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# State-Dependent Scaling Characterization for Interconnected Nonlinear Systems–Part I: Constructive Generalization of Stability Criteria¶§

Hiroshi Ito†‡

† Dept. of Systems Innovation and Informatics
Kyushu Institute of Technology
680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan
Phone: +81 948-29-7717, Fax: +81 948-29-7709
E-mail: hiroshi@ces.kyutech.ac.jp

Abstract: This paper is concerned with the problem of global stability and performance of nonlinear interconnected systems. The state-dependent scaling problem is proposed as a unified mathematical formulation whose solutions explicitly provide Lyapunov functions proving stability and dissipative properties of feedback and cascade systems. The framework covers nonlinear systems having diverse and strong nonlinearities represented by general supply rates. Previously existing stability theorems such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, the circle and Popov criteria and the input-to-state stable(ISS) small-gain theorem are extracted from the framework as special elementary cases. Indeed, those existing stability criteria are sufficient conditions for the existence of solutions to the state-dependent scaling problems, and the solutions can be shown explicitly. A notable benefit of the proposed formulation is that it actually yields useful answers for systems disagreeing with classical and ISS nonlinearities, which is much more than formal applicability. The paper focuses on generalization and unification, while the follow-up paper puts a special emphasis on theoretical demonstration of the effectiveness of the state-dependent scaling characterization.

Keywords: Nonlinear interconnected system, Global asymptotic stability, Lyapunov function, Stability criteria, Dissipative system, Supply rates, State-dependent scaling,

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‡Author for correspondence

# 1 Introduction

In the literature of nonlinear control theory, a great deal of effort has been put into the problem of finding useful formulations of conditions under which interconnected systems are stable. One of significant contributions is the stability theory developed in [1], which unifies previously known stability criteria and provides Lyapunov versions of input-output stability results such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, and the circle and Popov criteria. Another major development which presently plays an important role in nonlinear control analysis and design is the ISS small-gain theorem also known as the nonlinear small-gain theorem[2, 3]. A small-gain theorem which brought about the ISS small-gain theorem was originally formulated by Hill[4], and Mareels and Hill[5], and that was extended in the ISS framework by Jiang et al.[2] which was also further generalized by Teel[3]. The effectiveness of the ISS small-gain theorem is evident when systems have essential nonlinearities described by the input-to-state stable(ISS) property[6]. It is, however, known that there are systems for which ISS is too strong requirement. One has yet to develop a stability theory which encompasses much broader classes of systems in dealing with interconnections of nonlinear systems.

The main purpose of the paper is to propose a general approach to the stability and performance of nonlinear interconnected systems. We need a framework which is not limited to the settings of popular classical stability criteria and the ISS small-gain theorem. The goal of this paper is the development of a general framework from which the set of presently known important results can be easily extracted as special cases. To this end, this paper borrows an idea from the state-dependent scaling techniques which have been recently introduced by the author[7, 8, 9], and this paper generalizes it much further. The main product of this paper is the formulation of the state-dependent scaling problems. The state-dependent scaling problems are scalar inequalities we solve for parameters. The parameters are called state-dependent scaling functions in this paper. The points this paper addresses include three main issues. One is to show that problems of stability and performance analysis for interconnected dissipative systems can be reduced into the state-dependent scaling problems in a unified way. Another is to clarify when solutions of the state-dependent scaling problems exist. The other is to demonstrate how we are able to calculate the solutions explicitly.

The techniques employed by [7, 8, 9] can be considered as primitive versions and limited special cases of the general results developed in this paper. The usefulness of the idea of state-dependent scaling in constructing robust control Lyapunov functions has been demonstrated in [8, 9] for several design problems of robust nonlinear controllers. However, discussion about connections with other approaches has not been given in the literature. One of objectives of this paper is to clarify for the first time the relation between the state-dependent scaling formulation and the ISS small-gain condition[2, 3] as well as stability criteria for dissipative systems[1].

One of benefits from the developments in this paper is that the ISS small-gain theorem and the dissipative approach can be explained in a unified language. The ISS small-gain theorem and its proof are usually given in terms of trajectories of systems(or input-output-type formulation)[2, 3, 10]. A notable exception is [11] which presents non-smooth functions guaranteeing the existence of Lyapunov functions. However, the result has not yielded an explicit formula to directly obtain smooth Lyapunov functions which are useful for controller design. The ISS small-gain condition by itself does not show explicit information about how to construct Lyapunov functions. We sometimes prefer constructive tools since we rely on Lyapunov functions in many cases of nonlinear systems design when system models are given. The storage function which plays a key role in the dissipative analysis[12, 1] serves as a Lyapunov function. The storage function is an abstract notion of energy stored in a system. The energy increases when energy is supplied from outside. The supply rate determines the variation of the storage function. The idea proposed in [1] is to define the storage function of interconnected systems by summing up supply rates of individual systems. Although the ISS property of open-loop systems has been related to the existence of Lyapunov functions[13, 14, 10], little development has been made in the construction of Lyapunov functions for feedback systems. This background has created a gap between the dissipative approach and the ISS small-gain theorem. This paper offers a new avenue to the ISS small-gain theorem, and it closes the gap successfully. This paper comes at an idea of summing up supply rates nonlinearly for constructing a Lyapunov function for the ISS small-gain theorem. Calculation of nonlinear coefficients to combine supply rates for the ISS small-gain theorem and calculation of constant coefficients for the dissipative approach are unified into solutions to the state-dependent problems.

Another major advantage of the state-dependent scaling approach over the existing stability criteria is that it is applicable to systems whose nonlinearity disagrees with ISS and other classical nonlinearities. It is not only applicable, but also surely effective so that useful answers can be obtained for broader classes of essentially nonlinear systems. This paper demonstrates it through some examples. Theoretical demonstration of the effectiveness is a very important issue on which the authors put a special emphasis. The follow-up paper is devoted to the discussion and gives interesting evidences of the universality beyond formal applicability.

This paper is organized as follows. In Section 2, we presents a mathematical problem of state-dependent scaling which forms the main idea of this paper. Several variants of the state-dependent scaling problem are formulated by introducing little modification into the main problem. These problems are closely connected each other by a simple inclusive relation. In Section 3, we define a nonlinear interconnected system. A general configuration is employed to deal with feedback systems and cascade systems in a unified manner. Then, solutions to the state-dependent scaling problems are related explicitly to the construction of Lyapunov functions of the interconnected system. The central inequality of the state-dependent scaling problems is given an interpretation of the sum of nonlinearly scaled supply rates of dissipative systems. It is shown that properties of stability and disturbance attenuation can be established for the interconnected system once we have obtained a solution to the state-dependent scaling problems. Thereby, we propose a unified approach to analysis of stability and performance of interconnected dissipative systems. In Section 4, the idea and the effectiveness of the approach are illustrated through several examples involving nonlinearities which are not covered by the classical stability criteria and the ISS small-gain theorem. Section 5 is concentrated on supply rates which are popular in classical stability analysis such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, and the circle and Popov criteria. It is explained that those classical stability criteria are based on linear combination of supply rates, which are proved to be the easiest cases of the state-dependent scaling problems. Section 6 deals with supply rates of the ISS property. The ISS small-gain theorem is explained in the state-dependent scaling framework, and the interconnection of ISS systems is considered as an example for which nonlinear combination of supply rates is essential. Using Section 5 and Section 6, the author demonstrates that the classical stability theorems and the ISS small-gain theorem can be extracted as special cases of the state-dependent scaling formulation. Stability conditions provided by classical stability criteria and the ISS small-gain theorem are viewed as sufficient conditions for guaranteeing the existence of solutions to the state-dependent scaling problems. Finally, conclusions are drawn in Section 7. The follow-up paper [15] gives a deep insight into the effectiveness of the state-dependent scaling approach by theoretically addressing the question of how solutions to the state-dependent scaling problems are obtained for advanced types of nonlinearities which are not covered by popular classical and advanced stability criteria available previously.

This paper uses the following notations. The interval $[0, \infty)$ in the space of real numbers $\mathbb{R}$ is

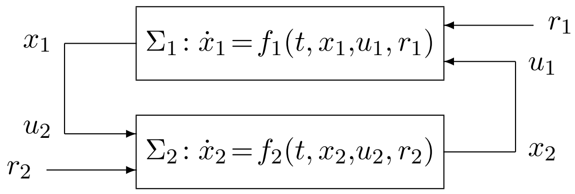



Figure 1: Feedback interconnected system $\Sigma$

denoted by $\mathbb{R}_+$. Euclidean norm of a vector in $\mathbb{R}^n$ of dimension $n$ is denoted by $|\cdot|$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be class $\mathcal{K}$ and written as $\gamma \in \mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be class $\mathcal{K}_\infty$ and written as $\gamma \in \mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty} \gamma(r) = \infty$.

## 2 State-dependent scaling problems

This section presents a main mathematical problem which plays a central role in this paper, and introduces several variants of the main problem. We refer to the mathematical problems as the state-dependent scaling problems. Interpretation and importance of the problems in nonlinear systems control will be discussed in subsequent sections.

The following defines our main mathematical problem which contains the primary idea of the state-dependent scaling framework.

**Problem 1** *Given continuously differentiable functions $V_i : (t, x_i) \in \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ and continuous functions $\rho_i : (x_i, x_j, r_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \times \mathbb{R}^{m_i} \to \mathbb{R}$ for $i = 1, 2$ and $j = \{1, 2\} \setminus \{i\}$, find continuous functions $\lambda_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\lambda_i(s) > 0 \quad \forall s \in (0, \infty) \tag{1}$$

$$\lim_{s\to 0^+} \lambda_i(s) < \infty \tag{2}$$

$$\int_1^\infty \lambda_i(s) ds = \infty \tag{3}$$

*for $i = 1, 2$ such that*

$$\lambda_1(V_1(t, x_1))\rho_1(x_1, x_2, r_1) + \lambda_2(V_2(t, x_2))\rho_2(x_2, x_1, r_2) \le \rho_e(x_1, x_2, r_1, r_2),$$
$$\forall x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{4}$$

*holds for some continuous function $\rho_e : (x_1, x_2, r_1, r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying*

$$\rho_e(x_1, x_2, 0, 0) < 0 \quad ,\forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \{(0, 0)\} \tag{5}$$

The following is a variant of Problem 1.

**Problem 2** *Given a continuously differentiable function $V_2 : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and continuous functions $\rho_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ and $\rho_2 : (x_2, z_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$, find continuous functions $\lambda_1 : (t, z_1, x_2, r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}_+$, $\lambda_2 : s \in \mathbb{R}_+ \to \mathbb{R}_+$, an increasing continuous function $\xi_1 : s \in [0, N] \to \mathbb{R}_+$ and a continuous function $\varphi_1 : (z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}_+$ satisfying*

$$\lambda_2(s) > 0 \quad \forall s \in (0, \infty) \tag{6}$$

$$\lim_{s\to 0^+} \lambda_2(s) < \infty \tag{7}$$

$$\int_1^\infty \lambda_2(s)ds = \infty \tag{8}$$

$$\xi_1(s) \geq 0 \quad \forall s \in [0, N] \tag{9}$$

$$\varphi_1(z_1, x_2, r_1) \geq 0, \quad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1} \tag{10}$$

*such that*

$$\begin{aligned}\lambda_1(t, z_1, x_2, r_1, r_2)\left[-\xi_1(\varphi_1(z_1, x_2, r_1)) + \xi_1(\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1))\right] + \\ \lambda_2(V_2(t, x_2))\rho_2(x_2, z_1, r_2) \leq \rho_e(x_2, r_1, r_2), \\ \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+\end{aligned} \tag{11}$$

*holds for some continuous function* $\rho_e : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ *satisfying*

$$\rho_e(x_2, 0, 0) < 0 \quad , \forall x \in \mathbb{R}^{n_2} \setminus \{0\} \tag{12}$$

*where* $N \in [0, \infty]$ *is defined by*

$$N = \sup_{(z_1, x_2, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1}} [\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1))] \tag{13}$$

In this paper, the functions $\lambda_i$ and $\xi_i$ are referred to as state-dependent scaling functions. The inequalities (4) and (11) are key formulas on which interpretations will be put later on. The reason of using the terminology "state-dependent scaling" will be also clear in the next section. Note that (2) and (7) are redundant mathematically since each $\lambda_i$ is supposed to be continuous on $\mathbb{R}_+ = [0, \infty)$. The explicit statement of (2) and (7) may be helpful to direct the readers' attention to it.

This paper calls a pair of $\lambda_1$ and $\lambda_2$ a solution to Problem 1 if the pair fulfills all requirements stated in Problem 1. In a similar manner, a quartet of $\lambda_1$, $\lambda_2$, $\xi_1$ and $\varphi_1$ fulfilling all requirements stated in Problem 2 is called a solution to Problem 2. If we take $\xi_1(s) = s$, the inequality (11) becomes

$$\begin{aligned}\lambda_1(t, z_1, x_2, r_1, r_2)\rho_1(z_1, x_2, r_1) + \lambda_2(V_2(t, x_2))\rho_2(x_2, z_1, r_2) \leq \rho_e(x_2, r_1, r_2), \\ \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+\end{aligned} \tag{14}$$

In the same way, whenever the function $\xi_1(s)$ is affine in $s$, the function $\varphi_1$ disappears from (11). Therefore, in the case of affine $\xi_1(s)$, a triplet of $\lambda_1$, $\lambda_2$ and $\xi_1$ is called a solution to Problem 2.

At first glance, Problem 2 is complicated since it has more parameters than Problem 1. It is, however, milder than Problem 1. In other words, Problem 1 has a solution only if so does Problem 2. It is verified by considering a choice $\xi_1(s) = s$ in Problem 2 with $p_1 = n_1$. Indeed, the claim can be obtain easily from (14) and the following lemma.

**Lemma 1** *Suppose that a continuous function* $\rho_e : (x_1, x_2, r_1, r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ *satisfies*

$$\rho_e(x_1, x_2, 0, 0) < 0 \quad , \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \setminus \{(0, 0)\} \tag{15}$$

*(i) If the function* $\rho_e$ *satisfies*

$$\sup_{x_1 \in \mathbb{R}^{n_1}} \rho_e(x_1, x_2, r_1, r_2) < +\infty, \quad \forall (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \tag{16}$$

$$\sup_{x_1 \in \mathbb{R}^{n_1}} \rho_e(x_1, x_2, 0, 0) < 0, \quad \forall x_2 \in \mathbb{R}^{n_2} \setminus \{0\} \tag{17}$$

*then there exists a continuous function* $\tilde{\rho}_e : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ *such that*

$$\tilde{\rho}_e(x_2, 0, 0) < 0 \quad , \forall x_2 \in \mathbb{R}^{n_2} \setminus \{0\} \tag{18}$$
$$\rho_e(x_1, x_2, r_1, r_2) \leq \tilde{\rho}_e(x_2, r_1, r_2), \quad \forall (x_1, x_2, r_1, r_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \tag{19}$$

*hold.*

*(ii) If the vector* $x_1$ *is a dependent variable, and there exists a continuous function* $g_{12} : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}_+$ *such that*

$$|x_1| \leq g_{12}(x_2, r_1, r_2) \tag{20}$$

*is satisfied, then there exists a continuous function* $\tilde{\rho}_e : (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ *such that (18) and*

$$\rho_e(x_1, x_2, r_1, r_2) \leq \tilde{\rho}_e(x_2, r_1, r_2), \quad \forall (x_2, r_1, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \tag{21}$$

*hold.*

Materials focused on by subsequent sections are the situations of (i) and (ii) in the above lemma.

In this paper, we also consider several other variants which relax Problem 1 and Problem 2. We define Problem 1' by replacing the condition (5) with

$$\rho_e(x_1, x_2, 0, 0) \leq 0 \quad , \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \tag{22}$$

Clearly, Problem 1 is solvable only if Problem 1' has a solution. In the same manner, we define Problem 2' by replacing the condition (12) with

$$\rho_e(x_2, 0, 0) \leq 0 \quad , \forall x_2 \in \mathbb{R}^{n_2} \tag{23}$$

Problem 2 is solvable only if Problem 2' has a solution. Again, it is easily seen from (14) that Problem 1' is solvable only if Problem 2' has a solution.

# 3 Construction of Lyapunov functions

This section demonstrates that stable properties of nonlinear interconnected systems are strongly related to the solutions of the state-dependent scaling problems introduced in Section 2. It is shown that the inequalities of the sum of scaled supply rates, which are (4) an (11), directly lead us to Lyapunov functions establishing the stability of interconnection of dissipative systems in a unified manner.

Consider the nonlinear interconnected system $\Sigma$ shown in Fig.1. Suppose that subsystems $\Sigma_1$ and $\Sigma_2$ are described by

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \tag{24}$$
$$\Sigma_2 : \ \dot{x}_2 = f_2(t, x_2, u_2, r_2) \tag{25}$$

These two dynamic systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. If $\Sigma_1$ is static, we suppose that $\Sigma_1$ is described by

$$\Sigma_1 : \ z_1 = h_1(t, u_1, r_1) \tag{26}$$

Then, $u_2 = x_1$ is replaced by $u_2 = z_1$. Assume that $f_1(t,0,0,0) = 0$, $f_2(t,0,0,0) = 0$ and $h_1(t,0,0,0) = 0$ hold for all $t \in [t_0,\infty)$, $t_0 \geq 0$. The functions $f_1$, $f_2$ and $h_1$ are supposed to be piecewise continuous in $t$, and locally Lipschitz in the other arguments. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. The state vector of the interconnected system $\Sigma$ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of $\Sigma_i$. When the "t"s are dropped in (24), (25) and (26), the system $\Sigma_i$ is said to be time-invariant.

In what follows we shall provide sufficient conditions for stable properties of the interconnected system $\Sigma$. Those results are obtained by making use of a Lyapunov function in the form of

$$V_{cl}(t,x) = \int_0^{V_1(t,x_1)} \lambda_1(s)ds + \int_0^{V_2(t,x_2)} \lambda_2(s)ds \tag{27}$$

where $V_i$ is the Lyapunov-like function of the $x_i$-subsystem $\Sigma_i$.

**Theorem 1** *Suppose that $\Sigma_1$ and $\Sigma_2$ are dynamic systems fulfilling the following.*

*(i) The system $\Sigma_1$ admits the existence of a $\mathbf{C}^1$ function $V_1 : (t,x_1) \in \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+$ such that it satisfies*

$$\underline{\alpha}_1(|x_1|) \leq V_1(t,x_1) \leq \bar{\alpha}_1(|x_1|) \tag{28}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1(t,x_1,u_1,r_1) \leq \rho_1(x_1,u_1,r_1) \tag{29}$$

*for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$, where $\underline{\alpha}_1$ and $\bar{\alpha}_1$ are class $\mathcal{K}_\infty$ functions, and $\rho_1 : (x_1,u_1,r_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \to \mathbb{R}$ is a continuous function satisfying $\rho_1(0,0,0) = 0$.*

*(ii) The system $\Sigma_2$ admits the existence of a $\mathbf{C}^1$ function $V_2 : (t,x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ such that it satisfies*

$$\underline{\alpha}_2(|x_2|) \leq V_2(t,x_2) \leq \bar{\alpha}_2(|x_2|) \tag{30}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t,x_2,u_2,r_2) \leq \rho_2(x_2,u_2,r_2) \tag{31}$$

*for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$, $r_2 \in \mathbb{R}^{m_2}$ and $t \in \mathbb{R}_+$, where $\underline{\alpha}_2$ and $\bar{\alpha}_2$ are class $\mathcal{K}_\infty$ functions, and $\rho_2 : (x_2,u_2,r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is a continuous function satisfying $\rho_2(0,0,0) = 0$.*

*If there is a solution $\{\lambda_1,\lambda_2\}$ to Problem 1, the equilibrium $x = [x_1^T, x_2^T]^T = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that*

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|), \quad \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{32}$$

*is satisfied and*

$$\frac{dV_{cl}}{dt} \leq \rho_e(x,r), \quad \forall x \in \mathbb{R}^n, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{33}$$

*holds along the trajectories of the system $\Sigma$.*

The inequality (33) represents disturbance attenuation properties. Integrating (33) in $t$ from $t_0 \geq 0$ to $T$ we obtain

$$\int_{t_0}^{T} \rho_e(x,r)dt \geq \underline{\alpha}_{cl}(x(T)) - \bar{\alpha}_{cl}(x(t_0)), \quad \forall T \in [t_0,\infty) \tag{34}$$

For instance, by choosing $\rho_e(x,r)$ as

$$\rho_e(x,r) = -|x|^p + \gamma^p|r|^p, \quad p>0$$

the inequality (34) with $x(t_0)=0$ becomes

$$\int_{t_0}^{T} \gamma^p|r|^p dt \geq \int_{t_0}^{T} |x|^p dt, \quad \forall T \in [t_0, \infty)$$

which represents $\mathcal{L}_p$ disturbance attenuation of level $\gamma$. The $\mathcal{L}_2$ disturbance attenuation is a popular performance index for linear systems and it is called $\mathcal{H}^\infty$ norm of stable linear systems.

The condition (3) requires certain growth order of the function $\lambda_i(s)$ with respect to $s$ toward $\infty$. For instance, $s^2$, $s$, 1 and $1/(s+1)$ are admitted. However, $1/(s^2+1)$ and $1/(s^3+1)$ do not meet the condition (3). If $\lambda_i(s)$ is continuous and satisfies

$$\lambda_i(s) \geq k_\infty s^{-1}, \quad \forall s \in [1, \infty)$$

for some $k_\infty > 0$, it fulfills (3). This growth order assumption ensures that the Lyapunov function $V_{cl}(t,x)$ constructed by (27) is radially unbounded, which leads us to the global stability of $\Sigma$.

If a system $\Sigma_i$ in Fig.1 is static, the growth order constraint (3) on $\lambda_i(s)$ is unnecessary. In addition, we can employ other flexibilities of functions $\xi_i$ and $\varphi_i$. Thereby, Problem 1 can be replaced by a weaker Problem 2 in the presence of a static system.

**Theorem 2** *Suppose that $\Sigma_1$ is a static system, and $\Sigma_2$ is a dynamic system fulfilling the following.*

*(i) The system $\Sigma_1$ satisfies*

$$\rho_1(z_1, u_1, r_1) \geq 0 \tag{35}$$

*for all $u_1 \in \mathbb{R}^{n_2}$ , $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$, where $\rho_1 : (z_1, u_1, r_1) \in \mathbb{R}^{p_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ is a continuous function satisfying $\rho_1(0,0,0)=0$.*

*(ii) The system $\Sigma_2$ satisfies (ii) of Theorem 1.*

*If there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem 2, the equilibrium $x = x_2 = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that*

$$\underline{\alpha}_{cl}(|x_2|) \leq V_{cl}(t, x_2) \leq \bar{\alpha}_{cl}(|x_2|), \quad \forall x_2 \in \mathbb{R}^{n_2}, t \in \mathbb{R}_+ \tag{36}$$

*is satisfied and*

$$\frac{dV_{cl}}{dt} \leq \rho_e(x_2, r), \quad \forall x_2 \in \mathbb{R}^{n_2}, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{37}$$

*holds along the trajectories of the system $\Sigma$.*

A Lyapunov function proving the above theorem is

$$V_{cl}(t, x_2) = \int_0^{V_2(t,x_2)} \lambda_2(s) ds \tag{38}$$

Using another type of Lyapunov function which is different from the previous theorems, we can also establish the stability in the following way.

**Theorem 3** *Suppose that $\Sigma_1$ is a static system, and $\Sigma_2$ is a dynamic system fulfilling the following.*

*(i) The system $\Sigma_1$ satisfies (i) of Theorem 2.*

*(ii) The system $\Sigma_2$ admits the existence of a $\mathbf{C}^1$ function $V_2 : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ such that it satisfies*

$$\underline{\alpha}_2(|x_2|) \leq V_2(t, x_2) \leq \bar{\alpha}_2(|x_2|) \tag{39}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2, u_2, r_2) \leq \rho_2(x_2, u_2, r_2) - \omega_2(\mu_2(x_2)) \frac{d\mu_2(x_2)}{dt} \tag{40}$$

*for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$, $r_2 \in \mathbb{R}^{m_2}$ and $t \in \mathbb{R}_+$, where $\underline{\alpha}_2$ and $\bar{\alpha}_2$ are class $\mathcal{K}_\infty$ functions, and $\rho_2 : (x_2, u_2, r_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ is a continuous function satisfying $\rho_2(0, 0, 0) = 0$, and $\mu_2 : x_2 \in \mathbb{R}^{n_2} \to \mathbb{R}$ is a $\mathbf{C}^1$ function satisfying $\mu_2(0) = 0$, and $\omega_2 : s \in \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $s\omega_2(s) \geq 0$ for all $s \in \mathbb{R}$.*

*If there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem 2 and $\lambda_2$ is constant, then the equilibrium $x = x_2 = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t, x_2) \in \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that*

$$\underline{\alpha}_{cl}(|x_2|) \leq V_{cl}(t, x_2) \leq \bar{\alpha}_{cl}(|x_2|), \quad \forall x_2 \in \mathbb{R}^{n_2}, t \in \mathbb{R}_+ \tag{41}$$

*is satisfied and*

$$\frac{dV_{cl}}{dt} \leq \rho_e(x_2, r), \quad \forall x_2 \in \mathbb{R}^{n_2}, r \in \mathbb{R}^m, t \in \mathbb{R}_+ \tag{42}$$

*holds along the trajectories of the system $\Sigma$.*

This theorem is based on the following Lyapunov function.

$$V_{cl}(t, x_2) = \lambda_2 V_2(t, x_2) + \lambda_2 \int_0^{\mu_2(x_2)} \omega_2(s) ds \tag{43}$$

A system $\Sigma_1$ satisfying (29) is said to be dissipative[12, 1, 16], and the function $\rho_1$ is referred to as the supply rate. In the rest of this paper, a system $\Sigma_i$ is said to accept a supply rate $\rho_i$ if there exists a $\mathbf{C}^1$ function $V_i(t, x_i)$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_i$, $\bar{\alpha}_i$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \bar{\alpha}_i(|x_i|) \tag{44}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i, u_i, r_i) \leq \rho_i(x_i, u_i, r_i) \tag{45}$$

hold for all $x_i$, $u_i$, $r_i$ and $t$. When $\Sigma_i$ is static, we replace the pair of (44) and (45) by the following single inequality.

$$\rho_i(z_i, u_i, r_i) \geq 0 \tag{46}$$

For convenience, we call the function $\rho_i$ for the static system the supply rate although energy is never stored by static systems.

The conditions (4) and (11) are given in terms of the relation between supply rates of two subsystems. It is stressed that the conditions are not in the form of linear combinations of supply rates. Functional coefficients $\lambda_1$, $\lambda_2$ and $\xi_1$ are introduced into the combinations of supply rates, and they scale supply rates of subsystems. The functionals also appear in (27), (38) and (43) to construct the Lyapunov functions. The use of the functionals $\lambda_1$ $\lambda_2$ and $\xi_1$ is contrasted with the early works on Lyapunov stability criteria for interconnected dissipative systems such as [12, 1, 16], where linear combinations of supply rates were employed without such functional coefficients. In (4) and (11), the parameters $\lambda_1$, $\lambda_2$ and $\xi_1$ are allowed to be functions of the state variables $x_1$ and $x_2$. This is why this paper refers to $\lambda_1$, $\lambda_2$ and $\xi_1$ as state-dependent scaling functions. State-dependence of scaling factors is emphasized to distinguish them from constant $\lambda_1$ and $\lambda_2$ and an identity function $\xi_1(s) = s$. The conditions (4) and (11) can be regarded as a general formulation of the state-dependent scaling technique [7, 8, 9]. The results in [7, 8, 9] were developed on the basis of special cases of (4) and (11) where the supply rates are finite $\mathcal{L}_2$-gain, ISS or a subset of integral ISS. Those papers [7, 8, 9] originally refer to $1/\lambda_i$ as the state-dependent scaling factors.

If both the systems $\Sigma_1$ and $\Sigma_2$ are time-invariant, stability of interconnected systems can be established by solving Problem 1' and Problem 2' which are milder than Problem 1 and Problem 2, respectively.

**Corollary 1** *Suppose that $\Sigma_1$ and $\Sigma_2$ are time-invariant dynamic systems satisfying (i) and (ii) of Theorem 1 with $V_1(x_1)$ and $V_2(x_2)$, respectively. If there is a solution $\{\lambda_1, \lambda_2\}$ to Problem 1' and*

$$\left.\begin{array}{l} r(t) = 0 \\ x(t) \in \mathcal{Z} \end{array}\right\} \forall t \in \mathbb{R}_+ \Rightarrow \lim_{t\to\infty} x(t) = 0 \tag{47}$$

*holds for the set $\mathcal{Z} \in \{x \in \mathbb{R}^n : \rho_e(x, 0) = 0\}$, then the equilibrium $x = [x_1^T, x_2^T]^T = 0$ of the time-invariant interconnected system $\Sigma$ is globally asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : x \in \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that (32) is satisfied and (33) holds along the trajectories of the system $\Sigma$.*

**Corollary 2** *Suppose that $\Sigma_1$ is a time-invariant static system satisfying (i) of Theorem 2, and $\Sigma_2$ is a time-invariant dynamic system satisfying (ii) of Theorem 2 with $V_2(x_2)$ . If there is a solution $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem 2' and either of*

*(I) For the set $\mathcal{Z}_2 = \{x_2 \in \mathbb{R}^{n_2} : \rho_e(x_2, 0) = 0\}$,*

$$\left.\begin{array}{l} r(t) = 0 \\ x_2(t) \in \mathcal{Z}_2 \end{array}\right\} \forall t \in \mathbb{R}_+ \Rightarrow \lim_{t\to\infty} x_2(t) = 0 \tag{48}$$

*(II) For the set $\hat{\mathcal{Z}}_2 = \{x_2 \in \mathbb{R}^{n_2} : \rho_1(h_1(x_2, 0), x_2, 0) = 0\}$,*

$$\left.\begin{array}{l} u_2(t) = h_1(x_2, 0) \\ r_2(t) = 0 \\ x_2(t) \in \hat{\mathcal{Z}}_2 \end{array}\right\} \forall t \in \mathbb{R}_+ \Rightarrow \lim_{t\to\infty} x_2(t) = 0 \tag{49}$$

*hold, and there exists a constant $\varepsilon$ such that*

$$\lambda_1(t, z_1, x_2, r_1, r_2) \geq \varepsilon > 0, \quad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}, r_2 \in \mathbb{R}^{m_2}, t \in \mathbb{R}_+ \tag{50}$$

*is fulfilled, then the equilibrium $x = x_2 = 0$ of the time-invariant interconnected system $\Sigma$ is globally asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : x_2 \in \mathbb{R}^{n_2} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that (36) is satisfied and (37) holds along the trajectories of the system $\Sigma$.*

**Corollary 3** *Suppose that $\Sigma_1$ is a time-invariant static system satisfying (i) of Theorem 3, and $\Sigma_2$ is a time-invariant dynamic system satisfying (ii) of Theorem 3 with $V_2(x_2)$ . If there is a solution*

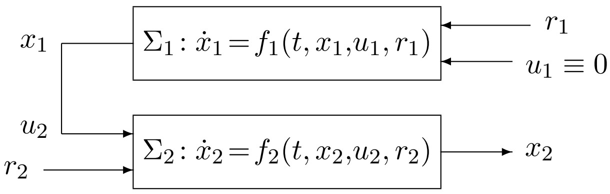



Figure 2: Cascade system $\Sigma_c$

*$\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ to Problem 2' with a constant $\lambda_2$, and if either of (I) and (II) of Corollary 2 is fulfilled, then the equilibrium $x = x_2 = 0$ of the time-invariant interconnected system $\Sigma$ is globally asymptotically stable for $r \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : x_2 \in \mathbb{R}^{n_2} \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that (41) is satisfied and (42) holds along the trajectories of the system $\Sigma$.*

**Remark 1** In the case of $\varphi_1(z_1, x_2, r_1) \equiv 0$, the function $\xi_1$ is not necessarily increasing, and

$$\xi_1(s) > 0, \quad \forall s \in (0, N] \cap (0, \infty) \tag{51}$$

is sufficient for proving Corollary 2 and Corollary 3 as it is seen from their proofs. Furthermore,

$$\xi_1(s) \geq 0, \quad \forall s \in [0, N] \tag{52}$$

is enough for proving the part (I) of Corollary 2 and Corollary 3 in the case of $\varphi_1(z_1, x_2, r_1) \equiv 0$.

Cascade systems are special cases of the discussion in this section. In other words, the state-dependent scaling problems establish stable properties of cascade connection of systems. Indeed, cascade connection can be obtained by simply removing a feedback path in Fig.1. If we replace $\rho_i(x_i, u_i, r_i)$ with $\rho_i(x_i, r_i)$, the path of $u_i$ is disconnected. The cascade system $\Sigma_c$ obtained by disconnecting $u_1$ is shown in Fig.2.

This section has shown that the state-dependent scaling problems are directly related to construction of Lyapunov functions, and they provide a unified approach to stability and performance of interconnected systems which are allowed to have supply rates in a general form. Clearly, a solution to a state-dependent scaling problem exists only if the interconnected system actually possesses the stable property required. This section has not mentioned how easy or difficult it is to find the solution. It is seen that Problem 1 and Problem 2 are jointly affine in the scaling functions $\lambda_1$ and $\lambda_2$. This affine property should be helpful in calculating solutions. In Section 5, we address the question of when and how we are able to obtain solutions to the state-dependent scaling problems successfully for popular supply rates. A major purpose of the second part [15] is to give answers to the question for more advanced types of supply rates.

## 4 Examples

This section illustrates the effectiveness and versatility of the state-dependent scaling characterization through several simple examples. It is shown how the state-dependent scaling analysis enables us to discover Lyapunov functions establishing stable properties for various nonlinearities in a unified manner.

**Example 1** Consider the interconnected system shown in Fig.1. Suppose that the individual systems are given by

$$\Sigma_1 : \ \dot{x}_1 = -\left(\frac{x_1}{x_1+1}\right)^2 + 3\left(\frac{x_2}{x_2+1}\right)^2, \quad x_1(0) \in \mathbb{R}_+ \tag{53}$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{4x_2}{x_2+1} + \frac{2x_1}{x_1+1} + 6r_2, \quad x_1(0) \in \mathbb{R}_+ \tag{54}$$

This interconnected system is defined for $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ and $r_2 \in \mathbb{R}_+$. Indeed, $x(0) \in \mathbb{R}_+^2$ and $r_2(t) \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$ imply that $x(t) \in \mathbb{R}_+^2, \forall t \in \mathbb{R}_+$. Although this example is for a compact illustration of theoretical development in this paper, it is motivated by models of biological processes which usually involve Monod nonlinearities and exhibit the non-negative property. It is verified that both the systems $\Sigma_1$ and $\Sigma_2$ are neither finite $\mathcal{L}_2$-gain nor ISS[6]. Due to the non-negative property, the simplest choices of Lyapunov functions for individual $\Sigma_1$ and $\Sigma_2$ are $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$. Supply rate functions are calculated for them as $\rho_1 = \dot{x}_1$ and $\rho_2 = \dot{x}_2$. It is not difficult at all to calculate scaling parameters $\lambda_1(x_1)$ and $\lambda_2(x_2)$ achieving the scalar inequality (4) of Problem 1 since (4) is affine in $\lambda_1$ and $\lambda_2$. For this example, the sum of scaled supply rates is

$$\begin{aligned} S(x, r_2) &= \lambda_1\rho_1 + \lambda_2\rho_2 \\ &= -\left[\lambda_1\left(\frac{x_1}{x_1+1}\right)^2 - 2\lambda_2\frac{x_1}{x_1+1}\right] - \left[4\lambda_2\frac{x_2}{x_2+1} - 3\lambda_1\left(\frac{x_2}{x_2+1}\right)^2\right] + 6\lambda_2 r_2 \end{aligned} \tag{55}$$

It is easily observed that there are no constants $\lambda_1, \lambda_2 > 0$ which render $S(x, 0)$ negative definite. Thus, in order to solve (4), we need to introduce a function to at least one of $\lambda_1$ and $\lambda_2$. Let $\lambda_2$ be a function and let $\lambda_1$ be a constant. Set $\lambda_1 = 1$ without any loss of generality. Then, the function (55) satisfies

$$S(x, 0) = \lambda_2\left[2\frac{x_1}{x_1+1} - 4\frac{x_2}{x_2+1}\right] + 3\left(\frac{x_2}{x_2+1}\right)^2 - \left(\frac{x_1}{x_1+1}\right)^2 + 6\lambda_2 r_2$$

Let $D(x_2)$ denote the unique number $x_1 \in \mathbb{R}_+$ solving

$$2\frac{x_1}{x_1+1} - 4\frac{x_2}{x_2+1} = 0$$

The following holds.

$$x_1 = D(x_2) \ \Rightarrow \ S(x, 0) = -\left(\frac{x_2}{x_2+1}\right)^2$$

By using this property with $D(0) = 0$ and defining

$$F(x_1, x_2) = \frac{\left(\frac{x_1}{x_1+1}\right)^2 - 3\left(\frac{x_2}{x_2+1}\right)^2}{2\frac{x_1}{x_1+1} - 4\frac{x_2}{x_2+1}}$$

we obtain

$$S(x, 0) < 0, \ \forall x \in \mathbb{R}_+^2 \setminus \{0\} \Leftrightarrow \begin{cases} \lambda_2 < F(x) & \text{for } x_1 > D(x_2) \\ \lambda_2 > F(x) & \text{for } x_1 < D(x_2) \end{cases}$$

Since simple calculation leads us to

$$\inf_{x_1 \in (D(x_2), \infty)} F(x) = 3\frac{x_2}{x_2+1}, \qquad \sup_{x_1 \in (0, D(x_2))} F(x) = \frac{x_2}{x_2+1}$$

the pair

$$\lambda_1 = 1, \quad \lambda_2 = bx_2/(x_2+1), \quad b \in (1,3) \tag{56}$$

achieves

$$S(x,0) < 0, \ \forall x \in \mathbb{R}_+^2 \setminus \{0\}$$

Note that the non-negativeness of $x$ guarantees that $\lambda_2$ is positive all over the state space except $x_2 = 0$ and it is nonsingular everywhere in the nonnegative domain. Hence, using Theorem 1 with the solutions in (56), we can conclude that $x = 0$ is globally asymptotically stable for $r_2 \equiv 0$. In the presence of the exogenous input $r_2$, we obtain

$$S(x, r_2) = S(x,0) + 6\lambda_2 r_2 \leq S(x,0) + 6br_2 = \rho_e(x,r)$$

so that (4) is solved. The inequality (33) in Theorem 1 implies that the interconnected system $\Sigma$ has the integral input-to-state stable property[17]. Theorem 1 also gives a Lyapunov function in the interesting form for $\Sigma$ as

$$\begin{aligned} V_{cl}(x_1,x_2) &= \int_0^{V_1} \lambda_1(s)ds + \int_0^{V_2} \lambda_2(s)ds \\ &= x_1 + b(x_2 - \log(x_2+1)), \quad b \in (1,3) \end{aligned} \tag{57}$$

For an illustration, the sum of scaled supply rates, i.e., $S(x,r_2) = \lambda_1\rho_1 + \lambda_2\rho_2$ with $\lambda_1(s) = 1$ and $\lambda_2(s) = 1.7s/(s+1)$ is plotted on the state space in Fig. 3(a). For a comparison, the function $S(x,r_2)$ is also plotted for $\lambda_1 = 1$ and $\lambda_2 = 1$ in Fig. 3(b). For visual simplicity, the surface of $S(x,0)$ is drawn by setting $r_2 = 0$. According to Theorem 1, the equilibrium $x = 0$ is globally asymptotically stable if the surface is below the horizontal plane of zero. The choice of state-dependent scalings $\lambda_1(s) = 1$ and $\lambda_2(s) = 1.7s/(s+1)$ fulfills this requirement, while the choice of constant scalings $\lambda_1 = 1$ and $\lambda_2 = 1$ does not. The absence of constants $\{\lambda_1, \lambda_2\}$ rendering the surface below the zero plane implies that state-dependently scaled combination of supply rates is crucial, and linear combination is useless for this example. It is reasonable that 'nonlinear' combination is often effective for 'nonlinear' systems.

**Example 2** Suppose that $\Sigma_1$ and $\Sigma_2$ in Fig.1 are given by

$$\Sigma_1 : \ \dot{x}_1 = -\frac{2x_1}{x_1+1} + \frac{x_2}{(x_1+1)(x_2+1)}, \quad x_1(0) \in \mathbb{R}_+ \tag{58}$$

$$\Sigma_2 : \ \dot{x}_2 = -\frac{4x_2}{x_2+1} + x_1, \quad x_2(0) \in \mathbb{R}_+ \tag{59}$$

Note that $x = [x_1, x_2]^T \in \mathbb{R}_+^2$ holds for all $t \in \mathbb{R}_+$. The choice $V_1(x_1) = x_1$ yields

$$\dot{V}_1 = \rho_1(x_1,x_2) \leq \frac{2x_1}{x_1+1} + \frac{x_2}{x_2+1}$$

This implies that $\Sigma_1$ is ISS[6]. The system $\Sigma_2$ is not ISS since we have $x_2 \to \infty$ as $t \to \infty$ for $x_1(t) \equiv 5$. The sum of scaled supply rates is calculated for $V_1(x_1) = x_1$ and $V_2(x_2) = x_2$ as

$$\begin{aligned} S(x) &= \lambda_1\rho_1 + \lambda_2\rho_2 \\ &= -\left[2\lambda_1\frac{x_1}{x_1+1} - \lambda_2 x_1\right] - \left[4\lambda_2 - \lambda_1\frac{1}{x_1+1}\right]\frac{x_2}{x_2+1} \end{aligned} \tag{60}$$

It is seen clearly that if both $\lambda_1$ and $\lambda_2$ are restricted to constants, the function $S(x)$ is never negative definite. Hence, we introduce a function to $\lambda_1$ and let $\lambda_2$ be a constant, i.e., $\lambda_2 = 1$. Then, we have

$$S(x) = \lambda_1 \left[ \frac{1}{x_1+1} \frac{x_2}{x_2+1} - 2\frac{x_1}{x_1+1} \right] - 4\frac{x_2}{x_2+1} + x_1$$

The following property can be verified.

$$\frac{x_2}{x_2+1} = 2x_1 \;\Rightarrow\; S(x) = -7x_1$$

In the case of $x_2/(x_2+1) = 2x_1$, the situation of $-7x_1 = 0$ is $x_1 = x_2 = 0$. Therefore, by defining

$$F(x_1, x_2) = \frac{4\dfrac{x_2}{x_2+1} - x_1}{\dfrac{1}{x_1+1}\dfrac{x_2}{x_2+1} - 2\dfrac{x_1}{x_1+1}}$$

we obtain

$$S(x) < 0, \; \forall x \in \mathbb{R}_+^2 \backslash \{0\} \Leftrightarrow \begin{cases} \lambda_1 < F(x) \;\text{ for }\; \dfrac{x_2}{x_2+1} > 2x_1 \\ \lambda_1 > F(x) \;\text{ for }\; \dfrac{x_2}{x_2+1} < 2x_1 \end{cases}$$

The local minimum and maximum are calculated easily as

$$\inf_{x_2 \in (D(x_1), \infty)} F(x) = F(x_1, \infty) = \frac{x_1 - 4}{2x_1 - 1}(x_1+1)$$
$$\sup_{x_2 \in (0, D(x_1))} F(x) = F(x_1, 0) = \frac{1}{2}(x_1+1)$$

where $D(x_1) \in \mathbb{R}_+$ denotes the unique number $x_2$ fulfilling $x_2/(x_2+1) = 2x_1$. Note that $x_2/(x_2+1) > 2x_1$ implies $1/2 > x_1$. Since

$$\inf_{x_1 \in (0, 1/2)} \frac{x_1 - 4}{2x_1 - 1} = 4$$

holds, the pair

$$\lambda_1(x_1) = b(x_1 + 1), \; b \in (1/2, 4), \quad \lambda_2(x_2) = 1 \tag{61}$$

solves (4). Due to Theorem 1, the origin $x = 0$ is globally asymptotically stable. From (61) a Lyapunov function proving the stability is obtained as

$$V_{cl}(x_1, x_2) = bx_1(x_1/2 + 1) + x_2, \quad b \in (1/2, 4) \tag{62}$$

The sum $S(x)$ with (61) is plotted for $b = 1$ in Fig. 4(a), while the sum with $\lambda_1 = \lambda_2 = 1$ is shown by Fig. 4(b).

**Example 3** The last example is a feedback interconnection of two systems described by

$$\Sigma_1: \; \frac{dx_1}{dt} = -2x_1 + x_2 \tag{63}$$

$$\Sigma_2: \; \frac{dx_2}{dt} = -2x_2^5 + x_2^3 x_1^2 \tag{64}$$

The state vector of the overall interconnected system is $x = [x_1, x_2]^T \in \mathbb{R}^2$. It is verified that $\Sigma_1$ and $\Sigma_2$ are ISS[6]. Let $V_1(x_1) = x_1^2$ and $V_2(x_2) = x_2^2$, and their supply rates are set to $\rho_1 = 2x_1\dot{x}_1$ and $\rho_2 = 2x_2\dot{x}_2$. Then, the sum of scaled supply rates in (4) is

$$\begin{aligned} S(x) &= \lambda_1\rho_1 + \lambda_2\rho_2 \\ &= -4\lambda_1 x_1^2 + 2\lambda_1 x_1 x_2 - 4\lambda_2 x_2^6 + 2\lambda_2 x_2^4 x_1^2 \end{aligned} \tag{65}$$

Comparing the growth order of $x_i$, we can see that the function $S(x)$ cannot be rendered negative definite by constants $\lambda_1$ and $\lambda_2$. Since the fact implies that at least one of $\lambda_1$ and $\lambda_2$ needs to be a function, we let $\lambda_1$ and $\lambda_2$ be a function and a constant, respectively. Without any loss of generality, we set $\lambda_2 = 1$ and obtain

$$S(x) = \lambda_1\left[2x_1x_2 - 4x_1^2\right] - 4x_2^6 + 2x_2^4x_1^2$$

We also have the following.

$$2x_1x_2 - 4x_1^2 = 0 \;\Rightarrow\; \left\{ \begin{array}{c} x_1 = 0 \quad\Rightarrow\quad S(x) = -4x_2^6 \\ \text{or} \\ x_2 = 2x_1 \;\Rightarrow\; S(x) = -224x_1^6 \end{array} \right.$$

From this property and the definition of

$$F(x_1, x_2) = \frac{4x_2^6 - 2x_2^4x_1^2}{2x_1x_2 - 4x_1^2}$$

we obtain

$$\begin{array}{c} S(x) < 0 \\ \forall x \in \mathbb{R}_+^2 \setminus \{0\} \end{array} \;\Leftrightarrow\; \left\{ \begin{array}{ll} \lambda_1 < F(x) \text{ for } & \begin{array}{c} x_2 \in (2x_1, \infty) \\ x_1 > 0 \end{array} \\ \lambda_1 < F(x) \text{ for } & \begin{array}{c} x_2 \in (-\infty, 2x_1) \\ x_1 < 0 \end{array} \\ \lambda_1 > F(x) \text{ for } & \begin{array}{c} x_2 \in (-\infty, 2x_1) \\ x_1 > 0 \end{array} \\ \lambda_1 > F(x) \text{ for } & \begin{array}{c} x_2 \in (2x_1, \infty) \\ x_1 < 0 \end{array} \end{array} \right.$$

By calculating $\partial F/\partial x_2$, local minima and maxima of $F(x)$ as a function of $x_2$ are obtained.

$$\begin{array}{lllr} x_2 = -0.572x_1 & : & \text{local maximum} & 0.01439x_1^4 \\ x_2 = 0 & : & \text{local minimum} & 0 \\ x_2 = 0.587x_1 & : & \text{local maximum} & 0.02612x_1^4 \\ x_2 = 2x_1 & : & \text{singular point} & \pm\infty \\ x_2 = 2.385x_1 & : & \text{local minimum} & 872.0x_1^4 \end{array}$$

The maximum and minimum values lead us to

$$\begin{aligned} \inf_{\substack{x_2 \in (2x_1, \infty) \\ x_1 > 0}} F(x) = \inf_{\substack{x_2 \in (-\infty, 2x_1) \\ x_1 < 0}} F(x) &= 872.0x_1^4 \\ \sup_{\substack{x_2 \in (-\infty, 2x_1) \\ x_1 > 0}} F(x) = \sup_{\substack{x_2 \in (2x_1, \infty) \\ x_1 < 0}} F(x) &= 0.02612x_1^4 \end{aligned}$$

Since $V_1(x_1) = x_1^2$ has been chosen, the inequality (4) of Problem 1 is satisfied for

$$\lambda_1(s) = bs^2, \; b \in (0.02612, 872.0), \quad \lambda_2(x_2) = 1 \tag{66}$$

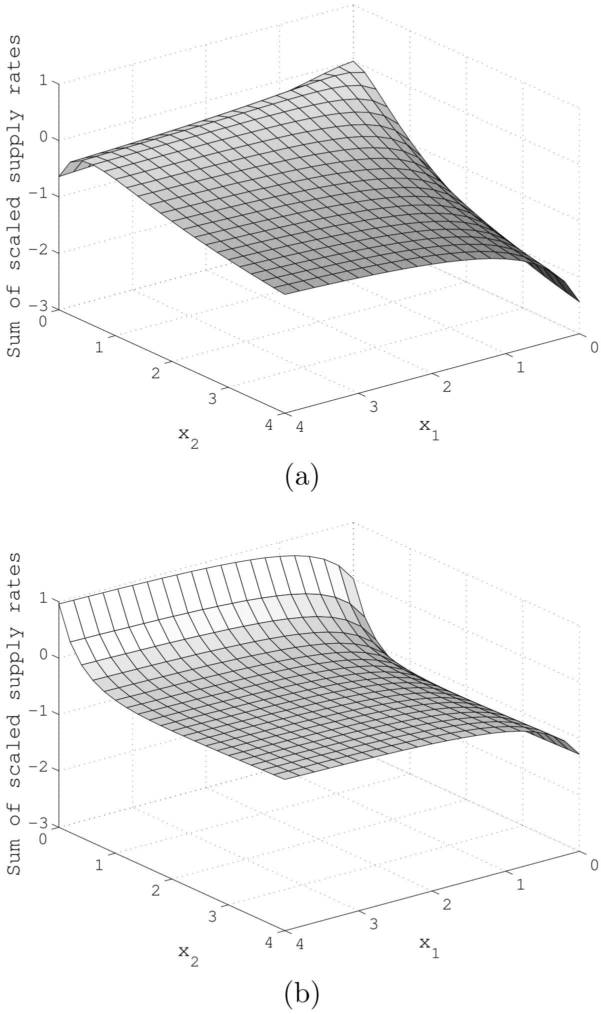



Figure 3: Example 1. (a) State-dependently scaled combination of supply rates with functions $\lambda_1$ and $\lambda_2$ calculated directly from (4). (b) Linear combination of supply rates with $\lambda_1 = 1$ and $\lambda_2 = 1$.

Theorem 1 not only guarantees the global asymptotic stability of $x = 0$, but also gives a Lyapunov function

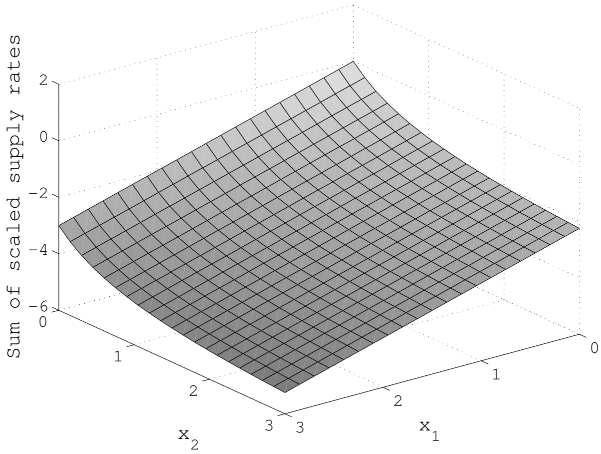
$$V_{cl}(x_1, x_2) = \frac{bx_1^6}{3} + x_2^2, \quad b \in (0.02612, 872.0) \tag{67}$$

establishing the stability of the feedback system. Figure 5(a) shows $S(x)$ with (66) for $b = 2$, The function $S(x)$ with $\lambda_1 = \lambda_2 = 1$ is shown in Fig. 5(b).

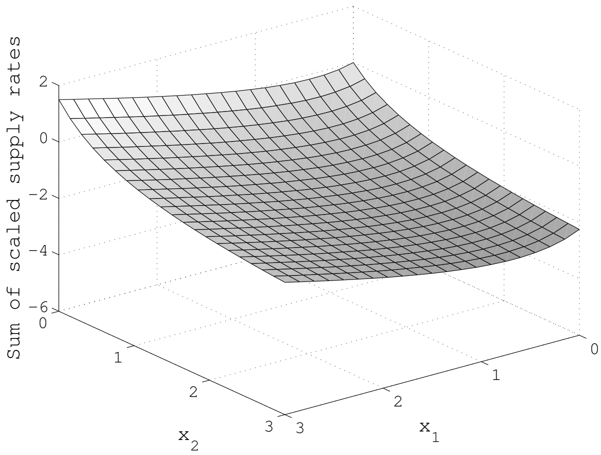
These three examples have suggested that the state-dependence of scaling functions, in other words 'nonlinear combination of individual supply rates' or 'nonlinear combination of individual storage functions', is vital for dealing with some strong nonlinearities.

# 5 Classical stability criteria

This section discusses the universality of the state-dependent scaling problems. The state-dependent scaling problems are able to deal with various properties represented by supply rates $\rho_i$ in a unified manner. As seen in Section 3, systems are allowed to be static as well as dynamic. For $\rho_i$ chosen from




(a)




(b)

Figure 4: Example 2: (a) State-dependently scaled combination of supply rates with functions $\lambda_1$ and $\lambda_2$ calculated directly from (4). (b) Linear combination of supply rates with $\lambda_1 = 1$ and $\lambda_2 = 1$.

functions popular in classical analysis, the state-dependent scaling problems reduce to well-known criteria for stability. This section clarifies that those classical criteria are sufficient conditions for the existence of solutions to the state-dependent scaling problems. Explicit selections of solutions are also shown.

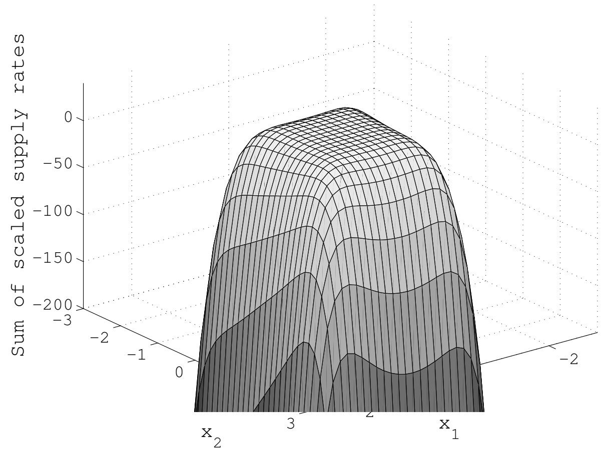
Let us review popular stability criteria presented in the famous paper [1] and see how they are extracted smoothly as special cases from the state-dependent scaling problems. Consider the interconnected system $\Sigma$ shown in Fig.6. Individual systems in Fig.6 are supposed to be described by

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(t, x_1, w) \\ z = h_1(t, x_1) \end{cases} \tag{68}$$

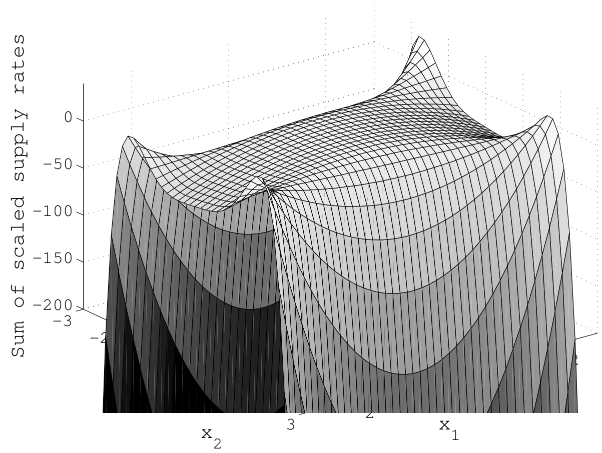
$$\Sigma_2 : \begin{cases} \dot{x}_2 = f_2(t, x_2, z) \\ w = h_2(t, x_2) \end{cases} \tag{69}$$

If $\Sigma_1$ is static, it is supposed to be described by

$$\Sigma_1 : z = h_1(t, w) \tag{70}$$




(a)




(b)

Figure 5: Example 3: (a) State-dependently scaled combination of supply rates with functions $\lambda_1$ and $\lambda_2$ calculated directly from (4). (b) Linear combination of supply rates with $\lambda_1 = 1$ and $\lambda_2 = 1$.

Assume that $f_i(t, 0, 0) = 0$ and $h_i(t, 0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$ and $i = 1, 2$. The functions $f_i$ and $h_i$ , i=1,2, are supposed to be piecewise continuous in $t$, and locally Lipschitz in the other arguments. The state vector of the interconnected system $\Sigma$ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of $\Sigma_i$. When "$t$"s are dropped in (68), (69) and (70), the system $\Sigma_i$ is said to be time-invariant.

We begin by addressing the $\mathcal{L}_2$ small-gain theorem.

**Proposition 1** *Suppose that $\gamma_1 > 0$ and $\gamma_2 > 0$ satisfy*

$$\gamma_1 \gamma_2 < 1 \tag{71}$$

*Then, there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that*

$$\lambda_1 \gamma_1^2 < \lambda_2 < \lambda_1 / \gamma_2^2 \tag{72}$$

*holds, and the following propositions are true.*

*(a.i) Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (72) solves Problem 1 defined with*

$$\rho_1(x_1, w, z) = -\beta_1(x_1) + \gamma_1^2 w^T w - z^T z$$
$$\rho_2(x_2, z, w) = -\beta_2(x_2) + \gamma_2^2 z^T z - w^T w$$

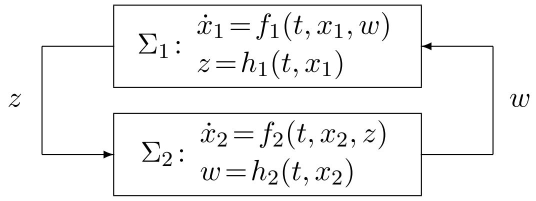



Figure 6: Feedback interconnected system $\Sigma$

*for any positive definite functions $\beta_1, \beta_2$.*

(a.ii) *If the systems $\Sigma_1$ and $\Sigma_2$ accept supply rates given in (a.i), the equilibrium $x = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

(b.i) *Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (72) solves Problem 1' defined with*

$$\rho_1(x_1, w, z) = \gamma_1^2 w^T w - z^T z, \;\; \rho_2(x_2, z, w) = \gamma_2^2 z^T z - w^T w$$

(b.ii) *If the systems $\Sigma_1$ and $\Sigma_2$ accept supply rates given in (b.i) and they are time-invariant and zero-state detectable, then the equilibrium $x = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

This proposition is verified straightforwardly. The stability is due to Theorem 1 and Corollary 1. The inequality (71) is often referred to as the $\mathcal{L}_2$ small-gain condition. In a similar manner, we can also obtain $\gamma_1\gamma_2 < 1$ as the $\mathcal{L}_p$ small-gain condition by replacing (72) with $\lambda_1\gamma_1^p < \lambda_2 < \lambda_1/\gamma_2^p$. We next consider the passivity theorems. The following state-space version of the passivity theorems can be verified easily in view of the state-dependent scaling problems.

**Proposition 2** *Suppose that constants $\lambda_1$ and $\lambda_2$ satisfy*

$$\lambda_1 = \lambda_2 > 0 \tag{73}$$

*Then, the following propositions are true.*

(a.i) *Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (73) solves Problem 1 defined with*

$$\rho_1(x_1, w, z) = -\beta_1(x_1) + w^T z, \;\; \rho_2(x_2, z, w) = -\beta_2(x_2) - z^T w$$

*for any positive definite functions $\beta_1, \beta_2$.*

(a.ii) *If the systems $\Sigma_1$ and $\Sigma_2$ accept supply rates given in (a.i), the equilibrium $x = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

(b.i) *Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (73) solves Problem 1' defined with*

$$\begin{aligned} \rho_1(x_1, w, z) &= w^T z - \epsilon_{w1} w^T w - \epsilon_{z1} z^T z \\ \rho_2(x_2, z, w) &= -z^T w - \epsilon_{z2} z^T z - \epsilon_{w2} w^T w \end{aligned}$$

*for any $\epsilon_{w1}, \epsilon_{z1}, \epsilon_{w2}, \epsilon_{z2} \in \mathbb{R}$ satisfying $\epsilon_{w1} + \epsilon_{w2} > 0$ and $\epsilon_{z1} + \epsilon_{z2} > 0$.*

(b.ii) *If the systems $\Sigma_1$ and $\Sigma_2$ accept supply rates given in (b.i) and they are time-invariant and zero-state detectable, then the equilibrium $x = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

*(c.i)* *Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (73), together with $\xi_1(s) = s$, solves Problem 2' defined with*

$$\rho_1(w, z) = w^T z, \quad \rho_2(x_2, z, w) = z^T w$$

*(c.ii)* *If $\Sigma_1$ is a time-invariant static system fulfilling*

$$w^T h_1(w) = 0 \Rightarrow w = 0$$

*and $\Sigma_2$ is a time-invariant zero-state detectable dynamic system and they accept supply rates given in (c.i), then the equilibrium $x = x_2 = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

It is easily verified that Proposition 2 remains valid even if $\epsilon_{wi}$ and $\epsilon_{zi}$ are replaced by $\epsilon_{wi}(w)$ and $\epsilon_{zi}(z)$, which is discussed in [18].

**Proposition 3** *Suppose that $\alpha, \beta, \nu \in \mathbb{R}$ satisfy*

$$0 \leq \alpha < \beta, \quad (\alpha + \beta)\nu \leq 1 \tag{74}$$

*Then, there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that*

$$\lambda_1(\alpha + \beta) - \lambda_2 = 0, \quad -\lambda_1 + \lambda_2 \nu \leq 0 \tag{75}$$

*hold, and the following propositions are true.*

*(i)* *Any pair of constants $\lambda_1$ and $\lambda_2$ satisfying (75), together with $\xi_1(s) = s$, solves Problem 2' defined with*

$$\begin{aligned}\rho_1(w, z) &= (\alpha + \beta) w^T z - z^T z - \alpha\beta w^T w \\ \rho_2(x_2, z, w) &= -z^T w + \nu z^T z\end{aligned}$$

*(ii)* *Suppose that $w$ and $z$ are scalar and*

$$\rho_1(w, h_1(w)) = 0 \Rightarrow w = 0 \tag{76}$$

*holds. If $\Sigma_1$ is a time-invariant static system accepting the supply rate $\rho_1$ given in (i), and $\Sigma_2$ is a time-invariant zero-state detectable dynamic system accepting a supply rate of*

$$\frac{\partial V_2}{\partial x_2} f_2 \leq -z^T w + \nu z^T z - \kappa \dot{w} z \tag{77}$$

*for some $\kappa \geq 0$, then the equilibrium $x = x_2 = 0$ of the interconnected system $\Sigma$ is globally uniformly asymptotically stable.*

The solvability of Problem 2' in the part of (i) is straightforward. The part (ii) of this proposition is obtained from Theorem 3 with $\mu_2 = w$ and $\omega_2(s) = \kappa h_1(s)$.

A static system $\Sigma_1$ which is time-invariant and single-input-single-output is said to belong to a sector $(\alpha, \beta)$ if

$$\alpha w^2 < w h_1(w) < \beta w^2, \quad \forall w \in \mathbb{R} \setminus \{0\}, \qquad 0 \leq \alpha < \beta \tag{78}$$

holds. Any static system $\Sigma_1$ belonging the sector $(\alpha, \beta)$ accepts the supply rate $\rho_1$ given in (i) of Proposition 3. It is clear that the sector $(\alpha, \beta)$ fulfills (76). For linear systems $\Sigma_2$, the stability criterion in (ii) for a sector $(0, \beta)$ yields the Popov criterion. In fact, the Lyapunov function used in the proof of Theorem 3 for $\mu_2 = w$ and $\omega_2(s) = \kappa h_1(s)$ reduces to a Lur'e function. The case of $\kappa = 0$ corresponds to the circle criterion.

Propositions in this section revisit the well-known classical stability criteria unified in [1] and presented in other textbooks of nonlinear systems control[19, 20]. It is not surprising that those classical criteria can be interpreted as simple special cases of the state-dependent scaling problems. It is, however, worth emphasizing that all those classical stability theorems are proved by simply using constant parameters $\lambda_1$ and $\lambda_2$ and an identity map $\xi_1$. In that situation, the state-dependent scaling problems are in the form of

$$\lambda_1\rho_1(x_1, x_2) + \lambda_2\rho_2(x_2, x_1) \leq \rho_e(x)$$

In other words, all the classical stability criteria in [1], such as the $\mathcal{L}_2$ small-gain theorem, the passivity theorems, the Popov and circle criteria, are proved by using linear combinations of supply rates. By contrast, in the previous section, the state-dependent scaling factors are required to be functions of state variables to establish the stable properties for the examples which are not covered by the classical stability criteria. In the next section, a stability criterion involving ISS systems is proved by making use of scaling functions $\lambda_i$ depending on state variables. This fact reveals the essential difference between the advanced stability theorem and the classical theorems in [1]. Providing more evidences of the effectiveness of the state-dependence of scaling for nonlinear systems is a purpose of the follow-up paper[15].

## 6 ISS small-gain theorem

This section concentrates on interconnected ISS systems. In the configuration of Fig.1, both the constituent systems are supposed to be ISS individually. This section derives a condition guaranteeing the existence of solutions to the corresponding state-dependent scaling problem. It is demonstrated that the state-dependent scaling formulation reduces to the ISS small-gain condition which has become popular recently in the area of nonlinear systems control. The development in this section is distinct from previous studies of the ISS small-gain theorems which are based on trajectories of systems. This section pursues explicit construction of Lyapunov functions.

In this section, we assume that, for each $\Sigma_i$, $i = 1, 2$ in Fig.1, there exists a $\mathbf{C}^1$ function $V_i$ : $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \bar{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{79}$$

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i, u_i, r_i) \leq -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \\ , \ \forall x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{n_{ui}}, r_i \in \mathbb{R}^{m_i}, t \in \mathbb{R}_+ \tag{80}$$

are satisfied for some $\underline{\alpha}_i$, $\bar{\alpha}_i$, $\alpha_i \in \mathcal{K}_\infty$ and some $\sigma_i$, $\sigma_{ri} \in \mathcal{K}$. To put it shortly, we assume that each $\Sigma_i$ is ISS with respect to input $(u_i, r_i)$ and state $x_i$. In the single input case, the second input $r_i$ is null, and the function $\sigma_{ri}$ vanishes. The function $V_i(t, x_i)$ is called a $\mathbf{C}^1$ ISS Lyapunov function[13]. The trajectory-based definition of ISS may be seen more often than the Lyapunov-based definition this paper adopts. The Lyapunov-based definition is more suitable for the state-space version of stability analysis. The two types of definition is equivalent in the sense that the existence of ISS Lyapunov functions is necessary and sufficient for ISS[13]. If one uses terminology introduced in Section 3, the system $\Sigma_i$ is assumed to accept the supply rate in the form of

$$\rho_i(x_i, u_i, r_i) = -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{81}$$

$$\alpha_i \in \mathcal{K}_\infty, \quad \sigma_i \in \mathcal{K}, \quad \sigma_{ri} \in \mathcal{K} \tag{82}$$

throughout this section.

We are able to obtain solutions to the state-dependent scaling problem for the ISS supply rates in the following way.

**Theorem 4** *If there exist $c_i > 1$, $i = 1, 2$ such that*

$$\underline{\alpha}_1^{-1} \circ \bar{\alpha}_1 \circ \alpha_1^{-1} \circ c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{83}$$

*is satisfied, there exist solutions $\{\lambda_1, \lambda_2\}$ to Problem 1 with respect to a continuous function $\rho_e(x, r)$ of the form*

$$\rho_e(x, r) = -\alpha_{cl}(|x|) + \sigma_{cl}(|r|), \ \alpha_{cl} \in \mathcal{K}_\infty, \ \sigma_{cl} \in \mathcal{K} \tag{84}$$

In the case of

$$\sigma_1 \in \mathcal{K}_\infty, \quad , (c_1 - 1)(c_2 - 1) > 1 \tag{85}$$

it is not very difficult to verify that the pair

$$\lambda_1(s) = \left[\frac{1}{c_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{c_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right] \tag{86}$$

$$\lambda_2(s) = c_2\sqrt{\frac{c_1 - 1}{c_2 - 1}} \left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^2 \tag{87}$$

solves Problem 1 on the assumption (83). The proof and the solution without the simplifying assumption (85) can be found in the follow-up paper[15]. The following is a direct corollary of Theorem 4.

**Corollary 4** *If there exist $c_i > 1$, $i = 1, 2$ such that (83) is satisfied, the interconnected system $\Sigma$ is ISS with respect to input $r$ and state $x$.*

Although the statement of Corollary 4 by itself is essentially the same as the ISS small-gain theorem presented in [2, 3], this paper proposes a new approach to the ISS small-gain theorem. The combination of Corollary 4 and Theorem 4 forms a state-dependent scaling version of the proof of the ISS small-gain theorem. The state-dependent scaling approach gives explicit information about how to construct a Lyapunov function to establish the ISS property of the feedback interconnected system. It contrasts sharply with the original ISS small-gain theorem[2, 3, 10] which are stated and proved by using trajectories of systems. In this sense, the state-dependent scaling proof is constructive in view of Lyapunov functions. The Lyapunov function which leads us to the ISS small-gain theorem is not necessarily unique. There is another type proof of the ISS small-gain theorem based on a different Lyapunov function. In [11], the existence of a smooth Lyapunov function is proved by presenting non-smooth functions which determine a Lyapunov function in an implicit manner. In contrast, this paper demonstrates that the equation (27) defined with state-dependent scaling functions $\{\lambda_1, \lambda_2\}$ given by Theorem 4 provides us with an explicit formula of the Lyapunov function. Another desirable feature of the state-dependent scaling is that it allows a smooth transition to stability criteria for more general systems. This paper has explained the ISS small-gain theorem as a special case of the state-dependent scaling problems.

## 7 Conclusions

This paper has proposed the state-dependent scaling approach to the analysis of stability and performance of interconnection of nonlinear dissipative systems. State-dependent scaling problems have been formulated so that they are applicable to general functions of supply rate in a unified manner. If we restrict our attention to popular supply rates, classical stability theorems can be extracted as special cases. Classical stability criteria have been viewed as sufficient conditions for guaranteeing the existence of solutions to the state-dependent scaling problems. The idea of the state-dependent scaling problems is formed by an inequality representing the sum of nonlinearly scaled supply rates of dissipative systems. The inequality is solved for parameters called scaling functions. The scaling functions lead us to Lyapunov functions of feedback and cascade connected systems explicitly. Under the framework of the state-dependent scaling problems, we do not have to distinguish the ISS small-gain theorem and the dissipative approach, and they can be explained in a unified language. The effectiveness of the state-dependent scaling approach is not limited to the settings of popular classical stability criteria and the ISS small-gain theorem. It is not only illustrated by the examples provided by this paper, but also demonstrated by the follow-up paper[15]. Indeed, a major purpose of the follow-up paper is to show explicit formulas of solutions to the state-dependent scaling problems for integral input-to-state stable(iISS) supply rates, and we are able to obtain small-gain-like theorems for feedbacks and cascades involving iISS systems.

The developments of this two-part paper have brought up some interesting issues. Further research is needed to pursue analytical formulas of solutions to the state-dependent scaling problems for various types of supply rate. It is worth stressing that calculating analytical solutions is not the only way to make use of the developments of this paper. Using increasing power of computers and softwares, we are able to seek solutions numerically. While the analytical investigation gives us guarantees of the existence of solutions for representative types of supply rate, the numerical computation allows us to try to find solutions for general supply rates. Problem 1 is jointly affine in the parameters $\lambda_1$ and $\lambda_2$, and Problem 2 is also jointly affine in these parameters. The affine property should be advantageous to numerical computation and optimization. It is an important and practical direction of future research to investigate optimization algorithm that is effective particularly in solving the state-dependent scaling problems.

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# Appendix

## A Preliminaries

### A.1 Proof of Lemma 1

*(i)* Suppose that (16) holds. The function $\tilde{\rho}_e$ defined by

$$\tilde{\rho}_e(x_2, r_1, r_2) = \sup_{x_1 \in \mathbb{R}^{n_1}} \rho_e(x_1, x_2, r_1, r_2)$$

fulfills (19). If (17) holds, the function $\tilde{\rho}_e$ also fulfills (18).
*(ii)* Suppose that (20) holds, which implies that the vector $x_1$ is bounded independently of $t$ whenever $(x_2, r_1, r_2)$ is bounded. The assumption (15) implies

$$\left.\begin{array}{l} \rho_e(x_1, x_2, 0, 0) = 0 \\ x_2 \in \mathbb{R}^{n_2} \end{array}\right\} \Rightarrow x_2 = 0 \tag{88}$$

under the condition (20). From the continuity of $\rho_e$ it follows that

$$\rho_e(x_1, x_2, 0, 0) \leq 0, \quad \forall x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2} \tag{89}$$

holds. The existence of $\tilde{\rho}_e$ satisfying (18) and (21) is guaranteed by (88) and (89).

# B Construction of Lyapunov functions

## B.1 Proof of Theorem 1

Using the solution $\{\lambda_1, \lambda_2\}$ of Problem 1, define $V_{cl}(t,x)$ by (27). The function $V_{cl}(t,x)$ is $\mathbf{C}^1$ since $\lambda_1$ and $\lambda_2$ are continuous. Under the assumption of (28) and (30), the inequalities (1)-(3) imply the existence of $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$ satisfying (32). The time-derivative of $V_{cl}(t,x)$ along the trajectory of $\Sigma$ is calculated as

$$\begin{aligned}&\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x_1} f_1(t,x_1,x_2,r_1) + \frac{\partial V_{cl}}{\partial x_2} f_2(t,x_2,x_1,r_2)\\ &= \lambda_1(V_1(t,x_1))\rho_1(x_1,x_2,r_1) + \lambda_2(V_2(t,x_2))\rho_2(x_2,x_1,r_2)\end{aligned}$$

Since the pair $\{\lambda_1, \lambda_2\}$ achieves (4), we obtain

$$\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x_1} f_1 + \frac{\partial V_{cl}}{\partial x_2} f_2 \le \rho_e(x,r)$$

Due to the property (5), the function $\rho_e(x,0)$ is positive definite in $x$, which implies the global uniform asymptotic stability of $x=0$ when $r \equiv 0$.

## B.2 Proof of Theorem 2

Since $\varphi_1$ satisfies (10), the inequality (35) implies

$$\varphi_1(z_1,x_2,r_1) + \rho_1(z_1,x_2,r_1) \ge 0, \quad \forall z_1 \in \mathbb{R}^{p_1}, x_2 \in \mathbb{R}^{n_2}, r_1 \in \mathbb{R}^{m_1}$$

The increasing property of $\xi_1$ yields

$$\xi_1(\varphi_1(z_1,x_2,r_1)) \le \xi_1(\varphi_1(z_1,x_2,r_1) + \rho_1(z_1,x_2,r_1))$$

Using $\lambda_2$ of the solution to Problem 2, define a $\mathbf{C}^1$ function $V_{cl}(t,x_2)$ by (38). The inequalities (6)-(8) guarantee the existence of $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$ satisfying (36). The time-derivative of $V_{cl}(t,x_2)$ along the trajectory of $\Sigma$ satisfies

$$\begin{aligned}\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x} f_2 &= \lambda_2(V_2(t,x_2))\rho_2(x_2,z_1,r_2)\\ &\le \lambda_1(t,z_1,x_2,r)\left[-\xi_1(\varphi_1) + \xi_1(\varphi_1+\rho_1)\right] + \lambda_2(V_2(t,x_2))\rho_2\end{aligned}$$

since the range of $\lambda_1$ is in $\mathbb{R}_+$. If the quartet of $\{\lambda_1, \lambda_2, \xi_1, \varphi_1\}$ achieves (11), we obtain

$$\frac{\partial V_{cl}}{\partial t} + \frac{\partial V_{cl}}{\partial x} f_2 \le \rho_e(x_2,r)$$

Due to the property (12), the equilibrium $x = x_2 = 0$ is globally uniformly asymptotically stable when $r \equiv 0$.

## B.3 Proof of Theorem 3

Define a $\mathbf{C}^1$ function by (43). Due to $\mu_2(0)=0$ and $s\omega_2(s) \ge 0$, there exists $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl} \in \mathcal{K}_\infty$ such that (41) holds. Since it holds that

$$\frac{d}{dt}\left(\int_0^{\mu_2(x_2)} \omega_2(s)ds\right) = \omega_1(\mu_2(x_2))\frac{d\mu_2(x_2)}{dt}$$

the remaining part is the same as that of Theorem 2

## B.4 Proof of Corollary 1

Following the proof of Theorem 1, we obtain (32) and arrive at

$$\frac{\partial V_{cl}}{\partial x_1} f_1 + \frac{\partial V_{cl}}{\partial x_2} f_2 \leq \rho_e(x, r) \tag{90}$$

for a function $\rho_e$ that satisfies (22). The inequality (90) is identical to (33), which proves that $x = 0$ is globally stable, and all trajectories are bounded. Let $x(t; x_0, t_0)$ denote the trajectory of $\Sigma$ starting from $x = x_0$ at $t = t_0$ for nil exogenous signal $r(t) \equiv 0$. Let $r(t) \equiv 0$ in the rest of the proof. Since (22) implies that $V_{cl}(x)$ is non-increasing continuous function of $t$ and $V_{cl}(x)$ is bounded from below by 0, we can define $c_\infty = \lim_{t\to\infty} V_{cl}(x(t; x_0, t_0))$ for any trajectory $x(t; x_0, t_0)$. with arbitrary initial condition $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$. Let $\Gamma(x_0)$ be the $\omega$ limit set of the trajectory. Note that $\Gamma(x_0)$ is nonempty and compact due to boundedness of $x(t; x_0, t_0)$. By definition of $\omega$ limit set, $V(p) = c_\infty$ holds for all $p \in \Gamma(x_0)$. Consider the trajectory $x(t; p, t_0)$ produced by initial condition $p \in \Gamma(x_0)$. Since the time-invariant interconnected system with $r \equiv 0$ is autonomous, $\Gamma(x_0)$ is an invariant set. Thus, $V_{cl}(x(t; p, t_0)) = c_\infty$ holds for all $t \in \mathbb{R}_+$. The inequalities (22) and (90) imply $\rho_e(x(t; p, t_0), 0) = 0$ for all $t \in \mathbb{R}_+$. Since (47) is assumed for the set $\mathcal{Z} = \{x \in \mathbb{R}^n : \rho_e(x, 0) = 0\}$, we have $\lim_{t\to\infty} x(t; p, t_0) = 0$, and $c_\infty = 0$ is obtained from $V_{cl}(0) = 0$. We arrive at $\lim_{t\to\infty} V_{cl}(x(t; x_0, t_0)) = 0$. Since $V_{cl}(x)$ vanishes only at $x = 0$, all trajectories $x(t; x_0, t_0)$ converges to zero as $t \to \infty$.

## B.5 Proof of Corollary 2

Following the proof of Theorem 2, we obtain (36) and

$$\frac{\partial V_{cl}}{\partial x_2} f_2 \leq \rho_e(x_2, r) \tag{91}$$

and the function $\rho_e$ satisfies (23). The above inequality (91) which is identical to (37) proves that $x = x_2 = 0$ is globally stable, and all trajectories are bounded for $r \equiv 0$. Let $r \equiv 0$ in the rest of the proof.

(I) Consider an arbitrary point $p_2 \in \Gamma(x_{2,0})$. From (23) and (91) it follows that $\rho_e(x_2(t; p_2, t_0), 0) = 0$ holds for all $t \in \mathbb{R}_+$. If (48) holds for the set $\mathcal{Z}_2 = \{x_2 \in \mathbb{R}^{n_2} : \rho_e(x_2, 0) = 0\}$, we have $\lim_{t\to\infty} x_2(t; p_2, t_0) = 0$. From $V_{cl}(0) = 0$, we obtain $\lim_{t\to\infty} V_{cl}(x(t; x_{2,0}, t_0)) = 0$ for any trajectory $x_2(t; x_{2,0}, t_0)$. Since $V_{cl}(x_2)$ vanishes only at $x_2 = 0$, all trajectories $x_2(t; x_{2,0}, t_0)$ converges to zero as $t \to \infty$.

(II) From (11) we obtain

$$\begin{aligned} \frac{\partial V_{cl}}{\partial x_2} f_2 \leq \rho_e(x_2, r) - \lambda_1(t, z_1, x_2, r) \Big[ &-\xi_1(\varphi_1(z_1, x_2, r_1)) + \\ &\xi_1(\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1)) \Big] \end{aligned} \tag{92}$$

Note that, due to (50), the range of $\lambda_1$ is in $(0, \infty)$ and (9) holds. Due to the increasing property of $\xi_1$, we have

$$-\xi_1(\varphi_1(z_1, x_2, r_1)) + \xi_1(\varphi_1(z_1, x_2, r_1) + \rho_1(z_1, x_2, r_1)) = 0$$

if and only if $\rho_1(z_1, x_2, r_1) = 0$ holds. Consider an arbitrary point $p_2 \in \Gamma(x_{2,0})$. From (23) and (92) it follows that $\rho_1(h_1(x_2(t; p_2, t_0), 0), x_2(t; p_2, t_0), 0) = 0$ holds for all $t \in \mathbb{R}_+$. If (49) holds for the set $\hat{\mathcal{Z}}_2 = \{x_2 \in \mathbb{R}^{n_2} : \rho_1(h_1(x_2, 0), x_2, 0) = 0\}$, we have $\lim_{t\to\infty} x_2(t; p_2, t_0) = 0$. The rest of the proof is the same as (I).

### B.6 Proof of Corollary 3

The claims are obtained by combining proofs of Theorem 3 and Corollary 2.