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A GENERALIZED POISSON ALGEBRA STRUCTURE ON MANIFOLD:
Conserved quantities in N-body problem

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1. Introduction

On an n-dimensional differentiable manifold $\mathfrak{M}$ with a second rank skew-symmetric differentiable tensor field, Berezin [1] introduced the algebras of Poisson product provided with an infinite dimensional Lie algebra structure on the ring $\mathfrak{R}$ (with respect to the operations of addition and multiplication) of differentiable functions on $\mathfrak{M}$. And moreover, Kirillov [4] generalized the product leaving the Leibniz law out of account and investigated the infinite dimensional Lie algebra structure with the couple (Λ, Ξ) of a bivector (contravariant skew-symmetric 2-tensor) field Λ and a vector field Ξ on $\mathfrak{M}$ satisfying certain conditions. Lichnerowicz [6] showed that a Lie algebra structure (so-called Jacobi structure) is equivalent to the existence of such a couple satisfying the conditions which were given in terms of the Schouten-Nijenhuis bracket for multivector fields (Nijenhuis [11], Schouten [12]). Whenever Λ is non-singular, the couple (Λ, Ξ) can be translated into a couple (Ω, ω) of a non-degenerate differential 2-form Ω and a differential closed 1-form ω $(d\omega = 0)$ satisfying $d\Omega + \omega \wedge \Omega = 0$. Such a form Ω (called a semiclosed form with ω [4]) defines an infinitesimally conformally symplectic structure on $\mathfrak{M}$ (Guédira and Lichnerowitz [2]). Mimura and Nôno [9] gave an alternative approach to the infinite dimensional Lie algebra structure with the couple (Ω, Ξ) of non-degenerate differential 2-form Ω and a vector field Ξ on $\mathfrak{M}$, where Ω was assumed to be a semiclosed form with $\omega = \Xi \rfloor \Omega$ (contraction of Ω by Ξ) so that $\Xi(\Omega) = 0$ (Lie derivative of Ω by Ξ). They [10] gave a futher consideration by replacing the differentiable couple (Ω, Ξ) with a couple $(d\omega, \Xi)$ of a constant rank $2r$ closed differential 2-form $d\omega$ and a vector field Ξ satisfying the conditions $\Xi \rfloor d\omega = 0$ and $\Xi \rfloor \omega = \delta$, where δ takes the respective value $\delta = 1$ or $\delta = 0$ according to the cases: $\omega \wedge (d\omega)^r$ is non-zero or equal to zero (cf. Libermann [5], Lichnerowitz [6], for the contact structure on the case $\omega \wedge (d\omega)^r \neq 0$ where $2r + 1 = n = \dim \mathfrak{M}$; Ikushima, Fujiwara and Mimura [3] detailed the discussion of [10]).

In this paper, it is first reviewd (section 2 through section 3) the main results for the construction of Poisson algebra structure in [3] (also [10]), and then applied (section 4) to define the Poisson product of conserved quantities in N-body problem in $\mathbf{R}^3$. It is well-known that the Euler-Lagrange equations in the problem have the ten conserved quantities. We show that suitable four conserved quantities of them can generate other six conserved quantities by means of the Poisson product given in [3].

2. Lie and Poisson algebra structures

We first review the Lie and Poisson algebra structures discussed in [3] (also [10]).

Let ω be a differential 1-form on $\mathfrak{M}$ such that $d\omega$ has constant rank $2r$ $(r \neq 0)$, i.e., $(d\omega)^r = d\omega \wedge \cdots \wedge d\omega \neq 0$ (r-factors) everywhere but $(d\omega)^{r+1} = 0$ on $\mathfrak{M}$. For the form ω, there are two cases

$$\text{(i)} \quad \omega \wedge (d\omega)^r \neq 0, \qquad \text{(ii)} \quad \omega \wedge (d\omega)^r = 0.$$

According to the cases, we can set up a vector field Ξ on $\mathfrak{M}$ satisfying the relations (cf. Mimura and Nôno [10]):

$$\text{(1)} \qquad \text{a)} \quad \Xi \lrcorner\, d\omega = 0, \qquad \text{b)} \quad \Xi \lrcorner\, \omega = \delta,$$

where δ takes the respective value $\delta = 1$ for (i) or $\delta = 0$ for (ii).

Let $\mathfrak{X}$ be a set of all vector fields and $\mathfrak{R}$ a set of all differentiable functions on $\mathfrak{M}$; and define a set $\mathfrak{C}_\omega$ of characteristics of both ω and $d\omega$:

$$\mathfrak{C}_\omega = \{X \in \mathfrak{X} \mid X \lrcorner\, \omega = 0 \text{ and } X \lrcorner\, d\omega = 0\},$$

and a set of integrals $\mathfrak{R}_\omega$ of characteristics:

$$\mathfrak{R}_\omega = \{f \in \mathfrak{R} \mid X(f) = 0 \text{ for all } X \in \mathfrak{C}_\omega\}.$$

The set $\mathfrak{R}$ forms a ring with respect to the usual addition and multiplication of functions, and the set $\mathfrak{R}_\omega$ forms a subring of the ring $\mathfrak{R}$. A set of vector fields $X_f \in \mathfrak{X}$ corresponding to $f \in \mathfrak{R}_\omega$ can be determined by the following rule (cf. [10], Theorem 1).

THEOREM 1. *A differentiable function $f \in \mathfrak{R}$ is an integral of the chracteristics of both ω and $d\omega$, i.e., $f \in \mathfrak{R}_\omega$ if and only if there exists a vector field $X_f \in \mathfrak{X}$ determined uniquely up to modulo $\mathfrak{C}_\omega$ by the following rules. For the case* (i)*:*

$$\text{(2)} \qquad X_f \lrcorner\, d\omega \equiv -df \qquad (\text{mod } \omega),$$

$$\text{(3)} \qquad X_f \lrcorner\, \omega = f;$$

or for the case (ii)*:*

$$\text{(4)} \qquad X_f \lrcorner\, d\omega = -df.$$

The rules (2) *and* (4) *are unified as*

$$\text{(5)} \qquad X_f \lrcorner\, d\omega = -df + \Xi(f)\omega,$$

which is, for the case (i) *or* (ii), *equivalent to* (2) *or* (4), *respectively.*

REMARK 1. Note that a relation $X \lrcorner d\omega = -df + g\omega$ guarantees that in which f satisfies $\Xi(f) = g$ for (i) or $\Xi(f) = 0$ for (ii).

REMARK 2. A contraction of the relation (5) by X_f yields $\Xi(f)(X_f \lrcorner \omega) = X_f \lrcorner df = X_f(f)$, which implies that X_f satisfies $X_f(f) = f\Xi(f)$ for (i) or $X_f(f) = 0$ for (ii).

REMARK 3. For the case (i), whenever a vector field X satisfies the relation (2), it can be modified as $\tilde{X} = X + (f - X \lrcorner \omega)\Xi$ so as to satisfy both relations (2) and (3).

Here keep in mind the relation

$$X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega) \qquad (X \in \mathfrak{X}), \tag{6}$$

to see that the rules (2) and (3) for (i), or (4) for (ii) imply respectively $X_f(\omega) = \Xi(f)\omega$, or $X_f(\omega) = d(X_f \lrcorner \omega - f)$ so that $X_f(d\omega) = 0$. Conversely for (i), let $X(\omega) = g\omega$ $(g \in \mathfrak{R})$, which leads by (6) to $X \lrcorner d\omega = -d(X \lrcorner \omega) + g\omega$. So by putting $X \lrcorner \omega = f$ that $X \lrcorner d\omega = -df + g\omega$, where $g = \Xi(f)$ by Remark 1. And for (ii), let $X(d\omega) = 0$. Then, $X(\omega)$ is written as $X(\omega) = dh$ $(h \in \mathfrak{R})$, so that $X \lrcorner d\omega = X(\omega) - d(X \lrcorner \omega) = -df$ where $f = X \lrcorner \omega - h$. Therefore, for both cases (i) and (ii), there exist $f \in \mathfrak{R}$ satisfying the relation $X \lrcorner d\omega = -df + \Xi(f)\omega$, while $\Xi(f) = 0$ for (ii). A contraction of the relation by an arbitrary $Y \in \mathfrak{C}_\omega$ satisfying $Y \lrcorner d\omega = 0$ and $Y \lrcorner \omega = 0$ implies $Y(f) = 0$, so that $f \in \mathfrak{R}_\omega$. Thus we have the following theorem (cf. [10], Theorem 2).

THEOREM 2. *The vector field $X_f \in \mathfrak{X}$ crresponding to $f \in \mathfrak{R}_\omega$ is, for the case* (i), *an infinitesimal $\Xi(f)$-conformal symmetry of ω:*

$$X_f(\omega) = \Xi(f)\omega; \tag{7}$$

or, for the case (ii), *an infinitesimal symmetry of $d\omega$:*

$$X_f(d\omega) = 0. \tag{8}$$

Conversely, according to the case (i) *or* (ii), *for an infinitesimal conformal symmetry X of ω or an infinitesimal symmetry X of $d\omega$; there exists $f \in \mathfrak{R}_\omega$ such that $X \equiv X_f$* (mod $\mathfrak{C}_\omega$).

REMARK 4. In view of Remark 1, any infinitesimal conformal symmetry X of ω satisfying $X \lrcorner \omega = f$ is to be an infinitesimal $\Xi(f)$-conformal symmetry of ω.

Let $\mathfrak{X}_\omega$ be a set of all vector fields X_f $(f \in \mathfrak{R}_\omega)$ satisfying the relations (2) and (3) for (i), or (4) for (ii). Then, $\mathfrak{C}_\omega \subset \mathfrak{X}_\omega$ is trivial, and $\Xi \in \mathfrak{X}_\omega$ follows from (1a) and (1b) (note that $\Xi \equiv X_1$ (mod $\mathfrak{C}_\omega$) for (i), where X_1 is a vector field corresponding to $f = 1$; while $\Xi \in \mathfrak{C}_\omega$ for (ii)).

For $X_f, X_g \in \mathfrak{X}_\omega$, since by (1a) and (5):

$$X_{f+g} \lrcorner d\omega = (X_f + X_g) \lrcorner d\omega, \qquad X_{fg} \lrcorner d\omega = (fX_g + gX_f - fg\Xi) \lrcorner d\omega,$$

and moreover for the case (i), since by (1b) with $\delta = 1$ and (3) (in the relating line of ([10], p. 61): (ii) (i.e., $\delta = 0$) should be read (i) (i.e., $\delta = 1$)):

$$X_{f+g} \rfloor \omega = (X_f + X_g) \rfloor \omega, \qquad X_{fg} \rfloor \omega = (fX_g + gX_f - fg\Xi) \rfloor \omega,$$

the following relations are valid:

$$\text{(9)} \qquad \text{a)} \ X_{f+g} \equiv X_f + X_g, \qquad \text{b)} \ X_{fg} \equiv fX_g + gX_f - fg\Xi, \qquad (\mathrm{mod}\ \mathfrak{C}_\omega).$$

Here recall the relation (6) together with

$$\text{(10)} \qquad [Y, X] \rfloor \omega = Y(X \rfloor \omega) - X \rfloor Y(\omega) \qquad (Y, X \in \mathfrak{X}).$$

Particularly for $X_f \in \mathfrak{X}_\omega$ and $Y \in \mathfrak{X}$ satisfying $Y(\omega) = 0$, (10) is reduced to $[Y, X_f] \rfloor \omega = Y(X_f \rfloor \omega)$, so for the case (i) with (3) that

$$\text{(11)} \qquad [Y, X_f] \rfloor \omega = Y(f).$$

Similarly for $X_f \in \mathfrak{X}_\omega$ and $Y \in \mathfrak{X}$ satisfying $Y(d\omega) = 0$, the form ω in (10) is replaced with $d\omega$ to see $[Y, X_f] \rfloor d\omega = Y(X_f \rfloor d\omega)$, for which (5) is substituted to derive

$$\text{(12)} \qquad [Y, X_f] \rfloor d\omega = -dY(f) + Y(\Xi(f))\omega + \Xi(f)Y(\omega).$$

Since $\Xi(\omega) = 0$, (12) is reduced for $Y = \Xi$ to

$$[\Xi, X_f] \rfloor d\omega = -d\Xi(f) + \Xi^2(f)\omega,$$

so, in view of (5) and (11) with $Y = \Xi$, that

$$\text{(13)} \qquad [\Xi, X_f] \equiv X_{\Xi(f)} \qquad (\mathrm{mod}\ \mathfrak{C}_\omega).$$

Let $Y \in \mathfrak{C}_\omega$. Then $Y \rfloor \omega = 0$ and $Y \rfloor d\omega = 0$, so that $Y(\omega) = 0$ as well as $\Xi(\omega) = 0$. Therefore, in view of (10), $[Y, \Xi] \rfloor \omega = 0$ and also $[Y, \Xi] \rfloor d\omega = 0$; accordingly $[Y, \Xi] \in \mathfrak{C}_\omega$, i.e., $[Y, \Xi](f) = 0$ for $f \in \mathfrak{R}_\omega$. Since $Y(f) = 0$, (11) and (12) are reduced respectively to $[Y, X_f] \rfloor \omega = 0$ and $[Y, X_f] \rfloor d\omega = Y(\Xi(f))\omega$ which vanishes also as $Y(\Xi(f)) = [Y, \Xi](f) - \Xi(Y(f)) = 0$. Thus it it concluded:

$$\text{(14)} \qquad [\mathfrak{C}_\omega, \mathfrak{X}_\omega] \subset \mathfrak{C}_\omega.$$

In terms of $X_f \in \mathfrak{X}_\omega$ corresponding uniquely to $f \in \mathfrak{R}_\omega$ up to modulo $\mathfrak{C}_\omega$, since $Y(g) = 0$ for $Y \in \mathfrak{C}_\omega$ and $g \in \mathfrak{R}_\omega$, a product $\{f, g\}$ on $\mathfrak{R}_\omega$ can be defined by

$$\text{(15)} \qquad \{f, g\} = X_f(g) - \Xi(f)g.$$

First show that $\{f, g\} \in \mathfrak{R}_\omega$ if $f, g \in \mathfrak{R}_\omega$. In fact for $Y \in \mathfrak{C}_\omega$, since $Y(f) = 0$, $Y(g) = 0$ and $Y(\Xi(f)) = 0$ (see above), it follows that

$$Y(\{f, g\}) = Y(X_f(g)) - Y(\Xi(f)g) = [Y, X_f](g),$$

which vanishes by (14), i.e., $[Y, X_f] \in \mathfrak{C}_\omega$; so that $\{f, g\} \in \mathfrak{R}_\omega$, i.e.,

$$\{\mathfrak{R}_\omega, \mathfrak{R}_\omega\} \subset \mathfrak{R}_\omega. \tag{16}$$

Let $X_f, X_g \in \mathfrak{X}_\omega$. Then, for the case (i), (3) and (7) are substituted for (10) with $Y = X_f$ and $X = X_g$ to see

$$\begin{aligned}[X_f, X_g] \rfloor \omega &= X_f(g) - X_g \rfloor (\Xi(f)\omega) \\ &= X_f(g) - \Xi(f)g = \{f, g\}.\end{aligned}$$

So, after arranging $[X_f, X_g](\omega) = X_f(X_g(\omega)) - X_g(X_f(\omega))$ by (7), the property in Remark 4 is used to see

$$\begin{aligned}[X_f, X_g](\omega) &= X_f(\Xi(g)\omega) - X_g(\Xi(f)\omega) \\ &= (X_f(\Xi(g)) - X_g(\Xi(f)))\omega = \Xi(\{f, g\})\omega.\end{aligned} \tag{17}$$

Therefore, in view of (6), it follows that

$$[X_f, X_g] \rfloor d\omega = -d\{f, g\} + \Xi(\{f, g\})\omega,$$

which will be valid for the case (ii). In fact, since $X_g(d\omega) = 0$ (see (8)) and $\Xi(f) = 0$ (see Remark 1), (12) with $Y = X_g$ implies that $[X_g, X_f] \rfloor d\omega = -dX_g(f) = -d\{g, f\}$, i.e., $[X_f, X_g] \rfloor d\omega = -d\{f, g\}$. Therefore the final relation is obtained:

$$[X_f, X_g] \equiv X_{\{f, g\}} \qquad (\text{mod } \mathfrak{C}_\omega). \tag{18}$$

Thus obtained Lie algebra structure on $\mathfrak{X}_\omega$ is summarised as follows.

THEOREM 3. *The set $\mathfrak{X}_\omega$ of all vector fields X_f ($f \in \mathfrak{R}_\omega$) satisfying the relations* (2) *and* (3) *for the case* (i), *or* (4) *for the case* (ii), *forms a subalgebra of the Lie algebra $\mathfrak{X}$ under the bracket* $[\,,]$. *And the set $\mathfrak{C}_\omega$ of all characteristic vector fields of both ω and $d\omega$ forms an ideal of $\mathfrak{X}_\omega$.*

Essential relations are now in hand to show a Lie algebra structure on $\mathfrak{R}_\omega$ under the product $\{\,,\}$. For the case (i), the product (15) of $f, g \in \mathfrak{R}_\omega$ is written by (3) and (5) as

$$\begin{aligned}\{f, g\} &= X_f \rfloor (-X_g \rfloor d\omega + \Xi(g)\omega) - \Xi(f)g \\ &= -X_f \rfloor X_g \rfloor d\omega + \Xi(g)f - \Xi(f)g;\end{aligned}$$

which is valid also for (ii), while $\Xi(f) = \Xi(g) = 0$. Therefore the product is anti-commutative:

$$\{f, g\} = -\{g, f\}. \tag{19}$$

Since (14) and (18) yield the relation for $f, g, h \in \mathfrak{R}_\omega$:

$$X_{\{\{f,g\},h\}} \equiv [X_{\{f,g\}}, X_h] \equiv [[X_f, X_g], X_h] \qquad (\mathrm{mod}\ \mathfrak{C}_\omega),$$

it follows for the case (i) that (see (3) and (16))

$$\{\{f,g\},h\} = X_{\{\{f,g\},h\}} \rfloor \omega = [[X_f, X_g], X_h] \rfloor \omega.$$

Therefore, for (i), the Jacobi identity on $\mathfrak{R}_\omega$:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \tag{20}$$

is guaranteed by that on $\mathfrak{X}_\omega$ under the bracket $[\,,]$.

For the case (ii), since $X_f(X_g(h)) = -X_f(\{g,h\}) = -\{f,\{g,h\}\}$ (note that $\Xi(f) = 0$ in (15)), it follows that

$$\begin{aligned}\{\{f,g\},h\} &= X_{\{f,g\}}(h) = [X_f, X_g](h)\\ &= X_f(X_g(h)) - X_g(X_f(h))\\ &= \{f,\{g,h\}\} - \{g,\{f,h\}\},\end{aligned}$$

which turns by (19) to the Jacobi identity (20).

Particularly consider a subring $\mathfrak{R}_\omega^\Xi$ of $\mathfrak{R}_\omega$:

$$\mathfrak{R}_\omega^\Xi = \{f \in \mathfrak{R}_\omega \mid \Xi(f) = 0\},$$

which satisfies by the last euality of (17) that

$$\{\mathfrak{R}_\omega^\Xi, \mathfrak{R}_\omega^\Xi\} \subset \mathfrak{R}_\omega^\Xi,$$

while $\mathfrak{R}_\omega^\Xi = \mathfrak{R}_\omega$ for the case (ii). On $\mathfrak{R}_\omega^\Xi$, since $\{f, gh\} = X_f(gh)$, the Leibniz law

$$\{f, gh\} = g\{f,h\} + h\{f,g\}$$

is valid. Thus the following structure on $\mathfrak{R}_\omega$ is deduced under the product $\{\,,\}$ (cf. [10], Theorem 4).

THEOREM 4. *The subring $\mathfrak{R}_\omega$ of $\mathfrak{R}$ forms an infinite dimensional Lie algebra under the product $\{\,,\}$. Moreover, the subalgebra $\mathfrak{R}_\omega^\Xi$ of $\mathfrak{R}_\omega$ for the case* (i), *or $\mathfrak{R}_\omega$ itself for the case* (ii), *forms a Poisson algebra, i.e., on which the Leibniz law is valid.*

3. Poisson algebra structure on conserved quantities

Adding the time-axis $\mathbf{R}$ to the tangent bundle TM of m-dimensional configuration manifold M, let $\mathfrak{M} = TM \times \mathbf{R}$ and $(\dot{q}, q, t) = (\dot{q}_i(t), q_i(t), t)$ $(i = 1, \dots, m)$ be its local coordinate system. On the setting, introduce the Poincaré-Cartan form Θ associated with a given Lagrangian $L(\dot{q}, q, t)$ (the summation convention is employed in what follows):

$$\Theta = \frac{\partial L}{\partial \dot{q}_i}\theta_i + L\,dt,$$

where $\theta_i = dq_i - \dot{q}_i\,dt$. Here the Lagrangian is assumed to be regular, i.e., $\det(W_{ij}) \neq 0$ where $W_{ij} = \partial^2 L/\partial\dot{q}_i\partial\dot{q}_j$, to put the Euler-Lagrange equations

(21)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \ddot{q}_j W_{ij} + \dot{q}_j\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} + \frac{\partial^2 L}{\partial \dot{q}_i \partial t} - \frac{\partial L}{\partial q_i} = 0$$

into the kinematical form:

(22)
$$\ddot{q}_i = F_i(\dot{q}, q, t).$$

Then, in terms of $\phi_i = d\dot{q}_i - F_i\,dt$ and $\theta_i = dq_i - \dot{q}_i\,dt$, the exterior derivative $d\Theta$ of Θ is written as

$$d\Theta = W_{ij}\phi_i \wedge \theta_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\theta_i \wedge \theta_j - \left(F_j W_{ij} + \dot{q}_j\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} + \frac{\partial^2 L}{\partial q_i \partial t} - \frac{\partial L}{\partial q_i}\right)\theta_i \wedge dt,$$

which, by the Euler-Lagrange equations (21) and its equivalent form (22), turns into

$$d\Theta = W_{ij}\phi_i \wedge \theta_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\theta_i \wedge \theta_j.$$

Since $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k} = 0$ $(k > m)$ and $\phi_{i_1} \wedge \cdots \wedge \phi_{i_m} = \epsilon_{i_1 \dots i_m}\phi_1 \wedge \cdots \wedge \phi_m$ ($\epsilon_{i_1 \dots i_m}$ is the Eddington's symbol), and similar relations are valid for $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ $(k > m$ and $k = m)$, it follows that

$$\begin{aligned}(d\Theta)^m &= (W_{i_1 j_1}\phi_{i_1} \wedge \theta_{j_1}) \wedge \cdots \wedge (W_{i_m j_m}\phi_{i_m} \wedge \theta_{j_m}) \\ &= \varepsilon_m m!\,\det(W_{ij})\phi_1 \wedge \cdots \wedge \phi_m \wedge \theta_1 \wedge \cdots \wedge \theta_m.\end{aligned}$$

Therefore $(d\Theta)^m \neq 0$, $(d\Theta)^{m+1} = 0$ and $\Theta \wedge (d\Theta)^m \neq 0$, so that Θ lies in the case (i).

Let Γ be the equation field of (22):

$$\Gamma = F_i\frac{\partial}{\partial \dot{q}_i} + \dot{q}_i\frac{\partial}{\partial q_i} + \frac{\partial}{\partial t},$$

and put $\Xi = L^{-1}\Gamma$. Then, since $\Gamma \lrcorner \phi_i = 0$ and $\Gamma \lrcorner \theta_i = 0$, Ξ satisfies the conditions (1a) and (1b) with $\delta = 1$ for the form $\omega = \Theta$. Locally, in terms of a basis $\{\partial/\partial\dot{q}_i, \partial/\partial q_i, \Gamma\}$ $(i = 1, \dots, m)$, a vector field $X \in \mathfrak{M} = TM \times \mathbf{R}$ can be expressed as

$$X = \eta_i\frac{\partial}{\partial \dot{q}_i} + \xi_i\frac{\partial}{\partial q_i} + \psi\Gamma,$$

where $\eta_i, \xi_i, \psi \in \mathfrak{R}$ (the set of all differentiable functions on $\mathfrak{M} = TM \times \mathbf{R}$). Then both $X \lrcorner \Theta = \xi_i \partial L/\partial\dot{q}_i + \psi L$ and

$$(23)\qquad X \rfloor d\Theta = -W_{ij}\xi_j\phi_i + \left(W_{ij}\eta_j + \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} - \frac{\partial^2 L}{\partial \dot{q}_j \partial q_i}\right)\xi_j\right)\theta_i$$

vanish if and only if $\eta_i = \xi_i = \psi = 0$, i.e., the set $\mathfrak{C}_\Theta$ of characteristics of both Θ and $d\Theta$ is $\mathfrak{C}_\Theta = \{0\}$, and so $\mathfrak{R}_\Theta = \mathfrak{R}$.

Now consider the subring $\mathfrak{R}_\Theta^\Xi = \{f \in \mathfrak{R} \mid \Xi(f) = 0\}$ which coincides with a ring of all conserved quantities $\mathfrak{R}^\Gamma$ for the Euler-Lagrange equations (21):

$$\mathfrak{R}^\Gamma = \{f \in \mathfrak{R} \mid \Gamma(f) = 0\}.$$

By replacing $d\dot{q}_i$ and dq_i with $\phi_i + F_i\,dt$ and $\theta_i + \dot{q}_i\,dt$ respectively, df $(f \in \mathfrak{R})$ can be put into

$$df = \frac{\partial f}{\partial \dot{q}_i}\phi_i + \frac{\partial f}{\partial q_i}\theta_i + \Gamma(f)dt,$$

which is combined with (23) to see that $X \rfloor d\Theta = -df$ $(f \in \mathfrak{R}^\Gamma)$ if and only if

$$W_{ij}\xi_j = \frac{\partial f}{\partial \dot{q}_i}, \qquad W_{ij}\eta_j = \left(\frac{\partial^2 L}{\partial \dot{q}_j \partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\right)\xi_j - \frac{\partial f}{\partial q_i}.$$

By the first equations of the above, ξ_j are determined as $\xi_j = W^{ij}\partial f/\partial \dot{q}_i$, where $(W^{ij}) = (W_{ij})^{-1}$; and after substituting the ξ_j for the second ones, η_j are also as

$$\eta_j = W^{ik}W^{js}\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\right)\frac{\partial f}{\partial \dot{q}_i} - W^{ij}\frac{\partial f}{\partial q_i}.$$

Consequently, $X_f \in \mathfrak{X}_\Theta$ corresponding to $f \in \mathfrak{R}^\Gamma$ is of the form $X_f = X_f^0 + \psi\Gamma$, where

$$X_f^0 = W^{ik}W^{js}\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\right)\frac{\partial f}{\partial \dot{q}_i}\frac{\partial}{\partial \dot{q}_j} + W^{ij}\left(\frac{\partial f}{\partial \dot{q}_i}\frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial \dot{q}_j}\right),$$

while ψ is determined by $X_f \rfloor \Theta = f$ as

$$\psi = \frac{1}{L}\left(f - W^{ij}\frac{\partial f}{\partial \dot{q}_i}\frac{\partial L}{\partial \dot{q}_j}\right).$$

Therefore the following product $\{f, g\} = X_f(g)$ for $f, g \in \mathfrak{R}^\Gamma$ is deduced:

$$(24)\qquad \{f, g\} = W^{ik}W^{js}\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\right)\frac{\partial f}{\partial \dot{q}_i}\frac{\partial g}{\partial \dot{q}_j} + W^{ij}\left(\frac{\partial f}{\partial \dot{q}_i}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial \dot{q}_j}\right)$$

$$= W^{ik}W^{js}\frac{\partial^2 L}{\partial \dot{q}_k \partial q_s}\left(\frac{\partial f}{\partial \dot{q}_i}\frac{\partial g}{\partial \dot{q}_j} - \frac{\partial f}{\partial \dot{q}_j}\frac{\partial g}{\partial \dot{q}_i}\right) + W^{ij}\left(\frac{\partial f}{\partial \dot{q}_i}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial \dot{q}_j}\right).$$

Thus the structure on $\mathfrak{R}_\omega^\Xi$ in Theorem 4 is translated into that on $\mathfrak{R}^\Gamma$ as follows (cf. [7], Theorem 9; [8], Theorem 10).

THEOREM 5. *The ring* $\mathfrak{R}^{\Gamma}$ *of all conserved quantities for the Euler-Lagrange equations* (21) *with regular Lagrangian forms a Poisson algebra under the product* (24).

4. Conserved quantities in N-body problem

Consider N particles P_k $(k = 1, \cdots, N)$ with masses m_k. Let $\boldsymbol{r}_k = (x_k, y_k, z_k)$ be the position vectors of P_k and G be the gravitational constant. Then the Lagrangian L in the problem is given as

$$L = \frac{1}{2}\sum_{k=1}^{N} m_k \|\dot{\boldsymbol{r}}_k\|^2 + \sum_{k \neq \ell} \frac{G m_k m_\ell}{\|\boldsymbol{r}_k - \boldsymbol{r}_\ell\|},$$

so that the Euler-Lagrange equations (21) have the appearance

$$\ddot{\boldsymbol{r}}_k = -\sum_{k \neq \ell} \frac{G m_\ell (\boldsymbol{r}_k - \boldsymbol{r}_\ell)}{\|\boldsymbol{r}_k - \boldsymbol{r}_\ell\|^3}.$$

It is well-known that the equations have the following ten conserved quantities:
(i) Total energy (Hamiltonian):

$$\begin{aligned} H &= \frac{1}{2}\sum_{k=1}^{N} m_k \|\dot{\boldsymbol{r}}_k\|^2 - \sum_{k>\ell} \frac{G m_k m_\ell}{\|\boldsymbol{r}_k - \boldsymbol{r}_\ell\|} \\ &= \frac{1}{2}\sum_{k=1}^{N} m_k (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) - \sum_{k>\ell} \frac{G m_k m_\ell}{\sqrt{(x_k - x_\ell)^2 + (y_k - y_\ell)^2 + (z_k - z_\ell)^2}}. \end{aligned}$$

(ii) Components of momentum vector $\sum_{k=1}^{N} m_k \dot{\boldsymbol{r}}_k$:

$$\Omega_1 = \sum_{k=1}^{N} m_k \dot{x}_k, \quad \Omega_2 = \sum_{k=1}^{N} m_k \dot{y}_k, \quad \Omega_3 = \sum_{k=1}^{N} m_k \dot{z}_k.$$

(iii) Components of the integration $\sum_{k=1}^{N} m_k (\boldsymbol{r}_k - t\dot{\boldsymbol{r}}_k)$ of (ii):

$$\Xi_1 = \sum_{k=1}^{N} m_k (x_k - t\dot{x}_k), \quad \Xi_2 = \sum_{k=1}^{N} m_k (y_k - t\dot{y}_k), \quad \Xi_3 = \sum_{k=1}^{N} m_k (z_k - t\dot{z}_k).$$

(iv) Components of angular momentum vector $\sum_{k=1}^{N} m_k (\boldsymbol{r}_k \times \dot{\boldsymbol{r}}_k)$:

$$\Lambda_1 = \sum_{k=1}^{N} m_k (y_k \dot{z}_k - \dot{y}_k z_k), \quad \Lambda_2 = \sum_{k=1}^{N} m_k (z_k \dot{x}_k - \dot{z}_k x_k), \quad \Lambda_3 = \sum_{k=1}^{N} m_k (x_k \dot{y}_k - \dot{x}_k y_k).$$

In view of the Lagrangian, since $\partial^2 L / \partial \dot{q}_i \partial q_j = 0$ and

$$(W^{ij}) = \begin{pmatrix} \frac{1}{m_1}E_3 & & & 0 \\ & \frac{1}{m_2}E_3 & & \\ & & \ddots & \\ 0 & & & \frac{1}{m_N}E_3 \end{pmatrix},$$

where $(q_1, q_2, \dots, q_{3N}) = (x_1, y_1, z_1, \dots, x_N, y_N, z_N)$ and E_3 is the unit matrix of third order; the Poisson product (24) is reduced to

$$\{f, g\} = \sum_{k=1}^{N} \frac{1}{m_k}\left(\frac{\partial f}{\partial \dot{x}_k}\frac{\partial g}{\partial x_k} + \frac{\partial f}{\partial \dot{y}_k}\frac{\partial g}{\partial y_k} + \frac{\partial f}{\partial \dot{z}_k}\frac{\partial g}{\partial z_k}\right)$$
$$- \sum_{k=1}^{N} \frac{1}{m_k}\left(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial \dot{x}_k} + \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial \dot{y}_k} + \frac{\partial f}{\partial z_k}\frac{\partial g}{\partial \dot{z}_k}\right).$$

For example, the Poisson product $\{H, \Omega_1\}$ is

$$\{H, \Omega_1\} = \sum_{k=1}^{N} \frac{1}{m_k}\left(\frac{\partial H}{\partial \dot{x}_k}\frac{\partial \Omega_1}{\partial x_k} + \frac{\partial H}{\partial \dot{y}_k}\frac{\partial \Omega_1}{\partial y_k} + \frac{\partial H}{\partial \dot{z}_k}\frac{\partial \Omega_1}{\partial z_k}\right)$$
$$- \sum_{k=1}^{N} \frac{1}{m_k}\left(\frac{\partial H}{\partial x_k}\frac{\partial \Omega_1}{\partial \dot{x}_k} + \frac{\partial H}{\partial y_k}\frac{\partial \Omega_1}{\partial \dot{y}_k} + \frac{\partial H}{\partial z_k}\frac{\partial \Omega_1}{\partial \dot{z}_k}\right)$$
$$= -\sum_{\ell=1}^{N} \frac{\partial U}{\partial x_\ell},$$

where

$$U = -\sum_{k>s} \frac{G m_k m_s}{\|\boldsymbol{r}_k - \boldsymbol{r}_s\|} = -\sum_{k>s} \frac{G m_k m_s}{\sqrt{(x_k - x_s)^2 + (y_k - y_s)^2 + (z_k - z_s)^2}}.$$

Since

$$\frac{\partial U}{\partial x_1} = -\sum_{k=2}^{N} \frac{G m_k m_1 (x_k - x_1)}{\|\boldsymbol{r}_k - \boldsymbol{r}_1\|},$$

$$\frac{\partial U}{\partial x_\ell} = \sum_{k=1}^{\ell-1} \frac{G m_\ell m_k (x_\ell - x_k)}{\|\boldsymbol{r}_\ell - \boldsymbol{r}_k\|} - \sum_{k=\ell+1}^{N} \frac{G m_k m_\ell (x_k - x_\ell)}{\|\boldsymbol{r}_k - \boldsymbol{r}_\ell\|} \qquad (\ell = 2, \cdots, N-1),$$

$$\frac{\partial U}{\partial x_N} = -\sum_{k=1}^{N-1} \frac{G m_N m_k (x_N - x_k)}{\|\boldsymbol{r}_N - \boldsymbol{r}_k\|},$$

the product $\{H, \Omega_1\}$ leads to zero:

$$
\begin{aligned}
\{H,\Omega_1\} &= \sum_{k=2}^{N}\frac{Gm_k m_1(x_k-x_1)}{\|\boldsymbol{r}_k-\boldsymbol{r}_1\|} - \sum_{\ell=2}^{N-1}\left(\sum_{k=1}^{\ell-1}\frac{Gm_\ell m_k(x_\ell-x_k)}{\|\boldsymbol{r}_\ell-\boldsymbol{r}_k\|} - \sum_{k=\ell+1}^{N}\frac{Gm_k m_\ell(x_k-x_\ell)}{\|\boldsymbol{r}_k-\boldsymbol{r}_\ell\|}\right) \\
&\quad - \sum_{k=1}^{N-1}\frac{Gm_N m_k(x_N-x_k)}{\|\boldsymbol{r}_N-\boldsymbol{r}_k\|} \\
&= \sum_{k=2}^{N}\frac{Gm_k m_1(x_k-x_1)}{\|\boldsymbol{r}_k-\boldsymbol{r}_1\|} - \left(\sum_{\ell=2}^{N-1}\frac{Gm_\ell m_1(x_\ell-x_1)}{\|\boldsymbol{r}_\ell-\boldsymbol{r}_1\|} + \sum_{\ell=3}^{N-1}\sum_{k=2}^{\ell-1}\frac{Gm_\ell m_k(x_\ell-x_k)}{\|\boldsymbol{r}_\ell-\boldsymbol{r}_k\|}\right) \\
&\quad + \sum_{\ell=2}^{N-1}\sum_{k=\ell+1}^{N}\frac{Gm_k m_\ell(x_k-x_\ell)}{\|\boldsymbol{r}_k-\boldsymbol{r}_\ell\|} - \left(\frac{Gm_N m_1(x_N-x_1)}{\|\boldsymbol{r}_N-\boldsymbol{r}_1\|} + \sum_{k=2}^{N-1}\frac{Gm_N m_k(x_N-x_k)}{\|\boldsymbol{r}_N-\boldsymbol{r}_k\|}\right) \\
&= -\sum_{\ell=3}^{N-1}\sum_{k=2}^{\ell-1}\frac{Gm_\ell m_k(x_\ell-x_k)}{\|\boldsymbol{r}_\ell-\boldsymbol{r}_k\|} + \sum_{\ell=2}^{N-1}\sum_{k=\ell+1}^{N}\frac{Gm_k m_\ell(x_k-x_\ell)}{\|\boldsymbol{r}_k-\boldsymbol{r}_\ell\|} \\
&\quad - \sum_{k=2}^{N-1}\frac{Gm_N m_k(x_N-x_k)}{\|\boldsymbol{r}_N-\boldsymbol{r}_k\|} \\
&= -\sum_{\ell=3}^{N-1}\sum_{k=2}^{\ell-1}\frac{Gm_\ell m_k(x_\ell-x_k)}{\|\boldsymbol{r}_\ell-\boldsymbol{r}_k\|} \\
&\quad + \left(\sum_{\ell=2}^{N-1}\frac{Gm_N m_\ell(x_N-x_\ell)}{\|\boldsymbol{r}_N-\boldsymbol{r}_\ell\|} + \sum_{\ell=2}^{N-2}\sum_{k=\ell+1}^{N-1}\frac{Gm_k m_\ell(x_k-x_\ell)}{\|\boldsymbol{r}_k-\boldsymbol{r}_\ell\|}\right) \\
&\quad - \sum_{k=2}^{N-1}\frac{Gm_N m_k(x_N-x_k)}{\|\boldsymbol{r}_N-\boldsymbol{r}_k\|} \\
&= 0
\end{aligned}
$$

For another example, the Poisson product $\{\Omega_2,\Xi_1\}$ leads to Ω_3:

$$
\begin{aligned}
\{\Omega_2,\Xi_1\} &= \sum_{k=1}^{N}\frac{1}{m_k}\left(\frac{\partial\Omega_2}{\partial\dot{x}_k}\frac{\partial\Xi_1}{\partial x_k} + \frac{\partial\Omega_2}{\partial\dot{y}_k}\frac{\partial\Xi_1}{\partial y_k} + \frac{\partial\Omega_2}{\partial\dot{z}_k}\frac{\partial\Xi_1}{\partial z_k}\right) \\
&\quad - \sum_{k=1}^{N}\frac{1}{m_k}\left(\frac{\partial\Omega_2}{\partial x_k}\frac{\partial\Xi_1}{\partial\dot{x}_k} + \frac{\partial\Omega_2}{\partial y_k}\frac{\partial\Xi_1}{\partial\dot{y}_k} + \frac{\partial\Omega_2}{\partial z_k}\frac{\partial\Xi_1}{\partial\dot{z}_k}\right) \\
&= \sum_{\ell=1}^{N} m_\ell \dot{z}_\ell \\
&= \Omega_3.
\end{aligned}
$$

Similar calculations of the Poisson products establish the following table.

$A \backslash B$	H	Ω_1	Ω_2	Ω_3	Ξ_1	Ξ_2	Ξ_3	Λ_1	Λ_2	Λ_3
H	0	0	0	0	Ω_1	Ω_2	Ω_3	0	0	0
Ω_1	0	0	0	0	m	0	0	0	$-\Omega_3$	Ω_2
Ω_2	0	0	0	0	0	m	0	Ω_3	0	$-\Omega_1$
Ω_3	0	0	0	0	0	0	m	$-\Omega_2$	Ω_1	0
Ξ_1	$-\Omega_1$	$-m$	0	0	0	0	0	0	$-\Xi_3$	Ξ_2
Ξ_2	$-\Omega_2$	0	$-m$	0	0	0	0	Ξ_3	0	$-\Xi_1$
Ξ_3	$-\Omega_3$	0	0	$-m$	0	0	0	$-\Xi_2$	Ξ_1	0
Λ_1	0	0	$-\Omega_3$	Ω_2	0	$-\Xi_3$	Ξ_2	0	$-\Lambda_3$	Λ_2
Λ_2	0	Ω_3	0	$-\Omega_1$	Ξ_3	0	$-\Xi_1$	Λ_3	0	$-\Lambda_1$
Λ_3	0	$-\Omega_2$	Ω_1	0	$-\Xi_2$	Ξ_1	0	$-\Lambda_2$	Λ_1	0

Poisson products $\{A, B\}$ $(m = \sum_{k=1}^{N} m_k)$

In this table, it can be observed that the conserved quantities H, Ξ_1, Λ_2, Λ_3 generate the other ones Λ_1, Ξ_2, Ξ_3, Ω_1, Ω_2, Ω_3. In fact, we can see that

$$\{H, \Xi_1\} = \Omega_1,$$
$$\{\Lambda_3, \Lambda_2\} = \Lambda_1,$$
$$\{\Xi_1, \Lambda_3\} = \Xi_2,$$
$$\{\Xi_2, \Lambda_1\} = \Xi_3,$$
$$\{H, \{\Xi_1, \Lambda_3\}\} = \{H, \Xi_2\} = \Omega_2,$$
$$\{H, \{\Xi_2, \Lambda_1\}\} = \{H, \Xi_3\} = \Omega_3.$$

Concluding remark. *A significance of the Poisson product given by* (24) *is that unknown conserved quantity may be discovered by the product of known two conserved ones.*

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