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A modified shell model for rotating turbulence is proposed. The effect of rotation is introduced by a randomized linear term. Randomization is shown to be important in correctly modeling the rotation effect. Numerical simulation shows that the exponent of the energy spectrum in the inertial range changes from \(-5/3\) to \(-2\) as rotation rate increases. The mechanism behind this change is explained by weak turbulence theory and supported by numerical results.

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I. INTRODUCTION

Statistical properties of rotating turbulence are less understood than those of homogeneous isotropic turbulence, although rotating turbulence is important in fluid motion both in the atmosphere and oceans and in turbomachinery. The equations of motion are the Navier-Stokes equations for an incompressible fluid in a rotating frame,

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + 2\Omega \times u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u, \quad (1)
\]

where \(\Omega\) is the angular velocity of system rotation. When the rotation is strong, or in other words, the Rossby number \(R_o = U/\Omega L \sim |u \cdot \nabla u|/|\Omega \times u|\) is much smaller than 1 and the Reynolds number is sufficiently large, the flow is considered to approach an essentially two-dimensional (quasi-2D) state as expected from the Taylor-Proudman theorem after a (possibly long) transient process. The small-scale turbulence is important not only in the transient process but also in the quasi-2D state where some kind of forcing injects energy so that the turbulent component is maintained. In the quasi-2D state, stretching and folding of vorticity lines in the direction of the rotation axis are suppressed; energy transfer is also suppressed. It should be noted that the quasi-2D state is usually different from the two-dimensional turbulence as discussed below.

There have been different predictions for the power law of the energy spectrum in the inertial range: the same argument as discussed below.

II. MODIFIED SHELL MODEL

A. Equation

Let us consider the following shell model for the rotating turbulence:
\[
d\frac{dZ_n}{dt} = i\alpha k_{n+1}Z_{n+1} + i\beta k_nZ_n + i\gamma k_{n-1}Z_{n-1} - \nu k_n^2 Z_n + i\bar{\Omega}_n(t)Z_n + f_n, \tag{3}
\]
where \(Z_n\) is a representative mode of \(u\) corresponding to the wave number \(k_n = k_0 \lambda^n\) and \(\alpha + \beta + \gamma = 0\). When \(\bar{\Omega}_n(t) = 0\), it is an improved shell model by L'vov et al. for nonrotating turbulence. The additional term \(i\bar{\Omega}_n(t)Z_n\) models the effect of rotation. The rotation rate \(\bar{\Omega}_n(t)\) is defined by

\[
\Omega_n = \Omega_{\Delta} + \Omega'_n, \quad \frac{d\Omega'_n}{dt} = -\frac{\Omega'_n}{T} + \bar{\Omega}_n(t), \tag{4}
\]

\[
\frac{d\bar{\Omega}_n}{dt} = -\frac{\bar{\Omega}_n}{\tau} + g_n(t), \tag{5}
\]
where \(g_n(t)\) is Gaussian white noise with

\[
\langle g_n(t) g_m(s) \rangle = \sigma_n^2 \delta(t-s). \tag{6}
\]

Thus \(\bar{\Omega}_n\) is a sum of the mean value \(\Omega_{\Delta}\) and the fluctuating part \(\Omega'_n\) which has the correlation time \(\tau\). In fact \(\Omega'_n\) is a filtered noise of \(\bar{\Omega}_n\), \(T^{-1}\) being the low-pass cutoff frequency; \(\bar{\Omega}_n\) is correlated as \(\langle \bar{\Omega}_n(t) \bar{\Omega}_n(s) \rangle \approx \exp(-|t-s|/\tau)\). Note that the fluctuating part has a significant role in correctly taking account of the rotational effect as shown later.

We may interpret the randomized linear term introduced above as follows. The inertial waves

\[
u = U \exp[i(k \cdot x - \omega t)],
\]

\[
k \cdot U = 0, \quad \omega = 2\Omega \frac{k}{k} = 2\Omega \cos \theta, \tag{7}
\]
are solutions of Eqs. (1) and (2) if we neglect the nonlinear and viscous terms. Here \(\theta\) is the angle between \(\Omega\) and \(k\). Since \(Z_n\) represents Fourier modes whose wave vectors satisfy \(k_0 \lambda^n \ll k < k_0 \lambda^{n+1}\), \(\theta\) should vary as \(Z_n\) moves among the Fourier modes. In other words, \(Z_n\) stochastically represents the corresponding set of Fourier modes. The corresponding variation of the angular frequency \(\omega\) is taken account into \(\bar{\Omega}_n\). The time scale of variation is \(\tau\); in the numerical computation we set \(\tau = \Omega^{-1}_n\) as it would be reasonable that these two time scales are in the same order of magnitude. The low-pass cutoff is introduced so that \(\bar{\Omega}_n\) is smooth; we set \(T = 0.1\Omega^{-1}_n\). The results below are insensitive to \(\tau\) and \(T\) as far as they are in reasonable ranges of values, while choosing too large \(\tau\) is essentially same with constant \(\Omega\).

The shell model equation (3) has two invariants,

\[
E = \sum_n |Z_n|^2, \quad H = \sum_n \left( \frac{\alpha}{\gamma} \right)^n |Z_n|^2, \tag{8}
\]
when \(\alpha + \beta + \gamma = 0\). In the following, the parameters are set to \(N = 26, \lambda = k_{n+1}/k_n = 2, \alpha = 1, \beta = -1, \gamma = -0.5\); we have confirmed that the results below are essentially unchanged by varying these parameters as long as the model stays in a chaotic regime. Forcing is given at \(n = 1\): \(f_1 = 0.5 + 0.5i, f_2 = 0(n \geq 2)\).
In particular, if $\Omega_n(t)$ were simply white noise so that $\tau=T=0$ in Eqs. (4) and (5), then
\[ R_n(t,s) = \sigma_n \delta(t-s), \]  
and
\[ G_n(t,s) = e^{-\sigma_n \gamma (t-s) / 2} H(t-s). \]  
If instead $R_n$ is given by Eq. (12), then
\[ G_n(t,s) = e^{-\sigma_n \gamma (t-s) / 2} \exp[\gamma (t-s) / 2] H(t-s). \]  
The applicability of weak turbulence theory to Eq. (9) is linked to the linear decorrelation mechanism provided by the random process $\Omega_n(t)$. First recall the crucial diagonal property of correlations in the shell model of L’vov et al. [7],
\[ \langle Z_n(t) Z_m(s) \rangle = \delta_{nm}. \]  
Define the correlations
\[ U_n(t,s) = \langle Z_n(t) Z_n^*(s) \rangle, \]
\[ U_n^*(t,s) = \langle Z_n(t) Z_n(s) \rangle. \]  
In the weak turbulence approximation, the dominance of linear decorrelation implies that the two-time and single-time correlations in shell $n$ are related by
\[ U_n(t,s) = \exp \left[ i \int_s^t d\tau \Omega_n(\tau) \right] Z_n(s) Z_n^*(s) = G_n(t,s) U_n(s), \]
\[ U_n^*(t,s) = \exp \left[ i \int_s^t d\tau \Omega_n(\tau) \right] Z_n(s) Z_n^*(s) = G_n(t,s) U_n^*(s), \]  
if $t \geq s$, where $U_n(s)$ denotes the single-time correlation $U_n(s)=U_n(s,s)$. Since $U_n^*(t,s)=0$ if $U_n^*(t,s)=0$, this correlation will be ignored in what follows.

The perturbation theory described in detail in Zhabarov et al. [9] or Benny and Saffman [10] leads to the governing equation for the single-time correlations $U_n(t)=U_n(t,t)$,
\[ \frac{dU_n}{dt} = \alpha \kappa_{n+1} \Theta_{n+2,n+1} \left[ \beta U_{n+1} U_n + \gamma \nu_{n+1} U_n + \alpha U_{n+1} U_{n+1} \right] + \beta \kappa_{n+1} \Theta_{n+1,n+1} \left[ \beta U_{n+1} U_{n+1} + \gamma \nu_{n+1} U_{n+1} + \alpha U_{n+1} U_{n+1} \right]
+ \gamma \kappa_n \Theta_{n,n-1} \left[ \beta U_{n-1} U_n + \gamma \nu_{n} U_{n-1} + \alpha U_{n-1} U_{n-1} \right], \]  
where
\[ \Theta_{m+1,m,m-1} = \int_{0}^{\infty} d\tau G_{m+1}(\tau) G_m(\tau) G_{m-1}(\tau). \]  

Time stationarity has been assumed, so that $G(t,s)=G(t-s)$ is a function of time difference only.

Defining
\[ \Xi_n = \Theta_{n+1,n+1,n+1} \kappa_{n+1,2} \left[ \alpha U_{n+1} U_n + \beta U_{n+1} U_{n-1} + \gamma \nu_{n} U_{n-1} \right], \]  
Eq. (26) takes the form
\[ \frac{dU_n}{dt} = \alpha \Xi_{n+1} + \beta \Xi_n + \gamma \Xi_{n-1} = \left[ \alpha \Xi_{n+1} - \gamma \Xi_n \right] \]  
\[ - \left[ \alpha \Xi_n - \gamma \Xi_{n-1} \right]. \]  
A constant flux solution of Eq. (29) is defined by
\[ \epsilon = \alpha \Xi_{n+1} - \gamma \Xi_n, \]
where $\epsilon$ denotes the energy flux. For the linear response function defined by Eq. (20),
\[ \Theta_{m+1,m,m-1} = \frac{1}{\sigma_{m+1} + \sigma_m + \sigma_{m-1}}. \]  
Equation (31) is also a good approximation for large shell indices $m$ for the more general response function Eq. (21).

Power counting in Eq. (30) shows that if
\[ \sigma_m \sim k_m^\mu, \]  
and if, as usual, the shells are in a geometric progression, $k_m=k_0\lambda^m$, then Eq. (26) admits the formal solution
\[ U_m \sim k_m^{-1+z\mu^2}. \]  
Standard arguments (compare L’vov et al. [9]) show that a solution of Eq. (26) exists with the scaling Eq. (33) in the “region of transparency” in which the unforced, inviscid equation Eq. (9) is valid.

Equation (33) corresponds to an energy spectrum scaling as
\[ E_m \sim k_m^{-2+z\mu^2}. \]  
In particular, for the case studied by Hattori and Ishizawa [4] in which $\mu=1$, $E_m \sim k_m^{-5/3}$, in agreement with their numerical simulations. In the present case with $\mu=0$, we have the theoretical prediction $E_m \sim k_m^{-2}$.

It was noted earlier that weak turbulence theory applies only when linear phase decorrelation dominates nonlinear phase decorrelation, so that
\[ \sigma_m \gg \sqrt{E_m k_m^3}. \]  
If weak turbulence generates an energy spectrum for which $k_m^3 E_m$ increases with $m$, the inequality in Eq. (35) can be satisfied for small $m$ but violated for sufficiently large $m$. For large $k_m$, linear decorrelation is a perturbation of nonlinear decorrelation, and Kolmogorov scaling $E_m \sim k_m^{-5/3}$ is recovered.

The phenomenological theory proposed by Zhou [5] and the closure theory of Canuto and Dubovikov [6] both predict the same picture of the spectral scaling in rotating turbulence, with $E(k) \sim k^{-2}$ at large scales, Kolmogorov scaling at small scales, and a transition between these two regimes when the rotation rate $\Omega$ satisfies $\Omega \approx \epsilon^{1/3} k^{5/3}$. These arguments perhaps apply better to the present problem, which is
effectively “isotropic,” than to rotating turbulence, in which anisotropy has an important role. The role of anisotropy in the spectral scaling of rotating turbulence remains an open question; “anisotropic” shell models in which the complex amplitude $Z_n$ is replaced by a higher dimensional geometric quantity may have a role in deciding this question.

It is noteworthy that weak turbulence scalings can be obtained from a problem in which resonant triads are entirely absent. The essential feature which permits weak turbulence scaling is the existence of the linear decorrelation mechanism provided by the random phase factors $V_{ns}$. But we stress that in a shell model, linear decorrelation only occurs if the phase factors are random; the introduction of deterministic phase factors cannot disrupt normal Kolmogorov scaling. This conclusion agrees with the numerical simulations of Hattori and Ishizawa [4] in which the scaling exponent depends on $n$ when, in effect, $\Omega_\tau$ is constant in each shell. The situation is of course entirely different for dispersive waves in two or three dimensions: although the linear decorrelation mechanism is deterministic, the interaction of waves with different wavevectors introduces the necessary phase randomization.

### III. NUMERICAL RESULTS

#### A. Energy spectrum

Figure 1 shows the energy spectrum obtained by the present shell model. The spectrum is multiplied by $k^2$ so that we can see subtle differences in the scaling exponent. In Fig. 1(a), the scaling law is close to the Kolmogorov law $E_n \approx k_n^{-5/3}$ for $\Omega_\tau=0$, the nonrotating case; actually the exponent

FIG. 1. Energy spectrum. (a) $\Omega_\tau=0, 1, 4, 7, 21, 35, \text{ and } 71$. (b) Comparison of constant and random large-scale effects. $\Omega_\tau=71$. Note that $k_n^{5/3}$ in (a) corresponds to the Kolmogorov 5/3 law.

is slightly smaller than $-5/3$. As $\Omega_\tau$ increases, there appears a region of $n$ for which the scaling exponent is close to $-2$. For $\Omega_\tau=71$, we see $E_n \approx k_n^{-2}$ in the entire inertial range. In Fig. 1(b), we compare the cases with and without the fluctuation $\Omega'_n$ in the rotation term. The energy spectrum does not show power law behavior for the case without fluctuation. Therefore the fluctuation is a key factor for the shell model to take account of strong rotation as discussed in the previous section.

#### B. Correlation

According to weak turbulence theory, the transition from the $k^{-2}$ spectrum to the $K41$ spectrum is caused by the change in the characteristic time scale. We can see this change in Fig. 2, where the correlation defined by

$$\text{Cor}(n; \tau) = \frac{\langle Z_n(t)Z_n(t+\tau) \rangle}{\langle |Z_n(t)|^2 \rangle} = \frac{U_n(t,t+\tau)}{U_n(t,t)},$$

FIG. 2. Correlation (see text for the definition). (a) $\Omega_\tau=0$, real part, (b) $\Omega_\tau=0$, imaginary part, (c) $\Omega_\tau=0$, real part, (d) $\Omega_\tau=71$, imaginary part.
C. Third-order structure function

Figure 3 shows the “third-order structure function”

$$S_3(k_n) = \text{Im}(Z_n Z_n Z_{n+1}^*),$$

introduced by L’vov et al. [7]. As in L’vov et al., we have

$$\frac{d}{dt}(|Z_n|^2) = 2k_n \left[ 2 \alpha S_3(k_{n+1}) + \beta S_3(k_n) + \frac{\gamma}{2} S_3(k_{n-1}) \right] + 2 \langle f_n Z_n^2 \rangle,$$

(37)

neglecting viscosity, which in stationary conditions leads to

$$S_3(k_n) = \frac{1}{k_n} \left[ A + B \left( \frac{\gamma}{\alpha} \right)^n \right],$$

(38)

with some constants $A$ and $B$. Thus $S_3 \propto k_n^{-1}$ for large $n$. Figure 3(a) confirms this relation. Note that the magnitude of $S_3$ is rather small for $\Omega_e = 71$. In other words, energy transfer is reduced owing to the rotation effect. This is relatively significant for small wave numbers as shown in Fig. 3(b). For $\Omega_e = 0$, $S_3/(S_2)^{3/2}$ (the “second-order structure function” $S_2$ is $\langle |Z_n|^2 \rangle$ as usual) is nearly constant in the inertial range; for $\Omega_e = 71$, it increases with $k_n$. As a result energy accumulates in small-wave-number modes, leading to the steeper energy spectrum $E_n \propto k_n^{-2}$.

D. PDF

The energy spectrum scales as $E_n \propto k_n^{-2}$ when the rotation effect is strong; the same power law is observed in the experiment by Baroud et al. [2]. Then it is of interest to see whether or not probability distribution function (PDF) behaves similarly. In order to construct velocity fields from shell variables, we employ the method used in Jensen [8]; that is,

$$u(x,t) = \sum_{n=1}^{N} c_n Z_n(t) \exp[i k_n \cdot x] + \text{c.c.},$$

where $k_n = k_n e_n$ and $e_n$ and $c_n$ are unit vectors in random directions. We impose

$$c_n \cdot e_n = 0, \quad n = 1, \ldots, N,$$

which is a sufficient condition for incompressibility. We average over a large number of sets $\{e_n, e_n\}$ to achieve a good statistical average.

Figure 4 shows PDF’s of the longitudinal velocity difference $\delta u$ obtained by the method described above. The PDF’s are not self-similar either for $\Omega_e = 0$ or for $\Omega_e = 71$. There is no significant difference between $\Omega_e = 0$ and $71$. This differs from the experiment which observed self-similar but non-Gaussian PDF’s for strong rotation.

These results raise the more general question of anomalous scaling in turbulent systems subject to external agencies. Recent investigations [12,13] of turbulence and shell models driven by forcing with power-law correlation $\langle f(k, t)f(-k, t) \rangle \sim ak^{-\gamma} \delta(t-s)$ (for simplicity, we only write the scalar amplitude of the force correlation; more complete definitions are found in the cited references) suggest that when the forcing dominates nonlinearity, higher order structure functions obey the “normal” scaling obtained by dimensional analysis, and velocity difference PDF’s over inertial range separations are self-similar. However, when the forcing decays sufficiently rapidly at large $k$, nonlinearity becomes dominant, and the non-self-similar velocity difference PDF’s of unforced turbulence are recovered.

A heuristic analogy between such forced systems and the present shell model might suggest that when linear decorrelation is dominant over all scales, as when $\Omega_e = 71$, self-similar velocity difference PDF’s should be observed, with increasingly dominant anomalous effects as $\Omega_e$ approaches zero. At this point, we cannot resolve the discrepancy between this expectation and our contrary computational re-
results: perhaps forcing has a more direct impact on shell models than modification of the time scale; alternatively, the linear decorrelation mechanism proposed here may simply be much weaker than the linear decorrelation of dispersive waves. Another possibility is that the subtle interplay between external agencies and nonlinearity is simply beyond the scope of our shell model. In any case, perhaps our results underscore the apparent independence of second order statistical properties like spectral scaling exponents from more refined details like anomalous scaling properties.

IV. CONCLUDING REMARKS

We have proposed a modified shell model for rotating turbulence. The transition from $k^{-5/3}$ spectrum for weak rotation to $k^{-2}$ spectrum for strong rotation is observed as predicted by weak turbulence theory. Although the energy spectrum obeys the same power law as the experiment by Baroud et al. [2], the results on PDF’s are different. The PDF’s obtained by the present model are non-Gaussian as in the experiment. However, unlike the self-similar PDF’s observed in the experiment, they are not self-similar. The direction of energy cascade may be responsible for this difference. The present model shows the normal cascade from small wave number to large wave number, while the experiment showed some evidence for the inverse cascade. The present model would be more closely related to the numerical simulation by Smith and Waleffe [1]; they observed $k^{-2}$ spectrum for $k > k_f$ with $k_f$ being the forcing wave number, which implies that the direction of energy cascade is normal. Simulations with higher Reynolds numbers are expected for more precise evaluation of the present model.

It is shown that introduction of the randomized rotation term leads to $k^{-2}$ spectrum. The same idea can be applied to other cases for which weak turbulence theory works, e.g., turbulence of gravity waves.