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ON THE RELATION BETWEEN THE WEAK PALAIS-SMALE CONDITION AND COERCIVITY BY ZHONG

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Abstract. In this paper, we discuss Zhong's result of that the weak Palais-Smale condition implies coercivity under some assumption in [Nonlinear Anal., 29 (1997), 1421–1431]. We also give a simple proof of Zhong's result. Further we generalize the result in Caklovic, Li and Willem [Differential Integral Equations, 3 (1990), 799–800].

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let f be a function from a Banach space X into $(-\infty, +\infty]$. We recall that f is called *Gâteaux differentiable* at $x \in X$ with $f(x) \in \mathbb{R}$ if there exists a continuous linear functional $f'(x)$ such that

$$\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \langle f'(x), y \rangle$$

holds for every $y \in X$. f is said to be *coercive* if

$$\lim_{r \to \infty} \inf_{\|x\| \geq r} f(x) = \infty$$

holds. Also, f is said to satisfy the *weak Palais-Smale condition* [17] if there exists a nondecreasing function h from $[0, \infty)$ into itself satisfying $\int_0^\infty (1/(1 + h(t)))dt = \infty$, and the following condition: Every sequence $\{x_n\}$ in X such that $\{f(x_n)\}$ is bounded and

$$\lim_{n \to \infty} \|f'(x_n)\| \big(1 + h(\|x_n\|)\big) = 0$$

contains a convergent subsequence. This definition seems to be weaker than the definition in [17]. However they are equivalent; see Section 5. In the case of $h(t) = 0$ for all $t \in [0, \infty)$, we call that f satisfies the *Palais-Smale condition.* In the case of $h(t) = t$ for all $t \in [0, \infty)$, we call that f satisfies the *Cerami-Palais-Smale condition* [4].

It is well known that the Palais-Smale condition implies coercivity under some assumption; see Brézis and Nirenberg [2], Caklovic, Li and Willem [3] and others. In 1997, Zhong [17] generalized these results and proved that the weak Palais-Smale condition implies coercivity. However the proof is slightly complicated.

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In this paper, we discuss Zhong's result and we also give a simple proof of it. Further we generalize the result in Caklovic, Li and Willem [3]. We also discuss the conditions of the continuity of h, $\int_0^\infty (1/(1+h(t)))dt = \infty$, and the completeness of X.

2. τ-Distance

In our discussion, the notion of τ-distance plays an important role.

Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X [10] if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;
(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$;
(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

We note that η is strictly increasing in its second variable. We also note that the metric d is a τ-distance on X. Many useful propositions and examples are stated in [7–16].

Though the following is a corollary of Proposition 2 in [12], we give a proof.

Proposition 1. *Let (X, d) be a metric space with a τ-distance p. Let q be a function from $X \times X$ into $[0, \infty)$. Suppose that*

(i) *q satisfies $(\tau 1)_q$, i.e., $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;*
(ii) *q is lower semicontinuous in its second variable;*
(iii) *$q(x, y) \geq p(x, y)$ for all $x, y \in X$.*

Then q is also a τ-distance on X.

Proof. Let η be a function satisfying (τ2)–(τ5). From the assumption (ii), $(\tau 3)_q$ clearly holds. We assume that $\lim_n \sup\{q(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$. Then from the assumption (iii), we have $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$. So by ($\tau$4), we obtain $\lim_n \eta(y_n, t_n) = 0$. This is $(\tau 4)_q$. Let us prove $(\tau 5)_q$. We assume that $\lim_n \eta\big(z_n, q(z_n, x_n)\big) = 0$ and $\lim_n \eta\big(z_n, q(z_n, y_n)\big) = 0$. Then from the assumption (iii) again, we have $\lim_n \eta\big(z_n, p(z_n, x_n)\big) = 0$ and $\lim_n \eta\big(z_n, p(z_n, y_n)\big) = 0$. So by ($\tau$5), we obtain $\lim_n d(x_n, y_n) = 0$. This completes the proof. □

Now, we give the following example.

Example 1. Let (X, d) be a metric space, and h a nondecreasing function from $[0, \infty)$ into itself such that $\int_0^\infty (1/(1+h(t)))dt = \infty$. Fix $z_0 \in X$. Then functions p and q from $X \times X$ into $[0, \infty)$ defined by

$$p(x, y) = \int_{d(z_0, x)}^{d(z_0, x) + d(x, y)} \frac{dt}{1 + h(t)} \quad \text{and} \quad q(x, y) = p(x, y) + p(y, x)$$

for all $x, y \in X$ are τ-distances on X.

Proof. We know that p is a τ-distance on X; see Proposition 4 in [10]. So, since p satisfies $(\tau 1)$, we have

$$\begin{aligned} q(x,z) &= p(x,z) + p(z,x) \\ &\le p(x,y) + p(y,z) + p(z,y) + p(y,x) \\ &= q(x,y) + q(y,z) \end{aligned}$$

for $x, y, z \in X$. This is $(\tau 1)_q$. It is obvious that q is continuous and $q(x,y) \ge p(x,y)$ for all $x, y \in X$. So by Proposition 1, we have q is a τ-distance on X. □

In [10], using the above p, the author gave the slight generalization and another proof of Zhong's variational principle [17, 18]. In this paper, we use the above q.

The following is Theorem 4 in [10], which is the τ-distance version of Ekeland's variational principle [5, 6]. Of course, this is one of the generalizations of the Banach contraction principle [1].

Theorem 1. *Let X be a complete metric space with a τ-distance p. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Then for $\varepsilon > 0$ and $u \in X$ with $p(u,u) = 0$, there exists $v \in X$ such that $f(v) \le f(u) - \varepsilon p(u,v)$ and $f(w) > f(v) - \varepsilon p(v,w)$ for all $w \in X$ with $w \ne v$.*

From Example 1 and Theorem 1, we obtain the following.

Theorem 2. *Let X, d, h, z_0 be as in Example 1. Suppose that X is complete. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Then for $\varepsilon > 0$ and $u \in X$, there exists $v \in X$ such that*

$$f(v) \le f(u) - \varepsilon \int_{d(z_0,u)}^{d(z_0,u)+d(u,v)} \frac{dt}{1+h(t)} - \varepsilon \int_{d(z_0,v)}^{d(z_0,v)+d(u,v)} \frac{dt}{1+h(t)}$$

and

$$f(w) > f(v) - \varepsilon \int_{d(z_0,v)}^{d(z_0,v)+d(v,w)} \frac{dt}{1+h(t)} - \varepsilon \int_{d(z_0,w)}^{d(z_0,w)+d(v,w)} \frac{dt}{1+h(t)}$$

for all $w \in X$ with $w \ne v$.

3. Zhong's Result

In this section, using Theorem 2, we can easily prove the following Zhong's result in [17]. Compare the proof with Zhong's. We use Theorem 2 only one time.

Theorem 3 (Zhong [17])**.** *Let X be a Banach space, and h a nondecreasing function from $[0, \infty)$ into itself such that $\int_0^\infty (1/(1+h(t)))dt = \infty$. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous. Assume that f is Gâteaux differentiable at every point $x \in X$ with $f(x) \in \mathbb{R}$. If*

$$\alpha := \lim_{r\to\infty} \inf_{\|x\|\ge r} f(x) \in \mathbb{R},$$

then there exists a sequence $\{x_n\}$ in X such that $\lim_n \|x_n\| = \infty$, $\lim_n f(x_n) = \alpha$, and

$$\lim_{n\to\infty} \|f'(x_n)\| \bigl(1 + h(\|x_n\|)\bigr) = 0.$$

Remark. In [17], the continuity of h is needed. We discuss this condition in Section 5.

In the proof of Theorem 3, we use the following lemma, which is well known.

Lemma 1. *Suppose that $c \geq 0$, $\delta > 0$, $v \in X$, $f(v) \in \mathbb{R}$ and either of the following holds:*

- $f(w) \geq f(v) - c\,\|v - w\|$ *for all* $w \in X$ *with* $0 < \|v - w\| < \delta$*; or*
- $f(w) \leq f(v) + c\,\|v - w\|$ *for all* $w \in X$ *with* $0 < \|v - w\| < \delta$*.*

Then $\|f'(v)\| \leq c$*.*

Proof of Theorem 3. We shall only show the following: For every $\varepsilon > 0$, there exists $v \in X$ satisfying $\|v\| \geq 1/\varepsilon$, $|f(v) - \alpha| \leq \varepsilon$, and $\|f'(v)\| \big(1 + h(\|v\|)\big) \leq \varepsilon$. Fix $\varepsilon > 0$. Define a function θ from $[0, \infty)$ into itself by

$$\theta(t) = 1 + 2\,h(t+1) \tag{1}$$

for $t \in [0, \infty)$. Then it is obvious that θ is nondecreasing, and we have

$$\int_0^\infty \frac{dt}{1 + \theta(t)} = \frac{1}{2} \int_0^\infty \frac{dt}{1 + h(t+1)} = \frac{1}{2} \int_1^\infty \frac{dt}{1 + h(t)} = \infty.$$

We also define a function g from X into $(-\infty, +\infty]$ by

$$g(x) = \max\big\{f(x),\ \alpha - 2\varepsilon\big\}$$

for $x \in X$. Then it is obvious that g is proper lower semicontinuous and bounded from below. We next choose $r, r' \in \mathbb{R}$ with $1/\varepsilon < r < r'$, $1 < r$,

$$\inf_{\|x\| \geq r} f(x) > \alpha - \varepsilon, \quad \text{and} \quad \int_r^{r'} \frac{dt}{1 + \theta(t)} = 3.$$

We also choose $u \in X$ with $\|u\| > r'$ and $f(u) < \alpha + \varepsilon$. We note that $g(u) = f(u)$ because of $\|u\| > r$. Then by Theorem 2, there exists $v \in X$ such that

$$g(v) \leq g(u) - \varepsilon \int_{\|u\|}^{\|u\| + \|u - v\|} \frac{dt}{1 + \theta(t)} - \varepsilon \int_{\|v\|}^{\|v\| + \|u - v\|} \frac{dt}{1 + \theta(t)} \tag{2}$$

and

$$g(w) > g(v) - \varepsilon \int_{\|v\|}^{\|v\| + \|v - w\|} \frac{dt}{1 + \theta(t)} - \varepsilon \int_{\|w\|}^{\|w\| + \|v - w\|} \frac{dt}{1 + \theta(t)} \tag{3}$$

for all $w \in X$ with $w \neq v$. Arguing by contradiction, we assume that $\|v\| < r$. From (2), we have

$$\begin{aligned}
\alpha - 2\varepsilon \leq g(v) &\leq g(u) - \varepsilon \int_{\|v\|}^{\|v\| + \|u - v\|} \frac{dt}{1 + \theta(t)} \\
&\leq g(u) - \varepsilon \int_{\|v\|}^{\|u\|} \frac{dt}{1 + \theta(t)} \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1 + \theta(t)} \\
&= f(u) - 3\varepsilon < \alpha - 2\varepsilon.
\end{aligned}$$

This is a contradiction. Therefore we obtain $\|v\| \geq r > 1/\varepsilon$. Thus we have $g(v) = f(v)$ and

$$\alpha - \varepsilon < \inf_{\|x\| \geq r} f(x) \leq f(v) \leq f(u) < \alpha + \varepsilon.$$

This implies $|f(v) - \alpha| \leq \varepsilon$. From (3) and nondecreasingness of θ, we have

$$g(w) > g(v) - \left(\frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|w\|)} \right) \|v - w\|$$

for $w \in X$ with $w \neq v$. Since f is lower semicontinuous and $f(v) > \alpha - 2\varepsilon$, there exists $\delta \in (0,1)$ such that $f(w) > \alpha - 2\varepsilon$ for $w \in X$ with $\|v - w\| < \delta$. Hence, for $w \in X$ with $0 < \|v - w\| < \delta$, since $g(w) = f(w)$ and

$$\|w\| \geq \|v\| - \|v - w\| > \|v\| - \delta > \|v\| - 1 > 0,$$

we have

$$\begin{aligned} f(w) &> f(v) - \left(\frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|v\| - 1)} \right) \|v - w\| \\ &\geq f(v) - \frac{2\,\varepsilon}{1 + \theta(\|v\| - 1)} \|v - w\| \\ &= f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|. \end{aligned}$$

So by Lemma 1, we have $\|f'(v)\| \big(1 + h(\|v\|)\big) \leq \varepsilon$. This completes the proof. □

As a direct consequence of Theorem 3, we obtain the following.

Theorem 4 (Zhong [17]). *Let X be a Banach space. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Assume that f is Gâteaux differentiable at every point $x \in X$ with $f(x) \in \mathbb{R}$, and f satisfies the weak Palais-Smale condition. Then f is coercive.*

Remark. We can weaken the condition that f satisfies the weak Palais-Smale condition as follows: Every sequence $\{x_n\}$ in X such that $\{f(x_n)\}$ is bounded and $\lim_n \|f'(x_n)\| \big(1 + h(\|x_n\|)\big) = 0$ contains a bounded subsequence.

4. Coercivity of $|f|$

In this section, we discuss the coercivity of $|f|$.

The following is a generalization of the result in Caklovic, Li and Willem [3].

Theorem 5. *Let X be a Banach space, and h a nondecreasing function from $[0, \infty)$ into itself such that $\int_0^\infty (1/(1 + h(t)))dt = \infty$. Let f be a continuous function from X into $\mathbb{R}$. Assume that f is Gâteaux differentiable at every point $x \in X$. If there exists $\gamma \in \mathbb{R}$ such that $\{x \in X : f(x) = \gamma\}$ is bounded, and*

$$\alpha := \lim_{r \to \infty} \inf_{\|x\| \geq r} |f(x) - \gamma| \in \mathbb{R},$$

then there exists a sequence $\{x_n\}$ in X such that $\lim_n \|x_n\| = \infty$, $\lim_n |f(x_n) - \gamma| = \alpha$, and

$$\lim_{n \to \infty} \|f'(x_n)\| \big(1 + h(\|x_n\|)\big) = 0.$$

Proof. We put $g(x) = |f(x) - \gamma|$ for all $x \in X$. We shall only show the following: For every $\varepsilon > 0$, there exists $v \in X$ satisfying $\|v\| \geq 1/\varepsilon$, $|g(v) - \alpha| \leq \varepsilon$, and $\|f'(v)\| \big(1 + h(\|v\|)\big) \leq \varepsilon$. Fix $\varepsilon > 0$. Define a function θ from $[0, \infty)$ into itself by (1). We next choose $r, r' \in \mathbb{R}$ with $1/\varepsilon < r < r'$, $1 < r$, $g(x) > 0$ for $x \in X$ with $\|x\| \geq r$,

$$\inf_{\|x\| \geq r} g(x) > \alpha - \varepsilon, \quad \text{and} \quad \int_r^{r'} \frac{dt}{1 + \theta(t)} = \frac{\alpha + \varepsilon}{\varepsilon}.$$

We also choose $u \in X$ with $\|u\| > r'$ and $g(u) < \alpha + \varepsilon$. Then by Theorem 2, there exists $v \in X$ with (2) and (3) for all $w \in X$ with $w \neq v$. Arguing by contradiction, we assume that $\|v\| < r$. From (2), we have

$$0 \leq g(v) \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1+\theta(t)} = g(u) - (\alpha + \varepsilon) < 0.$$

This is a contradiction. Therefore we obtain $\|v\| \geq r > 1/\varepsilon$ and hence $g(v) > 0$. We also have

$$\alpha - \varepsilon < \inf_{\|x\| \geq r} g(x) \leq g(v) \leq g(u) < \alpha + \varepsilon.$$

and hence $|g(v) - \alpha| \leq \varepsilon$. Since f is continuous and $g(v) > 0$, there exists $\delta \in (0,1)$ such that either of the following holds:

- $g(w) = +f(w) - \gamma$ for $w \in X$ with $\|v - w\| < \delta$; or
- $g(w) = -f(w) + \gamma$ for $w \in X$ with $\|v - w\| < \delta$.

As in the proof of Theorem 3, we have

$$g(w) > g(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|$$

for $w \in X$ with $0 < \|v - w\| < \delta$. In the former case, we obtain

$$f(w) > f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.$$

In the latter case, we obtain

$$f(w) < f(v) + \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.$$

So, by Lemma 1, we have $\|f'(v)\| \big(1 + h(\|v\|)\big) \leq \varepsilon$ in both cases. This completes the proof. □

As a direct consequence of Theorem 5, we obtain the following.

Theorem 6. *Let X be a Banach space. Let f be a continuous function from X into $\mathbb{R}$. Assume that f is Gâteaux differentiable at every point $x \in X$, and f satisfies the weak Palais-Smale condition. If there exists $\gamma \in \mathbb{R}$ such that $\{x \in X : f(x) = \gamma\}$ is bounded, then $|f|$ is coercive.*

Remark. We have the same remark of Theorem 4.

5. Continuity of h

In this section, we discuss the continuity of h.

Without the assumption of continuity of h, we can prove Theorem 3. However, Theorem 3 is not a generalization of Zhong's result because the following proposition holds. That is, Theorem 3 in this paper and Theorem 3.7 in [17] are equivalent. Also the two definitions of weak Palais-Smale condition in [17] and in this paper are equivalent.

Proposition 2. *Let h be a nondecreasing function from $[0, \infty)$ into itself such that $\int_0^\infty (1/(1 + h(t)))dt = \infty$. Then there exists a continuous nondecreasing function θ from $[0, \infty)$ into itself such that $\int_0^\infty (1/(1 + \theta(t)))dt = \infty$ and $h(t) \leq \theta(t)$ for all $t \in [0, \infty)$.*

Proof. For $t \in \mathbb{R}$, we denote by $[t]$ the maximum integer not exceeding t. Define a function θ from $[0,\infty)$ into itself by

$$\theta(t) = \big(1 - t + [t]\big)\, h\big([t]+1\big) + \big(t - [t]\big)\, h\big([t]+2\big)$$

for $t \in [0,\infty)$. Putting $k = [t]$ and $s = t - [t] \in [0,1)$, we have

$$\theta(k+s) = (1-s)\, h(k+1) + s\, h(k+2).$$

It is obvious that θ is continuous and nondecreasing. For $t \in [0,\infty)$, we have

$$\theta(t) \geq h\big([t]+1\big) \geq h(t)$$

because $t < [t]+1$. We also have

$$\int_0^\infty \frac{dt}{1+\theta(t)} \geq \int_0^\infty \frac{dt}{1+h\big([t]+2\big)} \geq \int_0^\infty \frac{dt}{1+h(t+2)} = \int_2^\infty \frac{dt}{1+h(t)} = \infty.$$

This completes the proof. □

Similarly, we can prove the following.

Proposition 3. *Let h be a nondecreasing function from $[0,\infty)$ into itself such that $\int_0^\infty (1/(1+h(t)))dt < \infty$. Then there exists a continuous nondecreasing function θ from $[0,\infty)$ into itself such that $\int_0^\infty (1/(1+\theta(t)))dt < \infty$ and $\theta(t) \leq h(t)$ for all $t \in [0,\infty)$.*

Proof. Define a function θ from $[0,\infty)$ into itself by

$$\theta(t) = \begin{cases} h(0), & \text{if } t \leq 1, \\ \big(1 - t + [t]\big)\, h\big([t]-1\big) + \big(t - [t]\big)\, h\big([t]\big), & \text{if } t \geq 1. \end{cases}$$

for $t \in [0,\infty)$. Then θ is continuous, nondecreasing, $\theta(t) \leq h([t]) \leq h(t)$ for $t \in [0,\infty)$, and $h(t-2) \leq h([t]-1) \leq \theta(t)$ for $t \in [2,\infty)$. Hence

$$\int_2^\infty \frac{dt}{1+\theta(t)} \leq \int_2^\infty \frac{dt}{1+h(t-2)} = \int_0^\infty \frac{dt}{1+h(t)} < \infty.$$

This completes the proof. □

6. Counterexamples

In this section, we give examples, which say that we use conditions $\int_0^\infty (1/(1+h(t)))dt = \infty$ and the completeness of X in Theorem 3 and others.

Example 2. Put $X := \mathbb{R}$ and let h be a nondecreasing function from $[0,\infty)$ into itself such that $\int_0^\infty (1/(1+h(t)))dt < \infty$. Then there exists a differentiable function f from X into $\mathbb{R}$ such that

$$\lim_{r\to\infty} \inf_{|x|\geq r} f(x) \in \mathbb{R} \quad \text{and} \quad |f'(x)|\, \big(1 + h(|x|)\big) \geq 1$$

for all $x \in X$.

Proof. By Proposition 3, there exists a continuous nondecreasing function θ from $[0,\infty)$ into itself such that $\int_0^\infty (1/(1+\theta(t)))dt < \infty$ and $\theta(t) \leq h(t)$ for all $t \in [0,\infty)$. Define a function f from X into $\mathbb{R}$ by

$$f(x) = \int_0^x \frac{-1}{1+\theta(\max\{t,0\})}\, dt$$

for $x \in X$. It is obvious that $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to +\infty} f(x) \in \mathbb{R}$. We also have

$$|f'(x)| \big(1 + h(|x|)\big) = \frac{1}{1 + \theta(\max\{x, 0\})} \big(1 + h(|x|)\big) \geq \frac{1 + h(|x|)}{1 + \theta(|x|)} \geq 1$$

for all $x \in X$. This completes the proof. $\square$

Example 3. Let X be the normed linear space consisting of all functions x from $\mathbb{N}$ into $\mathbb{R}$ (i.e., x is a real sequence) such that $\{n \in \mathbb{N} : \ x(n) \neq 0\}$ is a finite subset of $\mathbb{N}$. Define a norm $\|\cdot\|$ on X by $\|x\| = \sum_{n=1}^{\infty} |x(n)|$ for all $x \in X$. Define a lower semicontinuous (not continuous), convex, and Gâteaux differentiable function f from X into $\mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \exp\big(2^n\, x(n)\big)$$

for $x \in X$. Then

$$\lim_{r \to \infty} \inf_{\|x\| \geq r} f(x) = 0 \in \mathbb{R} \quad \text{and} \quad \|f'(x)\| \geq 1$$

for all $x \in X$.

Proof. It is obvious that f is convex and $\lim_{r \to \infty} \inf\{f(x) : \ \|x\| \geq r\} = 0$. By the definition of X, f is Gâteaux differentiable and its derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} \exp\big(2^n\, x(n)\big)\, e_n$$

for all $x \in X$, where $\{e_n\}$ is the canonical basis of X. Thus, we have

$$\|f'(x)\| = \sup\big\{ \exp\big(2^n\, x(n)\big) : \ n \in \mathbb{N}\big\} \geq \exp(0) = 1$$

for all $x \in X$. Fix $x \in X$ and define a sequence $\{x_n\}$ in X by

$$x_n(k) = \begin{cases} x(k), & \text{if } k \neq n, \\ 1/n, & \text{if } k = n \end{cases}$$

for $n \in \mathbb{N}$. Since $\|x - x_n\| = 1/n$ for large $n \in \mathbb{N}$, $\{x_n\}$ converges to x. Since

$$\frac{2^{n-1}}{n^2} \leq \frac{1}{2^n} \left(1 + \frac{2^n}{n} + \left(\frac{2^n}{n}\right)^2 /2 \right) \leq \frac{1}{2^n} \exp\left(\frac{2^n}{n}\right) \leq f(x_n)$$

for $n \in \mathbb{N}$, we have $\lim_n f(x_n) = \infty$. This implies f is not continuous everywhere. We finally show that f is lower semicontinuous. Let $\{x_n\}$ be a sequence in X converging to some $x \in X$. We fix $\varepsilon > 0$ and choose $\nu \in \mathbb{N}$ such that $2^{-\nu} < \varepsilon$ and $x(n) = 0$ for every $n \in \mathbb{N}$ with $n \geq \nu$. Define functions g and h from X into $(0, \infty)$ by

$$g(y) = \sum_{n=1}^{\nu} \frac{1}{2^n} \exp\big(2^n\, y(n)\big) \quad \text{and} \quad h(y) = \sum_{n=\nu+1}^{\infty} \frac{1}{2^n} \exp\big(2^n\, y(n)\big)$$

for $y \in X$. Then it is obvious that $f = g + h$, g is continuous and $h(x) = 2^{-\nu} < \varepsilon$. We have

$$f(x) = g(x) + h(x) \leq g(x) + \varepsilon = \lim_{n \to \infty} g(x_n) + \varepsilon \leq \liminf_{n \to \infty} f(x_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $f(x) \leq \liminf_n f(x_n)$. Therefore f is lower semicontinuous. This completes the proof. $\square$

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