(N+2)/2} \int_{\mathbb{R}^{N+2}} G(X - \tilde{X}, t - \tilde{t}) G(\tilde{X}, \tilde{t} + 1) d\tilde{X}$$

$$= \left\{ 4\pi(\tilde{t} + 1) \right\}^{(N+2)/2} G(X, t + 1)$$

$$= \left\{ \frac{\tilde{t} + 1}{t + 1} \right\}^{(N+2)/2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right). \quad (35)$$
Then, we have that
\[
\frac{\partial}{\partial t} g_{11,2}(X, t, \tilde{t}) = \left\{ -\frac{N + 2}{2(t + 1)} + \frac{|X|^2}{4(t + 1)^2} \right\} \left\{ \frac{\tilde{t} + 1}{t + 1} \right\}^{(N+2)/2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right).
\]
It follows from (35) that
\[
\frac{\partial}{\partial t} g_{11,2}(X, t, \tilde{t}) = \int_{R^{N+2}} \left\{ -\frac{N + 2}{2(t - \tilde{t})} + \frac{|X - \tilde{X}|^2}{4(t - \tilde{t})^2} \right\} g(X - \tilde{X}, t - \tilde{t}) \exp \left( -\frac{|\tilde{X}|^2}{4(t + 1)} \right) d\tilde{X}.
\]
By those and \( t_0 \leq \tilde{t} < t \leq t^* \), we obtain that
\[
\frac{1}{4} II_2(X, t, \tilde{t}) = \frac{N + 2}{2(t - \tilde{t})} \int_{R^{N+2}} g(X - \tilde{X}, t - \tilde{t}) \exp \left( -\frac{|\tilde{X}|^2}{4(t + 1)} \right) d\tilde{X}
\]
\[
+ \left\{ -\frac{N + 2}{2(t + 1)} + \frac{|X|^2}{4(t + 1)^2} \right\} \left\{ \frac{\tilde{t} + 1}{t + 1} \right\}^{(N+2)/2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right)
\]
\[
= \left\{ -\frac{N + 2}{2(t - \tilde{t})} + \frac{|X|^2}{4(t + 1)^2} \right\} \left\{ \frac{\tilde{t} + 1}{t + 1} \right\}^{(N+2)/2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right)
\]
\[
\leq \left\{ -\frac{N + 2}{2(t - \tilde{t})} + \frac{|X|^2}{4(t + 1)^2} \right\} \left\{ \frac{\tilde{t} + 1}{t + 1} \right\}^{(N+2)/2} \exp \left( -\frac{|X|^2}{4(t + 1)} \right).
\]
Combining this with (33) and (34) implies that
\[
|II(X^*, \delta^*)|
\]
\[
\leq \frac{7}{4} C_1 \int_{t_0}^{t^*} 2 \left\{ \sqrt{\frac{N + 2}{2(t^* - \tilde{t})}} + \frac{|X^*|}{2(t^* + 1)} \right\}^{(N+2)/4} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right) M(X^*, \delta^*; \delta^*) d\tilde{t}
\]
\[
\leq \frac{7}{2} C_1 \left\{ \sqrt{\frac{2(N + 2)}{|X^*|}} + \frac{2}{N + 6} \right\} \frac{1}{|X^*|^2} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right)
\]
\[
\leq \frac{7}{2} C_1 \left\{ \frac{\sqrt{2(N + 2)}}{(t^* + 1)\sigma} + \frac{2}{N + 6} \right\} \frac{1}{|X^*|^2} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right).
\]
Combining this with (27), (30), (32) and \( N \geq 11 \) implies that
\[
|M(|X^*|, \delta^*) - M_{\infty}(|X^*|)|
\]
\[
\leq \left\{ C_0 + \frac{7(N - 2)C_1}{2(t^* + 1)^{2\sigma}} + \frac{7C_1\sqrt{N + 2}}{2\sqrt{2(t^* + 1)^{2\sigma}}} + \frac{7C_1}{17} \right\} \frac{1}{|X^*|^2} \exp \left( -\frac{|X^*|^2}{4(t^* + 1)} \right).
\]
Take $t_0 \gg 1$. By $t^* \geq t_0$, we have that
$$|M(r^*, t^*) - M_\infty(r^*)| \leq \frac{16C_1}{17} \frac{1}{(r^*)^2} \exp \left( - \frac{(r^*)^2}{4(t^* + 1)} \right),$$
It contradicts the definition of $t^*$. Then, we obtain that $t^* = t_1$. Thus, we have this lemma. 
\[\square\]

Putting
$$W(\xi, \tau) = (T - t)M(r, t), \quad \xi = \frac{r}{\sqrt{T - t}}, \quad \tau = -\log(T - t), \quad \tau_0 = -\log T,$$
$W$ satisfies that
$$\begin{cases}
W_\tau = W_{\xi\xi} + \frac{N + 1}{\xi} W_\xi - \frac{\xi}{2} W_\xi - W + W(\xi W_\xi + NW) \\
W(0, \cdot) = 0 \quad \text{in } (\tau_0, \infty), \quad W(\xi, \tau_0) = TM(\sqrt{T}\xi, 0) \quad \text{in } \mathbb{R}_+.
\end{cases} \quad (36)$$
Putting $\psi(\xi, \tau) = W(\xi, \tau) - m_\infty(\xi)$, $\psi$ satisfies
$$\psi_\tau = -B\psi + H(\cdot, \psi), \quad (37)$$
where $m_\infty$ is the function in (13), $B = A - N/2$, $A$ is the operator in (8) and $H(\xi, \psi) = \xi \psi \psi_\xi + N\psi^2$.

Then, the $j$th eigenvalue $\mu_{j-1}$ of $B$ is given by
$$\mu_{j-1} = j + \nu \quad \text{for } j = 1, 2, 3, \ldots$$
with $\nu$ in (2) and $B$ has the same eigenfunctions as the operator $A$ defined by (8). In [11], Mizoguchi and the author obtained the following solution $W$ to (36) by the same method as in [9, 6].

**Theorem 3 ([11])** Let $N \geq 11$ and $J \geq 2$. There exists a solution $W$ to (36) satisfying the following.

(i) $Z(W(\cdot, \tau) - m_\infty) = J$ for $\tau \in [\tau_0, \infty)$, where $Z$ is in Lemma 2.2.
(ii) Let $\bar{\eta} = \mu_j / (-2\nu - 1)$, $\bar{R}_J = \{(\xi, \tau) : 0 < \xi < Ke^{-\bar{\eta} \tau}, \tau \geq \tau_0\}$,

$$\Psi_1(\xi, \tau) = \left(1 + \varepsilon K^{2\nu + 2} e^{-\bar{\eta} \tau} \right) \frac{e^{2\chi \bar{\eta} \tau}}{(e^{\chi \bar{\eta} \tau} \xi)^N} \int_0^{e^{\chi \bar{\eta} \tau} \xi} U_1(\zeta) \zeta^{N-1} d\zeta$$

and
$$\Psi_2(\xi, \tau) = \left(1 - \varepsilon K^{2\nu + 2} e^{-\bar{\eta} \tau} \right) \frac{e^{2\chi \bar{\eta} \tau}}{(e^{\chi \bar{\eta} \tau} \xi)^N} \int_0^{e^{\chi \bar{\eta} \tau} \xi} U_2(\zeta) \zeta^{N-1} d\zeta,$$
where $\overline{U}_1(y)$ and $\overline{U}_2(y)$ are in (16), and $\chi$ is in (15). For any sufficiently large $K > 0$ and any sufficiently small $\varepsilon > 0$, the solution $W$ satisfies that

$$
\Psi_1(\xi, \tau) < W(\xi, \tau) < \Psi_2(\xi, \tau) \quad \text{in } \tilde{\mathcal{R}}_J.
$$

(iii) For any sufficiently small $\varepsilon > 0$, the solution $W$ satisfies that

$$
|W(\xi, \tau) - m_\infty(\xi) + e^{-\mu_J T} \varphi_J(\xi)| \leq \varepsilon e^{-\mu_J T}(\xi^{2\nu} + \xi^{2J+2\nu})
$$

for $\xi \in [Ke^{-\eta T}, e^{\eta T}]$ and $\tau \geq \tau_0$, where $\kappa = 2\mu_J/(4\mu_J + 1)$.

Choose $K \gg \tilde{K} \gg 1$ and $\tilde{\kappa} = (1+\kappa)/2$, and take

$$
W(\xi, \tau) = \sum_{j=0}^{J-1} d_j \varphi_j(\xi) - e^{-\mu_J T} \hat{\varphi}_J(\xi)
$$

as follows:

(W1) $\sum_{j=0}^{J-1} |d_j| < \varepsilon \theta_1 e^{-\mu_J \tau_0}$, where $\theta_1$ is a constant in (18);

(W2) For $\xi \in [0, \tilde{K} e^{-\tilde{\eta} \tau_0}]$,

$$
\hat{\varphi}_J(\xi) = e^{\mu_J \tau_0} \left\{ m_\infty(\xi) - \left[ \hat{\theta} \Psi_1(\xi, \tau_0) + (1 - \hat{\theta}) \Psi_2(\xi, \tau_0) \right] + \sum_{j=0}^{J-1} d_j \varphi_j(\xi) \right\},
$$

where $\hat{\theta} \in [0, 1]$ is defined so that $\hat{\varphi}_J(\tilde{K} e^{-\tilde{\eta} \tau_0}) = \varphi_J(\tilde{K} e^{-\tilde{\eta} \tau_0})$;

(W3) For $\xi \in [Ke^{-\eta T}, e^{\eta T}]$, $\varphi_J(\xi) = \varphi_J(\xi)$;

(W4) $\mathcal{Z}(W(\cdot, \tau_0) - m_\infty) = J$.

For $d = (d_0, d_1, \cdots, d_{J-1}) \in \mathbb{R}^J$ satisfying (W1), we see that the above $W(\cdot, \tau_0)$ is determined uniquely. For $d \in \mathbb{R}^J$ satisfying (W1), $W(\cdot, \tau_0; d)$ denotes the above $W(\cdot, \tau_0)$, and $W(\cdot, \cdot; d)$ denotes the solution to (36) whose initial data is $W(\cdot, \tau_0; d)$.

For $\tau_1 \geq \tau_0$ and $\theta \in (0, 1]$, we say that $W(\cdot, \cdot; d) \in \mathcal{B}(\tau_0, \tau_1, \theta)$, if the solution $W(\cdot, \cdot; d)$ to (36) satisfies that

$$
|W(\xi, \tau; d) - m_\infty(\xi) + e^{-\mu_J T} \varphi_J(\xi)| \leq \theta \varepsilon e^{-\mu_J T}(\xi^{2\nu} + \xi^{2J+2\nu})
$$

for $\xi \in [Ke^{-\eta T}, e^{\eta T}]$ and $\tau \in [\tau_0, \tau_1]$.

Let $\mathcal{V}(\tau_0, \tau_1)$ be the set of $d \in \mathbb{R}^J$ with (W1) such that $W(\cdot, \cdot; d) \in \mathcal{B}(\tau_0, \tau_1; 1)$, where $W(\xi, \tau_0; d) = \psi(\xi, \tau_0; d) + m_\infty(\xi)$ is taken to satisfy (W1)-(W4).
In [11], Mizoguchi and the author show \( V(\tau_0, \infty) \neq \emptyset \). Then, taking \( d \in V(\tau_0, \infty) \), the corresponding solution \( W(\cdot, \cdot; d) \) to (36) is in \( \mathcal{B}(\tau_0, \infty; 1) \). By this and the comparison theorem, we have Theorem 3.

Combining Theorem 3 with the parabolic regularity implies the following.

**Corollary 1** For \( N \geq 11 \) and \( J \geq 2 \), let \((U, V)\) be the radial solution to (1) which corresponds to the solution \( W \) in Theorem 3. Then, the solution \((U, V)\) blows up at \( t = T \) and satisfies

\[
\lim_{t \to T} (T - t)^{-J/(\nu+1)}|U(\cdot, t)|_{\infty} \in (0, \infty).
\]

In order to prove (38), we divide \( \mathbb{R}^+ \) into \((0, K(T - t)^{(1+2\eta)/2}), (K(T - t)^{(1+2\eta)/2}, (T - t)^{1/2}) \) and \(( (T - t)^{1/2}, \infty) \), and show an estimate of the solution \((U, V)\) in each of those regions. Those estimates are shown by similar arguments. Then, we describe only the argument to establish the estimate in \((K(T - t)^{(1+2\eta)/2}, (T - t)^{1/2})\).

Fix \( \tau > \tau_0 + 1 \) and \( \xi \in [K e^{-\tilde{\eta} \tau}, 1] \). Let us put \( P(\rho, \theta) = W(\xi \rho, \xi^2 \theta + \tau) - m_{\infty}(\xi \rho) \), we have that

\[
P_{\theta} = P_{\rho\rho} + \left( \frac{N + 3}{\rho} - \frac{\xi^2}{2} \right) P_{\rho} - \xi^2 P + \frac{4(N - 1)}{\rho^2} P + \xi^2 P \left( \rho P_{\rho} + N P \right).
\]

It follows from Theorem 3 that

\[
|W(\xi, \tau; d) - m_{\infty}(\xi)| \leq C e^{-\mu \tau} (\xi^{2\nu} + \xi^{2J+2\nu})
\]

for \( \xi \in [(K/2) e^{-\tilde{\eta}(\tau+4)}, 2] \) and \( \tau > \tau_0 \).

Combining this with [8, Theorem 11.1 in Chapter III] and Theorem 3 implies that

\[
|P_{\rho}(1, 1)| \leq C \max_{1/2 \leq \rho \leq 2, 1/2 \leq \theta \leq 2} |P(\rho, \theta)| \leq C e^{-\mu \tau} (\xi^{2\nu} + \xi^{2J+2\nu}).
\]

By this, we have that

\[
|U(x, t) - \frac{2(N - 2)}{|x|^2}| \leq C e(T-t)^{J}|x|^{2\nu} \quad \text{for} \quad |x| \in [K(T-t)^{(2\eta+1)/2}, (T-t)^{1/2}].
\]

This is a desired estimate of the solution \((U, V)\) in \([K(T-t)^{(2\eta+1)/2}, (T-t)^{1/2}]\).

Then, Corollary 1 is an immediate conclusion of Theorem 3, and [11, Proposition 4.2] is important to show Theorem 3 or \( V(\tau_0, \infty) \neq \emptyset \). The roles of Lemma 2.2 and Proposition 2.2 in the proof of Theorem 1 correspond to those of Theorem 3 and [11, Proposition 4.2] in the proof of Corollary 1,
respectively. Namely, Theorem 1 is an immediate conclusion of Lemma 2.2, and Proposition 2.2 is important to show Lemma 2.2 or $\mathcal{U}(s_0, \infty) \neq \emptyset$.

Moreover, we can prove Proposition 2.2 by an argument similar to establish [11, Proposition 4.2]. Because, the equation (37) is essentially same as (7). That is to say, the eigenvalues and the eigenfunctions of the operator $\mathcal{B}$ in (37) are essentially same as those of the operator $\mathcal{A}$ in (8), since $\mathcal{B} = \mathcal{A} - N/2$. As for the estimate of the nonlinear term of (4) in $[0, e^{\sigma s}]$ and that of (36) in $[0, e^{\kappa \tau}]$, there is no crucial change between the proof of Proposition 2.2 and that of [11, Proposition 4.2]. As seen from [11], all we need about the nonlinear term in $[e^{\sigma s}, 1]$ is that

$$F(y, \phi(y, s)) \leq \frac{C}{y^k} \quad \text{for} \quad y \in [e^{\sigma s}, \infty) \quad \text{and} \quad s \in [s_0, s_1]$$

with some $C > 0$ and a sufficiently large $k > 0$. By Lemma 2.3, we get a desired estimate of $F$ in $[e^{\sigma s}, \infty)$. Thus, we see that an argument similar to the one in [11] works.

It is immediate that

$$\mathcal{U}(s_0, s_0) = \left\{ d \in \mathbb{R}^J, \sum_{j=0}^{J-1} |d_j| < \varepsilon \theta_1 e^{-\lambda_j s_0} \right\}.$$

**Proposition 2.3** Let $K \gg \tilde{K} \gg 1$ and $s_0 \gg 1$. If $\mathcal{U}(s_0, s_1) \neq \emptyset$ with $s_1 > s_0$, then

$$\deg(P(\cdot, s_0, s_1), 0, \mathcal{U}(s_0, s_1)) = 1$$

where $\deg(P(\cdot, s_0, s_1), 0, \mathcal{U}(s_0, s_1))$ denotes the degree of $P(\cdot, s_0, s_1)$ with respect to 0 in $\mathcal{U}(s_0, s_1)$.

**Proof.** We first see

$$p_j(d; s_0, s_0) = d_j - e^{-\lambda_j s_0} \langle \phi_j, \tilde{\phi}_J \rangle \quad \text{for} \quad j = 0, 1, \ldots, J - 1.$$

Using an argument similar to establish [11, (4.14)], for any sufficiently small $\delta > 0$ we can show that

$$|\langle \phi_j, \tilde{\phi}_J \rangle| < \delta e^{-\eta s_0} \quad \text{for} \quad j = 0, 1, \ldots, J - 1,$$

if $s_0 \gg 1$. Let $I$ be the identity mapping in $\mathbb{R}^J$. Then, it holds that

$$I(d) + \theta \left( P(d; s_0, s_0) - I(d) \right) \neq 0 \quad \text{on} \quad \partial \mathcal{U}(s_0, s_0) \quad \text{for} \quad \theta \in [0, 1],$$

if $s_0 \gg 1$. By the homotopy invariance of the degree, we have

$$\deg(P(\cdot, s_0, s_0), 0, \mathcal{U}(s_0, s_0)) = \deg(I, 0, \mathcal{U}(s_0, s_0)) = 1.$$

It follows from Proposition 2.2 that $P(\partial \mathcal{U}(s_0, s_1); s_0, s_1) \neq 0$ for $s_1 > s_0$. Therefore, the homotopy invariance of the degree implies the conclusion. \qed
Proposition 2.4 If $K \gg \tilde{K} \gg 1$ and $s_0 \gg 1$, for all $s > s_0$ it holds that $\mathcal{U}(s_0, s) \neq \emptyset$.

Proof. Putting

\[ s^* = \sup \{s > s_0 : \mathcal{U}(s_0, s) \neq \emptyset \}, \]

we see that $s^* > s_0$. Assume that $s^* < \infty$. Taking a sequence $\{s_n\}_{n=1}^{\infty} \subset [s_0, s^*)$ with $\lim_{n \to \infty} s_n = s^*$, for each $n$ we can take $d_n \in \mathcal{U}(s_0, s_n)$ such that $P(d_n; s_0, s_n) = 0$ by Proposition 2.3. Since $\{d_n\}_{n=1}^{\infty}$ is bounded, we may assume that $\lim_{n \to \infty} d_n = d^*$ for some $d^*$, without loss of generality. Then, it is immediate that $d^* \in \mathcal{U}(s_0, s^*)$ and $P(d^*, s_0, s^*) = 0$. It follows from Proposition 2.2 that $m(\cdot, \cdot; d^*) \in A(s_0, s^*; \theta)$ with some $\theta \in (0, 1)$. By the continuous dependence, we get $m(\cdot, \cdot; d^*) \in A(s_0, s^* + \delta; 1)$ for some $\delta > 0$. This and (21) contradict the definition of $s^*$. Thus, we get this proposition.

\[ \square \]

Proof of Lemma 2.2. Let $J$ be a positive even integer, and let $\{s_n\}_{n=1}^{\infty} \subset [s_0, \infty)$ be a sequence satisfying $\lim_{n \to \infty} s_n = \infty$. It follows from Propositions 2.2, 2.3 and 2.4 that $\mathcal{U}(s_0, s_n) \neq \emptyset$. Then, there exists a sequence $\{d_n\}_{n=1}^{\infty}$ satisfying $d_n \in \mathcal{U}(s_0, s_n)$. Since $\{d_n\}_{n=1}^{\infty}$ is bounded, we can assume that $d^* = \lim_{n \to \infty} d_n$ for some $d^*$, without loss of generality. Combining this with $m(\cdot, \cdot, d_n) \in A(s_0, s_n; 1)$ implies that $m(\cdot, \cdot; d^*) \in A(s_0, \infty; 1)$. By this and Proposition 2.1, we obtain that $m(\cdot, \cdot; d^*)$ is a desired solution. Thus, we have this lemma in the case where $J$ is a positive even integer.

In the case of $J = 0$, we take

\[ \phi(y, s_0) = e^{y^2/4} \{m(y, s_0) - m_\infty(y)\} = e^{-\lambda s_0} \varphi_0(y) \]

as follows:

$(\phi 1')$ For $y \in [0, \tilde{K}e^{-\eta s_0}]$,

\[ \varphi_0(y) = e^{\lambda s_0} e^{y^2/4} \{m_\infty(y) - [\theta'_1 m_1(y, s_0) + (1 - \theta'_1) m_2(y, s_0)]\}, \]

where $\theta'_1 \in [0, 1]$ is a constant satisfying

\[ \theta'_1 m_1(\tilde{K}e^{-\eta s_0}, s_0) + (1 - \theta'_1) m_2(\tilde{K}e^{-\eta s_0}, s_0) = m_\infty(\tilde{K}e^{-\eta s_0}) - e^{-\lambda s_0} \exp\left(-(\tilde{K}e^{-\eta s_0})^2/4\right) \varphi_0(\tilde{K}e^{-\eta s_0}). \]

$(\phi 2')$ For $y \in [K e^{-\eta s_0}, e^{\delta s_0}]$, $\varphi_0(y) = \varphi_0(y)$.

$(\phi 3')$ For $y \geq e^{\delta s_0}$,

\[ \phi(y, s_0) = e^{y^2/4} \min \{m'_E(s_0) - m_\infty(y), 0\}, \]
where

\[ m'(s_0) = m_\infty(e^{s_0}) - e^{-\lambda_0 s_0} \exp\left(-e^{2s_0}/4\right) \varphi_0(e^{s_0}). \]

Then, \( \phi(\cdot, s_0) \) satisfies that \( \phi(\cdot, s_0) \leq 0 \) in \( \mathbb{R}_+ \). Let \( \phi \) be the solution to (7) whose initial condition is the above \( \phi(\cdot, s_0) \). Then, \( \phi \) satisfies that

\[
\phi(\cdot, s) = e^{-(s-s_0)A} \phi(\cdot, s_0) + \int_{s_0}^{s} e^{-(s-\tilde{s})A} F(\cdot, \phi(\cdot, \tilde{s})) d\tilde{s}
\]

\[ = S_1(\cdot, s) + S_2(\cdot, s) + S_3(\cdot, s), \]

where

\[
S_1(\cdot, s) = -e^{-\lambda_0 s} \langle \tilde{\varphi}_0, \varphi_0 \rangle \varphi_0,
\]

\[
S_2(\cdot, s) = -\sum_{j=1}^{\infty} e^{-\lambda_0 s} \langle \tilde{\varphi}_0, \varphi_j \rangle e^{-\lambda_j (s-s_0)} \varphi_j
\]

\[
S_3(\cdot, s) = \int_{s_0}^{s} e^{-(s-\tilde{s})A} F(\cdot, \phi(\cdot, \tilde{s})) d\tilde{s}.
\]

Then, by using the quite same calculation as the case where \( J \) is a positive even integer, we can show that \( m = \phi + m_\infty \in \mathcal{A}(s_0, \infty; 1) \), since \( 0 < \lambda_0 < \lambda_1 < \cdots \). By this and an argument to establish this lemma in the case where \( J \) is a positive even integer, we get this lemma in the case of \( J = 0 \). Thus, we have this lemma.

\[ \square \]

**Proof of Theorem 1.** Using Lemma 2.2 and an argument similar to establish Corollary 1, and replacing \( t \) by \( t - t_0 \), we have this theorem.

\[ \square \]

**Proof of Theorem 2.** For any initial data \( M_0 \in L^\infty(\mathbb{R}_+) \) with \( M_0 \leq M_\infty \) in \( \mathbb{R}_+ \), the solution \( M \) to (11) satisfies \( M(r, t) \leq M_\infty(r) \) for \( (r, t) \in \mathbb{R}_+ \times (0, \infty) \). Thus we can choose a solution \( \hat{M}_h \) defined by \( \hat{M}_h(r, t) = h^2 \hat{M}(hr, h^2 t + t_0) \) with some \( t_0 \gg 1 \) and a solution \( \hat{M} \) obtained in Lemma 2.2 in the case of \( J = 0 \) and \( h > 0 \) such that \( M(r, t_1) \leq \hat{M}_h(r, t_1) \) in \( \mathbb{R}_+ \) for some \( t_1 \geq 0 \) by the choice of initial data of \( \hat{M} \). The comparison principle implies \( |M(\cdot, t)|_{L^\infty(\mathbb{R}_+)} \leq |\hat{M}(\cdot, t)|_{L^\infty(\mathbb{R}_+)} \) for any \( t \geq t_1 \). Combining this with Theorem 1 implies that this theorem.

\[ \square \]

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References


