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Separability of a Low-Momentum Effective Nucleon-Nucleon Potential

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A realistic nucleon-nucleon potential is transformed into a low-momentum effective one (LMNN) using the Okubo theory. The separable potentials are converted from the LMNN with a universal separable expansion method and a simple Legendre expansion. Through the calculation of the triton binding energies, the separability for the convergence of these ranks is evaluated. It is found that there is a tendency for the lower momentum cutoff parameter Λ of LMNN to gain good separability.

1. *Introduction* A unified theory is regarded as an *integration* of some independent Hilbert spaces, and an effective (equivalent) theory or renormalization is regarded as a *differentiation* into the physical space (P space) of interest and the remaining space (Q space). Theories of these two types are closely related, and they are the main themes of physics. The relation between a unified theory and an effective theory can be explicitly explained by the Okubo effective theory.[1] This theory is universally useful in many areas of physics. For example, it has a useful application to the recently developed chiral perturbation theory[2] in meson-nucleon systems. The original Lagrangian of the nucleon (N) and pion (π) fields generates a NN interaction. The bare NN interaction is connected with the πNN sector, we need to renormalize the sector into the effective NN interaction up to the πN threshold.

In the context of many-nucleon systems in nuclear physics, Suzuki and Okamoto extended the Okubo theory to a useful scheme called the unitary model operator approach (UMOA).[3,4] The UMOA is an approach to the study of many-body systems that considers the effective interactions in a nuclear medium, which are

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determined by solving the decoupling equation between the model space and its complement space. When the effective theory in the sense of the Okubo theory and UMOA is applied to a two-body system, it generates a low-momentum nucleon-nucleon (LMNN) potential by defining a model space (P space) and its complement (Q space). Bogner et al.[5),6)] suggested that their LMNN constructed with the G-matrix scheme is useful in application to many-body systems.

In order to determine the accuracy of LMNN, one needs to calculate the triton and the alpha-particle binding energies, where the Faddeev equation or the Yakubovsky equation gives the exact solutions for the given potential. It was concluded in Ref. 7) that in the case of realistic *NN* forces, e.g., Nijm-I[8)] and CD-Bonn[9)] potentials, the recommended cutoff parameter Λ, must at least be larger than 5 fm^{-1} in order to reproduce the exact values of the binding energies in these systems. The calculation of the ground state energy using the LMNN for the cutoff parameter $\Lambda \approx$ 2 fm^{-1} yields a considerably more attractive result (more binding energy) than the exact value. Variational principle is in possession a repulsion property (less binding energy) which its absolute value of the binding energy is less than the true one. There could be an accidental cancellation between the attraction caused by the short cutoff parameter Λ and the repulsion from the variational principle.

In addition to the above discussion, we would like to consider another property of the LMNN interaction. In general, when the Hilbert space is truncated into a small P space, the structure of the bases is expected to be very simple and regular. The *NN* interaction is expanded into a separable form. This greatly reduces the necessary computational time and memory size for these relatively heavy calculations in few-body systems.[10),11)] More precisely, we try to restrict the degrees of freedom for the continuous variables in the integral equations by introducing a separable potential. In the case of a three-nucleon system, the accuracy of the calculation has been examined using some benchmarks. The separable potential has the rank of the form factors that describe the behavior of the potential. The accuracy improves as the rank becomes larger but we need small rank. The simplicity of the P space is considered to be reflected in the convergence of the rank or the separability. In the application to few-body calculations it is interesting to consider whether the LMNN potential has the merit of separability. We believe that the LMNN potential possesses good separability and that it will reduce the computational time of numerical calculations.

In the next section, we introduce two kinds of separable expansions. The triton binding energies are calculated using these finite rank separable potentials in §3. We would like to investigate the convergence of the rank or the separability in order to show how the Hilbert space is effectively simplified. Discussion of the present result and the outlook for further developments is given in §4.

2. *Simple separable expansion and the universal separable expansion* In Ref. 7), the LMNN interaction was obtained using two kinds of methods. There, it was numerically confirmed that the methods proposed by Glöckle and Epelbaum[12)] and by Suzuki and Okamoto[3),4)] lead to the same LMNN interaction. The LMNN potential

$V(p,p')$ satisfies the following Lippmann-Schwinger equation at energy E:

$$T(p,p';E) = V(p,p') + \int_0^{\Lambda} V(p,p'')G_0(p'',E)T(p'',p';E)p''^2 dp''. \tag{1}$$

Here, T, V, G_0 and Λ are the transition matrix (t-matrix), the LMNN potential, the Green function of two free particles, and the cutoff parameter at low momentum, respectively.

Now, the momentum variable p in the integral is replaced by another variable x, defined through the relation

$$p = \frac{x+1}{2}\Lambda, \tag{2}$$

and the potential and the t-matrix are expanded into the separable forms

$$V(p,p')\; pp' \approx V^{sep}(p,p')\; pp' \equiv \sum_{i,j=1}^{n} g_i(x)\lambda_{i,j}g_j(x') \tag{3}$$

and

$$T(p,p';E)\; pp' \approx \sum_{i,j}^{n} g_i(x)\tau_{i,j}(E)g_j(x'), \tag{4}$$

where g and λ are the form factor and the coupling constant. Then, we rewrite Eq. (1) as

$$\tau_{i,j}(E) = \lambda_{i,j} + \sum_{k,l}^{n} \lambda_{i,k}I_{k,l}\tau_{l,j}(E), \tag{5}$$

with

$$I_{k,l} \equiv \frac{\Lambda}{2}\int_{-1}^{1} g_k(x)G_0(p(x))g_l(x)dx. \tag{6}$$

Equation (5) is algebraically solved using the matrix inversion method. The number n represents the rank of the separable expansion.

The interval of integration is finite, and the potential has no singularity. Therefore, we conjecture that the LMNN potential is easily expanded into simple polynomials. The Legendre function $P_i(x)$ can be naturally chosen for such polynomials:

$$g_i(x) = P_i(x) \tag{7}$$

and

$$\lambda_{i,j} = \frac{(2i+1)(2j+1)}{4}\int_{-1}^{1}\int_{-1}^{1} V(p,p')pp'P_i(x)P_j(x')dxdx'. \tag{8}$$

This expansion is not new, and the Hanover group has succeeded[13)] in carrying out accurate Faddeev calculations for proton-deuteron scattering by using Chebyshev polynomials. In §3, we call this the simple separable expansion (SSE).

In the present context, a well-developed separable expansion scheme has also been introduced.[10] In this scheme a new form factor g_i is defined as

$$g_i(p) = \langle p|g_i\rangle \equiv \langle p|V|P_i\rangle = \int_{-1}^{1} V(p, p'(x'))P_i(x')dx' \tag{9}$$

and

$$V^{USE}(p, p')\ pp' \equiv \langle p|g\rangle\lambda\langle g|p'\rangle = \sum_{i,j}^{n} g_i(p)\lambda_{i,j}^{USE} g_j(p') \tag{10}$$

with

$$\left[\lambda_{i,j}^{USE}\right]^{-1} = \int_{-1}^{1}\int_{-1}^{1} V(p, p')pp'P_i(x)P_j(x')dxdx', \tag{11}$$

where $[\ \ ]^{-1}$ in Eq. (11) represents matrix inversion. The polynomials (Legendre functions in this case) are required only for the linear independence, while Eq. (7) of the SSE requires orthonormality. Therefore, it is understood that this expansion is a more general method. We call it the universal separable expansion (USE). In the case of the Faddeev calculation for *nd* scattering, high convergence was emphasized,[10] but we would like to investigate the separability of the LMNN interaction by using the SSE and the USE.

3. *Calculation of the triton binding energies using the USE and the SSE* The dependence of the accuracy of the LMNN potential on the cutoff momentum Λ has already been investigated.[7),14] We are now interested in determining how the separability develops when Λ is changed. For example, we employ the CD-Bonn potential,[9] which is well known as a modern precise potential.

We calculated the triton binding energies using the USE and SSE for various values of the cutoff Λ as in Ref. 7). For the sake of simplicity, the calculation was performed only for the 5-channel coupled Faddeev equation. More specifically, the potential is used only for ${}^1\mathrm{S}_0$ and ${}^3\mathrm{S}_1$-${}^3\mathrm{D}_1$ states. The results are listed in Table I. In the second line, the exact values obtained using the stated finite values of Λ, calculated without the separable approximation, are listed. The true value (i.e., that for $\Lambda = \infty$) is -8.312 MeV.

The bold numbers in Table I perfectly agree with the exact ones for each value of the cutoff, $\Lambda = 3$, 5 and 10 fm^{-1}. Comparing the SSE and the USE, it is seen that the USE has good convergence, because in the low rank steps, the USE leads to the corresponding exact values. The effective potential tends to have better separability in the lower-rank separable form. Lower values of Λ result in better separability of the LMNN interaction.

4. *Discussion and outlook* We calculated the triton binding energies employing the LMNN CD-Bonn potential with a cutoff parameter Λ in the unitary-transformation method of the Okubo theory. We find that there is the tendency that the separability improves as the value of the cutoff parameter is decreased. The result becomes close to the exact value calculated with a high-rank separable potential, and the obtained exact value depends on the cutoff parameter.

Table I. The triton binding energies. The energies are in units of MeV. The exact values for the stated finite cutoff Λ appear in the second line. Here, "SSE" and "USE" denote the simple separable expansion and the universal separable expansion.

	$\Lambda = 3$ fm^{-1}		$\Lambda = 5$ fm^{-1}		$\Lambda = 10$ fm^{-1}		$\Lambda = 20$ fm^{-1}	
	−8.532		−8.355		−8.329		−8.322	
rank n	SSE	USE	SSE	USE	SSE	USE	SSE	USE
20	**−8.532**	**−8.532**	**−8.354**	**−8.355**	−8.319	**−8.329**	−8.317	−8.320
18	**−8.532**	**−8.532**	−8.353	**−8.355**	−8.319	−8.327	−8.308	−8.319
16	**−8.532**	**−8.532**	−8.349	**−8.354**	−8.253	−8.382	−7.762	−7.973
14	−8.531	**−8.532**	−8.346	**−8.354**	−7.798	−8.305	−6.833	−7.196
12	−8.526	**−8.532**	−8.328	−8.351	−6.605	−8.203	−5.733	−8.089
10	−8.523	**−8.532**	−7.963	−8.311	−5.645	−7.958		
8	−8.326	−8.509	−6.730	−8.125				
6	−7.182	−8.014	−6.417	−7.741				

It is well known that the binding energy consists of the positive expectation value from the kinetic part (~ 50 MeV) and the negative expectation value from the potential (~ -60 MeV). The difference of 43 keV from the true value in the case of $\Lambda = 5$ fm^{-1} represents an error of only 0.1% error of the potential expectation value. The magnitude of a cross section in the scattering process is estimated the potential expectation value. Therefore, such a 0.1% error could be negligibly small. The Faddeev three-body scattering calculation is obtained precisely without the separable expansion,[15)–17)] but, in the case of four-body scattering great effort is still needed to obtain precise and stable solutions. The separable scheme reduces the amount of memory and cpu time needed in the numerical computation. The contour deformation integral technique requires the analyticity of the form factor function to avoid logarithmic singularities that arise from the two-body t-matrix and the Green function in momentum space. Most schemes used to solve the Faddeev equation with the separable expansion method employ this contour-deformation technique.[19)]

The functional form of LMNN potential apparently has no analyticity, and therefore it is not easy to apply the contour-deformation technique to the few-body scattering problem. Recently, the complex energy method (CEM) was introduced.[18)] The CEM enables us the calculation without an analytical form factor, because the CEM exploits the analytic continuation of energy. The solution is obtained by using the complex analytic continuation from sample solutions of the complex energies near the on-energy shell. The idea of the complex energy was introduced in order to avoid dangerous singularities.

We plan to construct a finite-rank precise separable potential from the LMNN interaction. The existence of such a potential is guaranteed by the good separability, as shown in the present work. Precise calculations using the USE potential not only for the three-body scattering problem but also for the four-body scattering problem will be carried out using the CEM.

The numerical calculations were performed mainly on a IBM RS/6000SP at the Research Center of Nuclear Physics (Osaka University) in Japan, and partly on a

Hitachi SR8000 at the Leibnitz-Rechenzentrum (die München Hochschule) and a Cray SV1 at NIC (Jülich) in Germany.

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