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Small-Gain Conditions for iISS Systems: Some Proofs of Sufficiency and Necessity*§

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Abstract: This technical report addresses necessary conditions and sufficient conditions for stability of interconnection of nonlinear systems with external inputs. Integral input-to-state stable (iISS) systems and input-to-state stable (ISS) systems are treated in a unified manner using a Lyapunov formulation. The purpose of this technical report is to provide proofs of the following properties. One property this report proves is that, for the stability of the interconnected system in the presence of model uncertainty, at least one subsystem is necessarily ISS which is a stronger stability property in the set of iISS. Fulfillment of a small-gain-type property is also proved to be necessary. Finally, this report derives a common form of smooth Lyapunov functions which can establish the iISS and the ISS of the interconnection comprising iISS and ISS subsystems whenever the small-gain-type condition is satisfied. Instead of explaining implications of the properties, this technical report focuses on presenting their proofs.

Keywords: Nonlinear interconnected system, Stability criteria, Integral input-to-state stability, Input-to-state stability, Global asymptotic stability, Lyapunov function, Dissipative system,

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§This technical report presents detailed proofs and complementary materials for the results to be published in Hiroshi Ito and Zhong-Ping Jiang, "Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective", IEEE Trans. Automatic Control, Vol.54, 2009 in which some details are not presented due to the space limitation. For the sake of self-contained presentation, some results presented there are repeated.

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1 Introduction

For about a decade, the input-to-state stability (ISS) property has been accepted as a useful way to characterize a class of nonlinearities in view of stability[10]. The ISS small-gain theorem provides a sufficient condition for the stability of a feedback system comprised of ISS subsystems[9, 13, 8]. The theorem makes use of the idea of nonlinear loop gain. There is another important class of systems which are not necessarily ISS. This is characterized by the integral input-to-state stability (iISS) property[11, 2]. Those systems have finite nonlinear gain only in a very weak sense. In contrast to ISS systems, because of the weakness of gain, the cascade of iISS systems are not always stable[3, 4]. In spite of such a weak gain property, a stability criterion covering iISS systems in feedback configurations has been developed recently by one of the authors[5], which is a result of the Lyapunov constructive approach presented in [6]. The criterion gives a sufficient condition for iISS property of interconnected iISS systems in the form of a small-gain property. The possibility of establishing stability for the feedback interconnection of iISS systems by means of gain conditions is followed up by a nullcline approach[1] in the absence of external signals. Generalizing the proposed result of [1] to the case of external stability with respect to external signals is by no means easy. As a matter of fact, the relationship between the nullcline approach and the Lyapunov constructive approach has not been investigated yet.

The purpose of this technical report is to provide detailed proofs of the results presented in [7]. The contribution of [7] is mainly threefold. One is to derive necessary conditions for the stability of interconnected systems in order to show how reasonable small-gain-type criteria are. Another is to unify the treatment of iISS and ISS systems by merging the two types of small-gain conditions derived from the two types of Lyapunov functions dealing with iISS and ISS separately. The third objective is to provide Lyapunov functions in the situations considered by the nullcline approach to global asymptotic stability(GAS) in the absence of external signals. An emphasis is placed on Lyapunov functions to accomplish all the points and the development of a single unified formula applicable equally to iISS systems and ISS systems. The condition of a small-gain type proposed in [6] for iISS systems looks more complicated and more restrictive than the small-gain condition for ISS systems. The result in [7] not only merges the two small-gain-type conditions, but also removes the assumption of uniform contraction used in [6]. The unification and the generalization of Lyapunov functions also enable us to come to the point where the necessity of the small-gain condition holds. Furthermore, in [7], it is shown that at least one of the subsystems in the loop needs to be ISS with respect to feedback input.

In this report, let the symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. Negation is $\neg$. The interval $[0, \infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. The Euclidean norm of a vector in $\mathbb{R}^n$ is denoted by $|\cdot|$. The identity map on $\mathbb{R}$ is denoted by $\mathbf{Id}$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ and written as $\gamma \in \mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ and written as $\gamma \in \mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty} \gamma(r) = \infty$. We write $\gamma \in \mathcal{P}_0$ for a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ if it is a continuous function satisfying $\gamma(0) = 0$. The set of $\gamma \in \mathcal{P}_0$ satisfying $\gamma(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$ is denoted by $\gamma \in \mathcal{P}$. For a function $h \in \mathcal{P}$, we write $h \in \mathcal{O}(> L)$ with a non-negative number L if there exists a positive number $K > L$ such that $\limsup_{s\to 0+} h(s)/s^K < \infty$ holds. We write $h \in \mathcal{O}(L)$ when $K = L$. As for limiting value of functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, we use the simple notation $\lim f(s) = \lim g(s)$ to describe $\{\lim f(s) = \infty \ \wedge \ \lim g(s) = \infty\} \ \vee \ \{\infty > \lim f(s) = \lim g(s)\}$. In a similar manner, $\lim f(s) \geq \lim g(s)$ denotes $\{\lim f(s) = \infty \ \vee \ \infty > \lim f(s) \geq \lim g(s)\}$. A system $\dot{x} = f(x, r)$ is said to be 0-GAS if the 0-input system $\dot{x} = f(x, 0)$ has a unique equilibrium which is globally asymptotically stable.

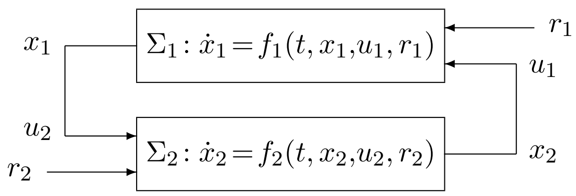

Figure 1: Interconnected system Σ

2 System Setup

Consider the interconnected system Σ shown in Fig.1. The subsystems Σ_1 and Σ_2 are connected with each other through $u_1 = x_2$ and $u_2 = x_1$. The state vector of Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$. The signals r_1 and r_2 are packed into $r = [r_1^T, r_2^T]^T \in \mathbb{R}^k$. In this report, we consider the following sets of Σ_i's as in [7].

Definition 1 *Given $\alpha_i \in \mathcal{P}$, $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{P}_0$ for $i = 1, 2$, let $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i = 1, 2$ denote the pair of sets containing systems Σ_i in the form of*

$$\dot{x}_i = f_i(t, x_i, u_i, r_i), \ x_i \in \mathbb{R}^{n_i}, \ u_i \in \mathbb{R}^{m_i}, \ r_i \in \mathbb{R}^{k_i} \tag{1}$$

$$f_i(t, 0, 0, 0) = 0, \qquad t \in \mathbb{R}_+ \tag{2}$$

$$f_i \textit{ is locally Lipschitz in } (x_i, u_i, r_i) \textit{ uniformly in } t \textit{ and piecewise continuous in } t \tag{3}$$

for which there exist $\mathbf{C}^1$ functions V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(t, x_i) \le \bar{\alpha}_i(|x_i|) \tag{4}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x_1} f_1 \le -\alpha_1(V_1(t, x_1)) + \sigma_1(V_2(t, x_2)) + \sigma_{r1}(|r_1|) \tag{5}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2 \le -\alpha_2(V_2(t, x_2)) + \sigma_2(V_1(t, x_1)) + \sigma_{r2}(|r_2|) \tag{6}$$

hold for all $x_i \in \mathbb{R}^{n_i}$, $r_i \in \mathbb{R}^{k_i}$ and $t \in \mathbb{R}_+$, $i = 1, 2$.

The integers m_i's are supposed to satisfy $m_1 = n_2$ and $m_2 = n_1$ so that the interconnection of Σ_1 and Σ_2 makes sense. The Lipschitzness imposed on f_i guarantees the existence of a unique maximal solution of Σ for locally essentially bounded $r_i(t)$. If the exogenous signal r_i is absent, the set of systems is denoted by $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i)$.

The inequalities (5) and (6) are often referred to as "dissipation inequalities", and their right hand sides are called supply rates. The individual system Σ_i fulfilling the above definition is said to be integral input-to-state stable (iISS)[11]. The function V_i is called a $\mathbf{C}^1$ iISS Lyapunov function[2]. Under a stronger assumption $\alpha_i \in \mathcal{K}_\infty$, the system Σ_i is said to be input-to-state stable (ISS)[10], and the function V_i is a $\mathbf{C}^1$ ISS Lyapunov function[12]. By definition, an ISS system is always iISS. The converse does not hold. The original notion of iISS and ISS is given in terms of trajectories and, in the context of time-invariant systems, is equivalent to the existence of $\mathbf{C}^1$ iISS and ISS Lyapunov functions, respectively[2, 12]. As we see on the right hand side of (5) and (6), the iISS and ISS properties we consider in this report are uniform in time t.

Definition 2 *Given $\alpha_i \in \mathcal{P}$, $\sigma_i \in \mathcal{K}$, $\sigma_{ri} \in \mathcal{P}_0$ and $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ for $i = 1, 2$, let $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ denote the set of systems Σ_i of the form (1), (2) and (3) which admit the existence of a $\mathbf{C}^1$ function*

V_i: $\mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}$ *satisfying (4) and*

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i \leq -\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|) \tag{7}$$

for all $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $r_i \in \mathbb{R}^{k_i}$ *and* $t \in \mathbb{R}_+$, $i = 1, 2$.

Definition 3 *Let* $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ *denote the set of* Σ_i *for which there exist* $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ *such that* $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ *holds.*

We write $\mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ and $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \underline{\alpha}_i, \bar{\alpha}_i)$ when we consider $r_i(t) \equiv 0$. Definitions 2 and 3 involve $|\cdot|$ to measure the magnitude of feedback signals in the dissipation inequalities. As we will see in the sequel, for the set $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ whose dissipation inequalities do not involve the Euclidean norm of feedback signals, stability criteria become simpler than those for $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$. The set $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ in Definition 3 naturally generalizes the notion of prescribed $\mathcal{L}^p$-gain systems. By comparison, the set $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ in Definition 2 includes the explicit information $\underline{\alpha}_i$, $\bar{\alpha}_i$ on the discrepancy between $|\cdot|$ and $V_i(\cdot)$, which is essential to the analysis of 0-GAS of the interconnection.

3 Small-Gain Conditions

This section briefly presents small-gain-type theorems developed in [7]. The following provides a necessary and sufficient condition for the uniform 0-GAS of a set of interconnected iISS systems as shown in Fig. 1. By uniform 0-GAS, we mean that the trivial solution of the interconnected system Σ without external inputs r_1 and r_2 is uniformly GAS.

Theorem 1 *Let* n_i *be a positive integer for each* $i = 1, 2$. *Assume that functions* $\alpha_i, \sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$ *,* $i = 1, 2$ *are* $\mathbf{C}^1$ *and satisfy*

$$\alpha_i \in \mathcal{O}(>1), \;\; \sigma_i \in \mathcal{O}(>0), \quad i = 1, 2 \tag{8}$$

$$\alpha_i \in \mathcal{K}, \quad i = 1, 2 \tag{9}$$

Suppose that there exists some integer $j \in \{1, 2\}$ *such that* α_1, α_2, σ_1 *and* σ_2 *satisfy*

$$\lim_{s\to\infty} \alpha_{3-j}(s) \geq \lim_{s\to\infty} \sigma_{3-j}(s) \tag{10}$$

and one of the following conditions

(G1) $\lim_{s\to\infty} \alpha_{3-j}(s) = \lim_{s\to\infty} \sigma_{3-j}(s)$

(G2) $\lim_{s\to\infty} \dfrac{\sigma_j \circ \alpha_{3-j}^{-1} \circ \sigma_{3-j}(s)}{\alpha_j(s)} \neq 1$

Then, the interconnected system Σ *is uniformly 0-GAS for all pairs* $\Sigma_i \in \mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i)$, $i = 1, 2$ *if and only if*

$$\alpha_j^{-1} \circ \sigma_j \circ \alpha_{3-j}^{-1} \circ \sigma_{3-j}(s) < s, \quad \forall s \in (0, \infty) \tag{11}$$

holds for the above j. *Furthermore, a Lyapunov function of* Σ *characterizing the uniform 0-GAS is given as*

$$V_{cl}(t, x) = \int_0^{V_1(t,x_1)} \lambda_1(s) ds + \int_0^{V_2(t,x_2)} \lambda_2(s) ds \tag{12}$$

for some non-decreasing continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$\lambda_1(s) > 0,\ \lambda_2(s) > 0, \quad s \in (0, \infty) \tag{13}$$

It is emphasized that j in (11) is the same as in any of *(G1)-(G2)*. The properties (9) and (10) are assumed beforehand only for simplicity of expressions. Their necessity will be proven in Theorem 4 and Theorem 5 of Section 4. It is stressed that *(G1)-(G2)* are not simultaneous constraints. Only one of them is required. Let the inequality (11) be referred to as a small-gain condition. It is mentioned here that the uniform 0-GAS in Theorem 1 is derived from

$$\exists \alpha_{cl} \in \mathcal{P} \quad \text{s.t.} \quad \dot{V}_{cl}(t, x) \leq -\alpha_{cl}(|x|), \quad \forall x \in \mathbb{R}^n \tag{14}$$

satisfied along the trajectories of the interconnected system Σ with $r_i(t) \equiv 0$, $i = 1, 2$. The "only if" part of Theorem 1 does not need the assumption *(G1)* $\vee$ *(G2)*. In other words, there always exists a pair of Σ_i, $i = 1, 2$ such that their interconnection is not 0-GAS when (11) is violated.

One can obtain iISS of a set of interconnected systems if amplification factors ω_i, $i = 1, 2$, are introduced to the small-gain condition. A stronger property, ISS, is a special case.

Theorem 2 *Assume that functions* $\alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ *satisfy (9). Suppose that there exists some integer* $j \in \{1, 2\}$ *such that one of the following conditions*

(H1) $\lim_{s\to\infty} \alpha_1(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_2(s) = \infty$

(H2) $\lim_{s\to\infty} \alpha_{3-j}(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_{3-j}(s) < \infty$

(H3) $\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

is satisfied. Then, the interconnected system Σ *is iISS with respect to input* r *and state* x *for all pairs* $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}, \underline{\alpha}_i, \bar{\alpha}_i)$ *with any positive integer* n_i, $i = 1, 2$ *if there exist* $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$ *such that*

$$\underline{\alpha}_j^{-1} \circ \bar{\alpha}_j \circ \alpha_j^{-1} \circ (\mathbf{Id} + \omega_j) \circ \sigma_j \circ \underline{\alpha}_{3-j}^{-1} \circ \bar{\alpha}_{3-j} \circ \alpha_{3-j}^{-1} \circ (\mathbf{Id} + \omega_{3-j}) \circ \sigma_{3-j}(s) \leq s,\ \forall s \in \mathbb{R}_+ \tag{15}$$

holds for the above j. *Furthermore, an iISS Lyapunov function of* Σ *is given as in (12) for some non-decreasing continuous functions* $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying (13). In the case of (H1), the function* V_{cl} *is also an ISS Lyapunov function.*

Note that the inverses of α_j and α_{3-j} in (11) and (15) are not necessarily well defined over $\mathbb{R}_+$. Instead, the fulfillment of (11) and (15) only requires the whole composite function on the left hand side of the inequality to be finite for finite s. Thus, $\lim_{s\to\infty} \alpha_j(s) \geq \lim_{s\to\infty} \sigma_j(s)$ is not necessary. The statement about a Lyapunov function in Theorem 2 claims that

$$\begin{aligned} &\exists \alpha_{cl} \in \mathcal{P},\ \sigma_{cl} \in \mathcal{P}_0 \quad \text{s.t.} \\ &\dot{V}_{cl}(t, x) \leq -\alpha_{cl}(|x|) + \sigma_{cl}(|r|),\ \forall x \in \mathbb{R}^n, r \in \mathbb{R}^k \end{aligned} \tag{16}$$

is satisfied along the trajectories of Σ. Since the above theorem only addresses the sufficiency of a small-gain condition for the stability, neither (8) nor the smoothness of α_i and σ_i is required. It is stressed that j in (15) is the same as in *(H2)*. It can be verified that

$$\textit{(G1)} \vee \textit{(G2)} \Leftarrow \textit{(H1)} \vee \textit{(H2)} \vee \textit{(H3)} \Leftarrow \textit{(H1)}$$

holds under the assumption that there exist $\omega_1, \omega_2 \in \mathcal{K}_\infty$ satisfying (15).

Theorem 3 *Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$ are $\mathbf{C}^1$ and satisfy (8), (9), (H1) and*

$$\sigma_{ri} \in \mathcal{K}_\infty, \quad i = 1, 2 \tag{17}$$

Then, the interconnected system Σ is ISS with respect to input r and state x for all pairs $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, $i = 1, 2$ if and only if there exist $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$ such that

$$\alpha_1^{-1} \circ (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \alpha_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \le s, \quad \forall s \in \mathbb{R}_+ \tag{18}$$

holds. Furthermore, an ISS Lyapunov function of Σ is given as in (12) for some non-decreasing continuous functions $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (13).

Theorem 3 indicates that there exists $\alpha_{cl} \in \mathcal{K}_\infty$ achieving (16). In contrast to Theorem 2 stated with $\sigma_{ri} \in \mathcal{P}_0$, Theorem 3 considers (17) which is narrower than $\mathcal{P}_0$. The assumption (17) is only for obtaining the "only if" part of Theorem 3. If the exogenous signals affect systems through sufficiently small $\sigma_{ri} \notin \mathcal{K}_\infty$, the condition (18) is not always required, while (11) is necessary. For sufficiently small σ_{r1} and σ_{r2}, none of *(H1)*, *(H2)* and *(H3)* is necessary (See Section 5.1).

It is stressed that (15) with $j = 1$ is not equivalent to (15) with $j = 2$ in general. The same remark applies to (11). The $j = 1$ case in (15) implies the $j = 2$ case if

$$\lim_{s\to\infty} \alpha_1(s) = \infty \ \vee \ \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) \tag{19}$$

Thus, the condition (15) is symmetric in terms of $j = 1$ and $j = 2$ when Σ_1 and Σ_2 are individually ISS with respect to the interacting inputs. When iISS subsystems are involved, we need to select $j \in \{1, 2\}$ or interchange Σ_1 and Σ_2 so that (15) or (11) can be fulfilled. Theorem 4 in Section 4 explains why the condition should be asymmetric.

Combining the materials in Sections 4 and 5 proves the theorems in this section.

4 Necessity

In this section, the necessity of the stability criteria presented in Section 3 is demonstrated. The issue of the necessity is important from the perspective of estimating stability margins for uncertain systems as well as the tightness of the stability criteria.

4.1 Destabilizing Perturbation

The following lemma provides a technique to construct destabilizing perturbations, which is the key to the proof of the necessity in Theorems 1 and 3.

Lemma 1 *Suppose that $\mathbf{C}^1$ functions $\alpha \in \mathcal{P}$, $\sigma \in \mathcal{K}$, real numbers $\delta \ge 0$, $\bar{\epsilon} > 0$ and integers $n > 0$, $m > 0$ are given. Assume that α and σ belong to $\mathcal{O}(> 1)$ and $\mathcal{O}(> 0)$, respectively. Then, there exist a locally Lipschitz function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, a $\mathbf{C}^1$ function $V\colon \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\bar{\alpha}$ and a real number $\epsilon \in [0, \bar{\epsilon}]$ such that*

$$f(0, 0) = 0 \tag{20}$$

$$\underline{\alpha}(|x|) = V(x) = \bar{\alpha}(|x|) \tag{21}$$

$$\frac{\partial V}{\partial x} f(x, u) \le -\alpha(|x|) + \sigma(|u|), \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{22}$$

$$\left.\begin{matrix} (1 + \delta)\alpha(|x|) < \sigma(|u|) \\ \epsilon \le |u| \end{matrix}\right\} \Rightarrow \frac{\partial V}{\partial x} f(x, u) > \delta\alpha(|x|) \tag{23}$$

The proof of this lemma given in Subsection 6.1 is constructive.

When $(1/N) + (1/J) < 1$, $\alpha \in \mathcal{O}(N)$ and $\sigma \in \mathcal{O}(J)$ are satisfied, the claim in Lemma 1 still holds for $\bar{\epsilon} = 0$.

The function $f(x, u)$ constructed in the proof of Lemma 1 satisfies

$$f_i(x, u)|_{x_i=0} = 0, \quad i = 1, 2, \ldots, n$$

where $f = [f_1, f_2, ..., f_n]^T$. This implies that each i-th scalar component of the solution vector $x(t) \in \mathbb{R}^n$ of the differential equation $\dot{x} = f(x, u)$ never changes signs, namely, for each $i = 1, 2, \ldots, n$,

$$x_i(0) \geq 0 \;\Rightarrow\; x_i(t) \geq 0, \; \forall t \in \mathbb{R}_+$$

holds. For such a positive system defined for initial conditions in the non-negative orthant, the $\mathbf{C}^1$ function $V(x)$ needs to be defined on only $\mathbb{R}^n_+$. Since $V(x) = |x|$ becomes eligible, Lemma 1 allows $\alpha \in \mathcal{O}(1)$ when one's attention is restricted to positive systems. Finally, it can be verified that all the results presented in [7] hold even for the interconnection of subsystems evolving on $\mathbb{R}^{n_i}_+$.

4.2 Necessary Conditions

Using Lemma 1, we can derive necessary conditions for the stability of the interconnected system Σ shown in Fig.1. The following addresses the existence of an integer $j \in \{1, 2\}$ satisfying (10).

Theorem 4 *Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$, and satisfy*

$$\alpha_i \in \mathcal{O}(>1), \quad \sigma_i, \sigma_{ri} \in \mathcal{O}(>0), \quad i = 1, 2 \tag{24}$$

Then, for the pair

$$S_i = \{\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri}) : \\ f_i(t, x_i, u_i, r_i) = f_i(0, x_i, u_i, r_i), \; \forall t \in \mathbb{R}_+\} \;, i = 1, 2 \tag{25}$$

and the pair

$$S_i = \{\Sigma_i \in \mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri}) : \\ f_i(t, x_i, u_i, r_i) = f_i(0, x_i, u_i, r_i), \; \forall t \in \mathbb{R}_+\} \;, i = 1, 2 \tag{26}$$

the following facts hold.

(i) The interconnected system Σ is 0-GAS for all $\Sigma_i \in S_i$, $i = 1, 2$, only if

$$\liminf_{s \to \infty} \alpha_i(s) \geq \lim_{s \to \infty} \sigma_i(s) \tag{27}$$

holds for at least one of $i = 1, 2$.

(ii) The interconnected system Σ is ISS with respect to input r and state x for all $\Sigma_i \in S_i$, $i = 1, 2$, only if

$$\liminf_{s \to \infty} \alpha_i(s) \geq \lim_{s \to \infty} \sigma_i(s) + \sup_{s \in \mathbb{R}_+} \sigma_{ri}(s) \tag{28}$$

holds for at least one of $i = 1, 2$.

The necessary condition (28) and (17) justify either of the two requirements in *(H1)* of Theorem 3. The use of the sets (25) and (26) illustrates that the necessity holds for sets of time-invariant systems which are narrower than $\mathcal{S}_i(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ and $\mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$, respectively. Note that (27) is also necessary for iISS of Σ since iISS implies 0-GAS. The property (28) indicates that Σ_i is ISS with respect to input (u_i, r_i) and state x_i if $\sigma_{ri} \in \mathcal{K}$. The property (27) implies that Σ_i is ISS with respect to input u_i and state x_i. It is worth noting that $\limsup_{s\to\infty} \sigma_{ri}(s) < \infty$ is not necessary for the iISS property of the interconnected system Σ even if $\liminf_{s\to\infty} \alpha_i(s) < \infty$. This fact can be understood naturally. In fact, a system is iISS if and only if it is 0-GAS and zero-output smoothly dissipative[2].

We can establish the necessity of the small-gain condition.

Theorem 5 *Let n_i be a positive integer for each $i = 1, 2$. Assume that functions $\alpha_i, \sigma_i, \sigma_{ri} : \mathbb{R}_+ \to \mathbb{R}_+$ are $\mathbf{C}^1$, and satisfy (24). Suppose*

$$\liminf_{s\to\infty} \alpha_2(s) = \infty \ \vee \ \begin{cases} \liminf_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \sigma_2(s) & \text{if } 2 \notin \mathbf{D} \\ \liminf_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) & \text{if } 2 \in \mathbf{D} \end{cases} \tag{29}$$

holds, where $\mathbf{D} := \{i \in \{1, 2\} : \sigma_{ri} \in \mathcal{K}_\infty\}$. Then, the following facts hold for the pairs S_1, S_2 defined in (25) and (26).

(i) The interconnected system Σ is 0-GAS for all $\Sigma_i \in S_i$, $i = 1, 2$ only if there exist $\tilde{\alpha}_i \in \mathcal{K}$, $i = 1, 2$, such that

$$\tilde{\alpha}_1^{-1} \circ \sigma_1 \circ \tilde{\alpha}_2^{-1} \circ \sigma_2(s) < s \quad \forall s \in (0, \infty) \tag{30}$$

$$\tilde{\alpha}_i(s) \leq \alpha_i(s), \quad \forall s \in \mathbb{R}_+ \tag{31}$$

(ii) The interconnected system Σ is ISS with respect to input r and state x for all $\Sigma_i \in S_i$, $i = 1, 2$ only if there exist

$$\omega_i \begin{cases} \in \mathcal{K}_\infty & \text{if } i \in \mathbf{D} \\ = 0 & \text{if } i \notin \mathbf{D} \end{cases} \tag{32}$$

and $\tilde{\alpha}_i \in \mathcal{K}$ for $i = 1, 2$ such that

$$\tilde{\alpha}_1^{-1} \circ (\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \tilde{\alpha}_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{33}$$

and (31) are satisfied.

Note that we can take $\tilde{\alpha}_i = \alpha_i$ if α_i is of class $\mathcal{K}$. Theorem 5 suggests that we can assume $\alpha_i \in \mathcal{K}$, $i = 1, 2$, without loss of generality in the stability analysis. The next lemma indicates that the assumption of $\alpha_i \in \mathcal{O}(> 1)$ and $\sigma_i \in \mathcal{O}(> 0)$ is reasonable.

Lemma 2 *For $n_i > 0$, the following holds.*

(i) If $\partial V_i/\partial x_i$ and $\partial V_i/\partial t$ are Hölder continuous of some order $a > 0$ and $b > 1$, respectively, in x_i at $x_i = 0$, then $\mathcal{S}_i(n_i, \alpha_i, \sigma_i) \neq \emptyset$ implies $\alpha_i \in \mathcal{O}(> 1)$.

(ii) For each $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$, there exists $\hat{\sigma}_i \in \mathcal{K}$ such that $\hat{\sigma}_i \in \mathcal{O}(> 0)$ and $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \hat{\sigma}_i) \subseteq \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ holds.

5 Sufficiency

In this section, the sufficiency of the stability criteria presented in Section 3 is derived. This section gives a pair of $\{\lambda_1, \lambda_2\}$ with which the composite Lyapunov function V_{cl} in (12) fulfills (14) and (16).

5.1 A common form of Lyapunov function

Consider the set of the quadruplets $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying

$$\alpha_1, \alpha_2, \sigma_1, \sigma_2 \in \mathcal{K}, \tag{34}$$

$$\lim_{s\to\infty} \alpha_2(s) \geq \lim_{s\to\infty} \sigma_2(s) \tag{35}$$

Define the following seven situations for $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$:

(M1) $\displaystyle\lim_{s\to\infty} \frac{\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\sigma_2(s)}{\alpha_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)} \leq 1 \quad \wedge \quad \lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s)$

(M2) $\displaystyle\lim_{s\to\infty} \frac{\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\sigma_2(s)}{\alpha_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)} < 1 \quad \wedge \quad \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$

(M3) $\displaystyle\lim_{s\to\infty} \frac{\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\sigma_2(s)}{\alpha_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)} = 1 \quad \wedge \quad \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$

(J1) $\displaystyle\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty$

(J2) $\displaystyle\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

(J3) $\displaystyle\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

(J4) $\displaystyle\exists k \in \{1,2\} \text{ s.t. } \{\lim_{s\to\infty} \alpha_k(s) < \infty \ \wedge \ \lim_{s\to\infty} \sigma_{3-k}(s) = \infty\}$

The pair of $\{\lambda_1, \lambda_2\}$ for the Lyapunov function V_{cl} can be constructed from the functions in the small-gain conditions (11), (15) and (18). The following lemma provides the functions to be used in $\{\lambda_1, \lambda_2\}$ directly.

Lemma 3 *Assume that*

$$\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1\circ\alpha_1^{-1}\circ\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\sigma_2(s) < s, \quad \forall s \in (0,\infty) \tag{36}$$

holds for a pair $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathcal{K}_\infty$ and a quadruplet $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying (34), (35) and (M1)∨(M2). Then, there exist $\hat{\alpha}_1, \hat{\sigma}_1 \in \mathcal{K}$, $\hat{\omega}_1$, $\hat{\omega}_2 \in \mathcal{P}_0$ and $\hat{\tau}_1, \hat{\tau}_2 \in \mathcal{K}_\infty$ such that

$$(\mathbf{Id}+\hat{\omega}_1)\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ(\mathbf{Id}+\hat{\omega}_2)\circ\sigma_2(s) \leq \hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s),\ \forall s\in\mathbb{R}_+ \tag{37}$$

$$\sigma_1(s) \leq \hat{\sigma}_1(s), \quad \hat{\alpha}_1(s) \leq \alpha_1(s),\ \forall s \in \mathbb{R}_+ \tag{38}$$

$$\lim_{s\to\infty} \hat{\sigma}_1(s) \geq \lim_{s\to\infty} \hat{\alpha}_1(s) \tag{39}$$

$$\lim_{s\to\infty} \alpha_2(s) = \infty \ \Rightarrow \ \hat{\alpha}_1 = \alpha_1 \tag{40}$$

$$\lim_{s\to\infty} \alpha_2(s) < \infty \ \Rightarrow \ \hat{\sigma}_1 = \sigma_1 \tag{41}$$

$$\lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) \ \Rightarrow \ \begin{cases} \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty}(\mathbf{Id}+\hat{\omega}_2)\circ\sigma_2(s) \\ \lim_{s\to\infty} \hat{\sigma}_1(s) > \lim_{s\to\infty} \hat{\alpha}_1(s) \\ \hat{\omega}_1, \hat{\omega}_2 \in \mathcal{K} \end{cases} \tag{42}$$

$$\lim_{s\to\infty}\sigma_2(s)=\lim_{s\to\infty}\alpha_2(s)\ \Rightarrow\ \begin{cases}\lim_{s\to\infty}\alpha_2(s)\geq\lim_{s\to\infty}(\mathbf{Id}+\hat{\omega}_2)\circ\sigma_2(s)\\ \lim_{s\to\infty}(\mathbf{Id}+\hat{\omega}_1)\circ\hat{\sigma}_1(s)=\lim_{s\to\infty}\hat{\alpha}_1(s)\\ \hat{\omega}_1\circ\hat{\sigma}_1(s)>0,\ \hat{\omega}_2\circ\sigma_2(s)>0,\ \forall s\in(0,\infty)\end{cases} \tag{43}$$

$$\hat{\tau}_i=\mathbf{Id}+\hat{\omega}_i,\quad i=1,2 \tag{44}$$

Furthermore, the claim can be fulfilled by $\hat{\omega}_1$, $\hat{\omega}_2\in\mathcal{K}_\infty$ if there exist $\omega_1,\omega_2\in\mathcal{K}_\infty$ such that

$$\underline{\alpha}_1^{-1}\circ\bar{\alpha}_1^{-1}\circ\alpha_1\circ(\mathbf{Id}+\omega_1)\circ\sigma_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ(\mathbf{Id}+\omega_2)\circ\sigma_2(s)\leq s,\quad\forall s\in\mathbb{R}_+ \tag{45}$$

is satisfied under the assumption of

$$\lim_{s\to\infty}\alpha_2(s)=\infty\quad\vee\quad\lim_{s\to\infty}\alpha_2(s)>\lim_{s\to\infty}\sigma_2(s) \tag{46}$$

Using the functions given in Lemma 3 and $L:=\lim_{s\to\infty}\hat{\sigma}_1(s)$, we define continuous functions $\lambda_1,\lambda_2:\mathbb{R}_+\to\mathbb{R}_+$ as

$$\begin{aligned}\lambda_1(s):=&\big[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1\\ &\circ\bar{\alpha}_1^{-1}(s)\big]\big[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\big]\big[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\big]\end{aligned} \tag{47}$$

$$\lambda_2(s):=\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\big[\nu\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\big]\big[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}(s)\big] \tag{48}$$

where δ_i and τ_1 are any class $\mathcal{K}_\infty$ functions satisfying

$$\mathbf{Id}-\delta_i\in\mathcal{K}_\infty,\quad i=1,2 \tag{49}$$

$$\tau_1=\mathbf{Id}+k\hat{\omega}_1 \tag{50}$$

for some $k\in(0,1)$, and $\nu,\psi:(0,L)\to\mathbb{R}_+$ are any continuous functions which satisfy

$$0<\nu(s)<\infty,\ 0<\psi(s)<\infty,\quad\forall s\in(0,L) \tag{51}$$

and fulfill

$$\big[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2(s)\big]\big[\nu\circ\hat{\sigma}_1(s)\big]\big[\psi\circ\hat{\sigma}_1(s)\big]\quad:\text{ non-decreasing} \tag{52}$$

$$\hat{\sigma}_1(s)\big[\nu\circ\hat{\sigma}_1(s)\big]\big[\psi\circ\hat{\sigma}_1(s)\big]\quad:\text{ non-decreasing} \tag{53}$$

$$\big[\delta_2\circ\hat{\omega}_2\circ\hat{\tau}_2^{-1}\circ\alpha_2\circ\bar{\alpha}_2^{-1}\circ\underline{\alpha}_2\circ\hat{\sigma}_1^{-1}\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\big]\big[\nu\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}(s)\big]\ :\text{ non-decreasing} \tag{54}$$

$$\begin{aligned}&\big[\psi\circ\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\hat{\tau}_2\circ\sigma_2(s)\big]\big[\hat{\sigma}_1\circ\underline{\alpha}_2^{-1}\circ\bar{\alpha}_2\circ\alpha_2^{-1}\circ\hat{\tau}_2\circ\sigma_2(s)\big]\sigma_2(s)\\ &\quad\leq\big[\psi\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\big]\big[\delta_2\circ\hat{\omega}_2\circ\sigma_2(s)\big]\big[\delta_1\circ k\hat{\omega}_1\circ\tau_1^{-1}\circ\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\big]\end{aligned} \tag{55}$$

for all $s\in\mathbb{R}_+$. Note that $\tau_1\in\mathcal{K}_\infty$ holds since $s+k\hat{\omega}_1(s)=k(s+\hat{\omega}_1(s))+(1-k)s$ and $\hat{\tau}_1\in\mathcal{K}_\infty$.

The following demonstrates that the pair of $\{\lambda_1,\lambda_2\}$ in (47) and (48) yields a Lyapunov function V_{cl} establishing the 0-GAS, iISS and ISS of the interconnected system Σ under appropriate small-gain conditions.

Theorem 6 *Consider $\sigma_{r1},\sigma_{r2}\in\mathcal{P}_0$, a quadruplet $(\alpha_1,\alpha_2,\sigma_1,\sigma_2)$ satisfying (34) and (35), and $V_i:(t,x_i)\in\mathbb{R}_+\times\mathbb{R}^{n_i}\to\mathbb{R}_+$, $i=1,2$, satisfying (4) for some $\underline{\alpha}_i$, $\bar{\alpha}_i\in\mathcal{K}_\infty$. Then, we have the following.*

(i) Suppose that $\sigma_{r1}(s)\equiv 0$, $\sigma_{r2}(s)\equiv 0$ and (M1) $\vee$ (M2) hold. If (36) is satisfied, the functions (47) and (48) satisfy

$$\sum_{i=1}^{2}\lambda_i(V_i(t,x_i))\{-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|)\}\leq\sum_{i=1}^{2}-\alpha_{cl,i}(|x_i|)+\sigma_{cl,i}(|r_i|),$$
$$\forall x_1\in\mathbb{R}^{n_1},x_2\in\mathbb{R}^{n_2},r_1\in\mathbb{R}^{m_1},r_2\in\mathbb{R}^{m_2},t\in\mathbb{R}_+ \tag{56}$$

for some $\alpha_{cl,1},\alpha_{cl,2}\in\mathcal{P}$ and $\sigma_{cl,1}(s)=\sigma_{cl,2}(s)\equiv 0$.

(ii) Suppose that (J1) $\vee$ (J2) $\vee$ (J3) and

$$L < \infty \Rightarrow \lim_{s\to L} \nu(s) < \infty, \ \lim_{s\to L} \psi(s) < \infty \tag{57}$$

hold. If there exist $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$ such that (45) is satisfied, the functions (47) and (48) with $\hat{\omega}_1$, $\hat{\omega}_2 \in \mathcal{K}_\infty$ satisfy (56) for some $\alpha_{cl,1}, \alpha_{cl,2} \in \mathcal{K}$ and some $\sigma_{cl,1}, \sigma_{cl,2} \in \mathcal{P}_0$ fulfilling

$$\alpha_1, \alpha_2 \in \mathcal{K}_\infty \ \Rightarrow \ \alpha_{cl,1}, \alpha_{cl,2} \in \mathcal{K}_\infty \tag{58}$$

$$\sigma_{r,i}(s) \equiv 0 \ \Rightarrow \ \sigma_{cl,i}(s) \equiv 0 \tag{59}$$

There always exist functions ν and ψ fulfilling (51), (52), (53), (54), (55) and (57). The existence and the construction are addressed in Subsection 5.2. The task of finding a pair $\{\lambda_1, \lambda_2\}$ which solves (56) is referred to as a state-dependent scaling problem in [6]. In Theorem 6 *(ii)*, the property (56) resulting in (16) yields the iISS of the interconnected system Σ. Theorem 6 *(i)* demonstrates that the amplification factors ω_1, ω_2 in the small-gain condition (45) can be replaced by a strict inequality sign as far as 0-GAS is concerned. Note that the existence of $\omega_1, \omega_2 \in \mathcal{K}_\infty$ achieving (45) implies not only (36), but also *(M1)* $\vee$ *(M2)*.

In order to understand the idea of the assumption *(M1)* $\vee$ *(M2)* for 0-GAS and the assumption *(J1)* $\vee$ *(J2)* $\vee$ *(J3)* for iISS, the following lemma is useful.

Lemma 4 *Given $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2$, the following propositions hold true for each quadruplet $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying (34), (35):*

$$\{\{ \text{ (M1)} \vee \text{(M2) } \} \ \Leftrightarrow \ \neg\text{(M3) } \} \ \textit{if } (36)$$

$$\{ \text{ (J1)} \vee \text{(J2)} \vee \text{(J3) } \} \ \Leftrightarrow \ \neg\text{(J4)}$$

The case of *(J4)* allows $\infty = \limsup_{s\to\infty} \sigma_{rk}(s) > \lim_{s\to\infty} \alpha_k(s)$. Notice that $\infty = \limsup_{s\to\infty} \sigma_{rk}(s)$ implies the unbounded influence of r_k on Σ_k. In this situation, *(J4)* implies that the underdamped state x_k of Σ_k affects Σ_{3-k} through the unbounded function σ_{3-k}. If the influence of r_k is small enough, we can still obtain iISS of Σ in the case of *(J4)*. In fact, there exists $\epsilon > 0$ for which we can obtain $\alpha_{cl,1}, \alpha_{cl,2} \in \mathcal{K}$ and $\sigma_{cl,1}, \sigma_{cl,2} \in \mathcal{P}_0$ if (45) holds with $\omega_1, \omega_2 \in \mathcal{K}_\infty$ and $\lim_{s\to\infty} \sigma_{rk}(s) \le \epsilon$.

5.2 Construction of ψ

Once a function ψ satisfying (51), (55) and (57), is given, we can always select a function ν required in Theorem 6 straightforwardly. Such a desired function ψ is constructed as follows: First, define

$$Q(t) = \begin{cases} \dfrac{1}{m(t)-t}\left(\dfrac{\hat{d}(t)}{\hat{b}(t)}-1\right) & , \ t \in (0, S) \\ \dfrac{1}{m(S)-S}\left(\limsup\limits_{s\to S} \dfrac{\hat{d}(s)}{\hat{b}(s)}-1\right) & , \ t \in [S, R) \end{cases} \tag{60}$$

$$\hat{b}(s) = b \circ \eta^{-1}(s), \ \hat{d}(s) = d \circ \eta^{-1}(s) \tag{61}$$

$$m(s) = \tau_1^{-1} \circ \alpha \circ \eta^{-1}(s) \tag{62}$$

$$S = \lim_{s\to\infty} \eta(s), \quad R = \lim_{s\to\infty} \tau_1^{-1} \circ \alpha(s)$$

for a real number $k \in (0, 1)$, where

$$\alpha(s) = \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)$$

$$
\begin{aligned}
b(s) &= \left[\delta_1 \circ k\hat{\omega}_1 \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] [\delta_2 \circ \hat{\omega}_2 \circ \sigma_2(s)] \\
d(s) &= \left[\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \hat{\tau}_2 \circ \sigma_2(s)\right] \sigma_2(s) \\
\eta(s) &= \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \hat{\tau}_2 \circ \sigma_2(s)
\end{aligned}
$$

We can always pick a non-decreasing function $\bar{Q} : (0, R) \to \mathbb{R}$ satisfying

$$\bar{Q}(s) \geq \max\{Q(s), 0\}, \quad \forall s \in (0, R) \tag{63}$$

In the case of $\limsup_{s\to 0} Q(s) = \infty$, let $\bar{Q}$ be of the form

$$\bar{Q}(s) = \frac{1}{\int_0^s \xi(r)dr}, \quad s \in (0, R) \tag{64}$$

and we can pick a function $\xi : (0, R) \to \mathbb{R}$ satisfying

$$\xi(s) \leq 1, \quad 0 < \int_0^s \xi(r)dr \leq \frac{1}{\max\{Q(s), 0\}}, \quad \forall s \in (0, R) \tag{65}$$

Then, for arbitrary $C > 0$ and $T \in (0, R)$, define ψ by

$$\psi(s) = Ce^{\int_T^s \bar{Q}(t)dt}, \quad s \in (0, R) \tag{66}$$

$$\psi(s) = \lim_{r \nearrow R} \psi(r), \quad s \in [R, \infty) \tag{67}$$

Note that (37) implies $S \leq R$.

Lemma 5 *Assume that (36) holds for a pair $\underline{\alpha}_i$, $\bar{\alpha}_i \in \mathcal{K}_\infty$ and a quadruplet $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying (34), (35) and (M1) $\vee$ (M2). Then, the function ψ defined by (66) and (67) is a continuous mapping from $(0, L)$ to $\mathbb{R}_+$ and satisfies*

$$0 < \psi(s) < \infty, \quad \forall s \in (0, L) \tag{68}$$

$$[\psi \circ \eta(s)]\, d(s) \leq \left[\psi \circ \tau_1^{-1} \circ \alpha(s)\right] b(s), \ \forall s \in (0, \infty) \tag{69}$$

Moreover, the function ψ is a continuous mapping from $(0, \infty)$ to $\mathbb{R}_+$ and satisfies

$$0 < \psi(s) < \infty, \quad \forall s \in (0, \infty) \tag{70}$$

if

$$\lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) \tag{71}$$

holds additionally.

The inequality (69) corresponds to (55). The property (68) ensures (51) in terms of ψ. As for (57) in *(ii)* of Theorem 6, the following lemma assures that (70) guarantees (57) in terms of ψ.

Lemma 6 *Suppose that the quadruplet $(\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ satisfying (34) and (35) fulfills (J1) $\vee$ (J2) $\vee$ (J3). If there exist $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$ such that (45) is satisfied, then $L < \infty$ implies (71)*

It is stressed that, when supply rates for Σ_i, $i = 1, 2$ are given by

$$-\alpha_i(V_i(t, x_i)) + \sigma_i(V_{3-i}(t, x_{3-i})) + \sigma_{ri}(|r_i|)$$

instead of $-\alpha_i(|x_i|) + \sigma_i(|u_i|) + \sigma_{ri}(|r_i|)$, all developments in this Section 5 remain valid by replacing $\underline{\alpha}_i$ and $\bar{\alpha}_i$ with the identity map, and replacing $|x_i|$ with V_i.

If $Q(s) \leq 0$ holds for all $s \in (0, R)$, the choice $\bar{Q}(s) = 0$ fulfills (63), which yields $\psi(s) = C > 0$. If there exists $K \in (-\infty, 0) \cup [1, \infty)$ such that

$$\sup_{t \in (0,R)} t\, Q(t) \leq K \tag{72}$$

holds, the choice $\bar{Q}(s) = K/s$ yields $\psi(s) = Cs^K$. In the case of uniform contraction where ω_1 and ω_2 are linear, there exist a sufficiently large $K < \infty$ such that (72) holds. When we take $\psi(s) = Cs^K$, the functions λ_1 and λ_2 reduce to the ones used in earlier results [6] dealing with uniformly contractive loop gain for ISS systems.

6 Proofs

6.1 Proof of Lemma 1

By assumption, there exist $N > 1$ and $J > 0$ such that $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ are written in the form of

$$\alpha(|x|) = \hat{\alpha}(|x|)|x|^N, \quad \sigma(|u|) = \hat{\sigma}(|u|)|u|^J$$

with some functions $\hat{\alpha}(s)$ and $\hat{\sigma}(s)$ which are continuous on $[0, \infty)$. The class $\mathbf{C}^1$ property of α and σ also implies that $\hat{\alpha}$ and $\hat{\sigma}$ are $\mathbf{C}^1$ in $(0, \infty)$. Pick a real number $Q \geq 1$ so that

$$\frac{1}{N} + \frac{1}{JQ} < 1$$

is satisfied. Let $\epsilon \in (0, \bar{\epsilon}]$. In the case of $(1/N) + (1/J) < 1$, let $Q = 1$ and $\epsilon = 0$. Define $\theta \in \mathcal{K}_\infty$ as

$$\theta(s) = \begin{cases} \sigma(\epsilon)\,(s/\sigma(\epsilon))^Q & , \text{ for } s \in [0, \sigma(\epsilon)) \\ s & , \text{ for } s \in [\sigma(\epsilon), \infty) \end{cases}$$

The class $\mathcal{K}$ function $\tilde{\sigma}$ given by $\tilde{\sigma}(s) = \theta \circ \sigma(s)$ satisfies

$$\begin{array}{ll} \tilde{\sigma}(s) = \sigma(s) = 0 & , \ s = 0 \\ \tilde{\sigma}(s) < \sigma(s) & , \ \forall s \in (0, \epsilon) \\ \tilde{\sigma}(s) = \sigma(s) & , \ \forall s \in [\epsilon, \infty) \end{array} \tag{73}$$

Define $p > 1$ by

$$\frac{1}{p} = 1 - \frac{1}{JQ} \tag{74}$$

Let $q = JQ$ so that $(1/p) + (1/q) = 1$ holds. Define

$$V(x) = \underline{\alpha}(|x|) = \bar{\alpha}(|x|) = |x|^{N/p} \tag{75}$$

$$f(x, u) = f_A(x) + f_B(x, u) \tag{76}$$

$$f_A = \frac{-\mu p}{N}\hat{\alpha}(|x|)|x|^{N/q}x, \quad \mu = \frac{q}{p}(1 + \delta) + 1 \tag{77}$$

$$f_B = \frac{p}{N}(q(1 + \delta)\hat{\alpha}(|x|))^{1/p}(q\tilde{\sigma}(|u|))^{1/q}x \tag{78}$$

Then, we have

$$\begin{aligned} \frac{\partial V}{\partial x} f =& \frac{N}{p}|x|^{\frac{N}{p}-2}x^T f \\ =& -\mu\hat{\alpha}(|x|)|x|^N + \left(p\frac{q}{p}(1 + \delta)\hat{\alpha}(|x|)|x|^N\right)^{1/p}(q\tilde{\sigma}(|u|))^{1/q} \end{aligned}$$

Applying Young's inequality to the right-hand side, we obtain

$$\frac{\partial V}{\partial x}f \leq -\left(\mu - \frac{q}{p}(1+\delta)\right)\hat{\alpha}(|x|)|x|^N + \tilde{\sigma}(|u|) \leq -\alpha(|x|) + \sigma(|u|)$$

Since $q(1+\delta) - \mu = \delta$ holds, we arrive at

$$(1+\delta)\alpha(|x|) = \tilde{\sigma}(|u|) \ \Rightarrow\ \frac{\partial V}{\partial x}f = -\mu\alpha(|x|) + q\tilde{\sigma}(|u|) = (q(1+\delta)-\mu)\alpha(|x|) = \delta\alpha(|x|)$$
$$(1+\delta)\alpha(|x|) < \tilde{\sigma}(|u|) \ \Rightarrow\ \frac{\partial V}{\partial x}f > (q(1+\delta)-\mu)\alpha(|x|) = \delta\alpha(|x|)$$

Thus, we have (23) by virtue of (73). The choice (74) of p implies $N/p > 1$, so that V given by (75) is $\mathbf{C}^1$. The function f_A is Lipschitz at each point in $\mathbb{R}^n$ due to $N/q \geq 0$ and the class $\mathbf{C}^1$ property of $\hat{\alpha}$ on $(0,\infty)$. The function f_B is also locally Lipschitz in x on $\mathbb{R}^n$ since $\hat{\alpha}(s)^{1/p}$ is $\mathbf{C}^1$ on $(0,\infty)$ and bounded on $\mathbb{R}_+$. To verify the local Lipschitzness with respect to $u \in \mathbb{R}^m$, we first obtain $JQ = q$ from (74). Next,

$$\tilde{\sigma}(s)^{1/q} = \sigma(\epsilon)^{1/q}(\hat{\sigma}(s)/\sigma(\epsilon))^{Q/q}|s| \ ,\ \forall s \in [0,\epsilon]$$

follows from $\sigma \in \mathcal{O}(>0)$. This function $\tilde{\sigma}(s)^{1/q}$ is continuously differentiable in the interval $(0,\epsilon]$ since $\hat{\sigma}(s)^{Q/q}$ is class $\mathbf{C}^1$. The function $\tilde{\sigma}(s)^{1/q}$ is also Lipschitz at zero since $Q/q > 0$. The identity

$$\tilde{\sigma}(s)^{1/q} = \hat{\sigma}(s)^{1/q}|s|^{J/q} \ ,\ \forall s \in [\epsilon,\infty)$$

together with $q > 1$ and $J > 0$ guarantees that $\tilde{\sigma}(s)^{1/q}$ is $\mathbf{C}^1$ at each $s \in [\epsilon,\infty)$ due to the continuous differentiability of $\hat{\sigma}(s)^{1/q}$. Hence, the function f_B is locally Lipschitz at all $u \in \mathbb{R}^m$.

6.2 Proof of Theorem 4

We first deal with S_1 and S_2 given by (25) and we begin with proving *(ii)*.
(ii) Suppose that (28) is not satisfied for each $i = 1,2$. This assumption is equivalent to

$$\liminf_{s\to\infty}\alpha_i(s) < \infty \ \wedge\ \liminf_{s\to\infty}\alpha_i(s) < \lim_{s\to\infty}\sigma_i(s) + \sup_{s\in\mathbb{R}_+}\sigma_{ri}(s)$$

for $i = 1,2$. Due to $\sigma_i \in \mathcal{K}$ and $\sigma_{ri} \in \mathcal{P}_0$, there exist $v_i > 0$, $w_i > 0$ and $\delta_i > 0$ for $i = 1,2$ such that

$$(1+\delta_1)\alpha_1(s) < \sigma_1(w_2) + \sigma_{r1}(v_1), \quad \forall s \in \{h_{11}, h_{12}, ...\}$$
$$(1+\delta_2)\alpha_2(s) < \sigma_2(w_1) + \sigma_{r2}(v_2), \quad \forall s \in \{h_{21}, h_{22}, ...\}$$

hold for some increasing sequences $h_{1n} \to \infty$ and $h_{2n} \to \infty$ satisfying h_{11}, $h_{21} \geq 0$, respectively. For all integers j and k satisfying $h_{1j} \geq w_1$ and $h_{2k} \geq w_2$, the properties

$$|x_1| = h_{1j},\ |x_2| \geq h_{2k} \Rightarrow (1+\delta_1)\alpha_1(|x_1|) < \sigma_1(|x_2|) + \sigma_{r1}(|r_1|)$$
$$|x_1| \geq h_{1j},\ |x_2| = h_{2k} \Rightarrow (1+\delta_2)\alpha_2(|x_2|) < \sigma_2(|x_1|) + \sigma_{r2}(|r_2|)$$

hold as long as r_1 and r_2 satisfy $|r_1| \geq v_1, |r_2| \geq v_2$. Lemma 1 with replacement of $\sigma(|u|)$ with $\sigma_i(|u_i|) + \sigma_{ri}(|r_i|)$ guarantees the existence of $f_1(x_1,u_1,r_1)$, $f_2(x_2,u_2,r_2)$: $\mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{k_i} \to \mathbb{R}$, $\mathbf{C}^1$ functions V_1, V_2: $\mathbb{R}^{n_i} \to \mathbb{R}$ and $\underline{\alpha}_1$, $\bar{\alpha}_1$, $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $\Sigma_i \in \mathcal{S}_i(n_i,\alpha_i,\sigma_i,\sigma_{ri})$ and

$$\underline{\alpha}_i(|x_i|) = V_i(x_i) = \bar{\alpha}_i(|x_i|) \tag{79}$$
$$(1+\delta_i)\alpha_i(|x_i|) < \sigma_i(|x_{3-i}|) + \sigma_{ri}(|r_i|) \ \Rightarrow\ \frac{\partial V_i}{\partial x_i}f_i > \delta_i\alpha_i(|x_i|)$$

hold for $i = 1, 2$. These systems Σ_1 and Σ_2 satisfy

$$|x_1| = h_{1j},\ |x_2| \geq h_{2k} \Rightarrow \frac{\partial V_1}{\partial x_1} f_1 > \delta_i \alpha_1(|x_1|) \tag{80}$$

$$|x_1| \geq h_{1j},\ |x_2| = h_{2k} \Rightarrow \frac{\partial V_2}{\partial x_2} f_2 > \delta_i \alpha_2(|x_2|) \tag{81}$$

for all $|r_1| \geq v_1, |r_2| \geq v_2$. Define

$$\mathbf{U}(l_1, l_2) = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : V_i(x_i) \geq \bar{\alpha}_i(l_i), i = 1, 2\} \tag{82}$$

Due to (79), the pair of (80) and (81) implies that trajectories starting from $(x_1(0), x_2(0)) \in \mathbf{U}(h_{1j}, h_{2k})$ stay in $\mathbf{U}(h_{1j}, h_{2k})$ forever if $|r_1| = v_1$ and $|r_2| = v_2$ hold for all $t \in \mathbb{R}_+$. The trajectories remain in $\mathbf{U}(h_{1j}, h_{2k})$ for the same r_1 and r_2 however large h_{1j} and h_{2k} are. This invariance property implies that the interconnected system does not have finite gain in terms of ISS[10].
(i) Suppose that (27) does not hold for $i = 1, 2$. There exist $w_i, \delta_i > 0$ for $i = 1, 2$ such that

$$(1 + \delta_1)\alpha_1(s) < \sigma_1(w_2), \quad \forall s \in \{h_{11}, h_{12}, ...\}$$
$$(1 + \delta_2)\alpha_2(s) < \sigma_2(w_1), \quad \forall s \in \{h_{21}, h_{22}, ...\}$$

are satisfied for some increasing sequences $h_{1n} \to \infty$ and $h_{2n} \to \infty$ satisfying $h_{11},\ h_{21} \geq 0$, respectively. Lemma 1 guarantees the existence of $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$, $i = 1, 2$, such that (80) and (81) hold. Trajectories starting from $\mathbf{U}(h_{1j}, h_{2k})$ remain in $\mathbf{U}(h_{1j}, h_{2k})$ for arbitrary h_{1j} and h_{2k}. Therefore, the interconnection is not 0-GAS.

In the case of (26), by assumption there exist $M_i > 1$ and $L_i > 0$ such that $\alpha_i \in \mathcal{O}(M_i)$ and $\sigma_i \in \mathcal{O}(L_i)$ hold for $i = 1, 2$. Define $\breve{\alpha}_i = \alpha_i(s^{K_i})$ and $\breve{\sigma}_i = \sigma_i(s^{K_{3-i}})$ for some $K_i > 1$, $i = 1, 2$. Then, there exist continuous functions $\hat{\alpha}_i, \hat{\sigma}_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\breve{\alpha}_i(|x_i|) = \hat{\alpha}_i(|x_i|)|x_i|^{N_i},\ N_i = K_i M_i > 1$$
$$\breve{\sigma}_i(|x_{3-i}|) = \hat{\sigma}_i(|x_{3-i}|)|x_{3-i}|^{J_i},\ J_i = K_{3-i} L_i > 0$$

hold for $i = 1, 2$. Since α_i and σ_i are $\mathbf{C}^1$, the functions $\hat{\alpha}_i$ and $\hat{\sigma}_i$ are also $\mathbf{C}^1$ on $(0, \infty)$. Lemma 1 yields a Lipschitz continuous time-invariant system $\Sigma_i \in \mathcal{S}_i(n_i, \breve{\alpha}_i, \breve{\sigma}_i, \sigma_{ri})$ with $V_i(x_i) = |x_i|^{K_i}$ for each $i = 1, 2$. The property $\mathcal{S}_i(n_i, \breve{\alpha}_i, \breve{\sigma}_i, \sigma_{ri}) = \mathcal{S}_{Vi}(n_i, \alpha_i, \sigma_i, \sigma_{ri})$ completes the proof.

6.3 Proof of Theorem 5

The following deals with (25). The technique to deal with (26) is the same as Theorem 4.
(i): Assume $\alpha_i \in \mathcal{K}$ temporarily and let $\tilde{\alpha}_i = \alpha_i$, $i = 1, 2$. Suppose that there exists $l_1 \in (0, \infty)$ such that

$$\sigma_1 \circ \alpha_2^{-1} \circ \sigma_2(l_1) \geq \alpha_1(l_1) \tag{83}$$

holds. Pick $l_2 \in (0, \infty)$ so that $\alpha_2^{-1} \circ \sigma_2(l_1) \geq l_2 \geq \sigma_1^{-1} \circ \alpha_1(l_1)$ is satisfied. Using $\alpha_2, \sigma_1 \in \mathcal{K}$, we obtain $\alpha_2(l_2) \leq \sigma_2(l_1)$ and $\alpha_1(l_1) \leq \sigma_1(l_2)$. Suppose $|r_1(t)| = |r_2(t)| = 0$ for all $t \in \mathbb{R}_+$. Lemma 1 guarantees the existence of two time-invariant systems $\Sigma_1 \in \mathcal{S}_1(n_1, \alpha_1, \sigma_1, \sigma_{r1})$ and $\Sigma_2 \in \mathcal{S}_2(n_2, \alpha_2, \sigma_2, \sigma_{r2})$ achieving (79) and

$$\alpha_i(|x_i|) \leq \sigma_i(|x_{3-i}|) \Rightarrow \frac{\partial V_i}{\partial x_i} f_i \geq 0$$

for $i = 1, 2$. This leads to the following:

$$|x_1| = l_1,\ |x_2| \geq l_2 \Rightarrow \frac{\partial V_1}{\partial x_1} f_1 \geq 0 \tag{84}$$

$$|x_1| \geq l_1,\ |x_2| = l_2 \Rightarrow \frac{\partial V_2}{\partial x_2} f_2 \geq 0 \tag{85}$$

Define $\mathbf{U}(l_1, l_2)$ as in (82). Due to (79), the property characterized by (84) and (85) implies that trajectories starting from $x(0) \in \mathbf{U}(l_1, l_2)$ remain in $\mathbf{U}(l_1, l_2)$. This invariance contradicts the 0-GAS. Next, consider the case of $\alpha_i \in \mathcal{P} \setminus \mathcal{K}$. Suppose that

$$\begin{aligned} &\alpha_{i.1},\ \alpha_{i.2} \in \mathcal{K}, \quad i = 1, 2 \\ &\alpha_{i.1}(s) \geq \alpha_{i.2}(s),\ \forall s \in \mathbb{R}_+, \quad i = 1, 2 \end{aligned}$$

hold. Then, if there exists $l_1 \in (0, \infty)$ such that

$$\sigma_1 \circ \alpha_{2.k}^{-1} \circ \sigma_2(l_1) \geq \alpha_{1.k}(l_1) \tag{86}$$

holds for $k = 1$, the same l_1 also satisfies (86) for $k = 2$. This property implies that the negation of (30) implies the existence of $l_1 \in (0, \infty)$ satisfying

$$\sigma_1 \circ \tilde{\alpha}_2^{-1} \circ \sigma_2(l_1) \geq \tilde{\alpha}_1(l_1) \tag{87}$$

for all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31). Define $l_2 \in (0, \infty)$ satisfying $\tilde{\alpha}_2^{-1} \circ \sigma_2(l_1) \geq l_2 \geq \sigma_1^{-1} \circ \tilde{\alpha}_1(l_1)$. If

$$\tilde{\alpha}_i(l_i) = \alpha_i(l_i), \quad i = 1, 2 \tag{88}$$

holds, the argument given above for $\alpha_i \in \mathcal{K}$, $i = 1, 2$ leads to the existence of a pair of systems whose interconnection is not 0-GAS. Suppose that there exists $l_1 \in (0, \infty)$ such that

$$\tilde{\alpha}_1(l_1) < \alpha_1(l_1) \tag{89}$$

and (87) hold for "all" $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31). Then, $\tilde{\alpha}_1(l_1) \leq \sigma_1(l_2)$ holds with any $\tilde{\alpha}_2 \in \mathcal{K}$ fulfilling (31), which implies that there exists $\bar{l}_1 \in (l_1, \infty)$ such that $\alpha_1(\bar{l}_1) \leq \sigma_1(l_2)$ holds. If (87) and

$$\tilde{\alpha}_2(l_2) < \alpha_2(l_2) \tag{90}$$

hold for all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31), there exists $\bar{l}_2 \in (l_2, \infty)$ such that $\alpha_2(\bar{l}_2) \leq \sigma_2(l_1)$ holds. If (89) and (90) are satisfied simultaneously, we have $\alpha_1(\bar{l}_1) \leq \sigma_1(\bar{l}_2)$ and $\alpha_2(\bar{l}_2) \leq \sigma_2(\bar{l}_1)$. Hence, the rest of the proof is the same as the case of $\alpha_i \in \mathcal{K}$, $i = 1, 2$.

(ii): Consider the case of $\sigma_{r1}, \sigma_{r2} \in \mathcal{K}_\infty$. In order to prove the claim by contradiction, assume that (33) is violated for all pairs of $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$. First, suppose that there exists $l_1 \in (0, \infty)$ such that (83) holds with all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31). Then, the claim *(i)* proves that $x = 0$ is not guaranteed to be GAS, which implies that the interconnection is not ISS. Next, we suppose that there are no $l_1 \in (0, \infty)$ and no $\tilde{\alpha}_i \in \mathcal{K}$ satisfying (83) and (31). Since all pair of $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2$ violate (33), there exist continuous functions $\omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a non-empty set $\mathbf{Y}$ such that

$$(\mathbf{Id} + \omega_1) \circ \sigma_1 \circ \tilde{\alpha}_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(s) = \tilde{\alpha}_1(s), \qquad \forall s \in \mathbb{R}_+ \tag{91}$$

$$\mathbf{Id} + \omega_1, \mathbf{Id} + \omega_2 \in \mathcal{K}_\infty$$

$$\lim_{s \to \infty} \omega_j(s) < \infty, \quad \forall j \in \mathbf{Y} \subset \{1, 2\} \tag{92}$$

are satisfied for some $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31). The property (92) yields

$$\lim_{s\to\infty} \omega_j \circ \sigma_j(s) < \infty, \tag{93}$$

Since σ_{r1} and σ_{r2} are of class $\mathcal{K}_\infty$, there exists $R_j \in (0, \infty)$ such that

$$\lim_{s\to\infty} \omega_j \circ \sigma_j(s) \le \sigma_{rj}(|r_j|),\ \forall s \in \mathbb{R}_+,\ \forall |r_j| \ge R_j \tag{94}$$

hold for all $j \in \mathbf{Y}$. Let l_1 be a real number in $(0, \infty)$, which is now given arbitrarily in contrast to the *(i)* case. Define $l_2(l_1) = \tilde{\alpha}_2^{-1} \circ (\mathbf{Id} + \omega_2) \circ \sigma_2(l_1)$ which is of class $\mathcal{K}$. Due to (91), we have $\tilde{\alpha}_2(l_2(l_1)) = (\mathbf{Id}+\omega_2) \circ \sigma_2(l_1)$ and $\tilde{\alpha}_1(l_1) = (\mathbf{Id}+\omega_1) \circ \sigma_1(l_2(l_1))$. By replacing σ with $\sigma_i + \sigma_{ri}$ in Lemma 1 , we obtain $\Sigma_i \in \mathcal{S}_i(n_i, \tilde{\alpha}_i, \sigma_i, \sigma_{ri})$, such that (79) and

$$\tilde{\alpha}_i(|x_i|) \le \sigma_i(|x_{3-i}|) + \sigma_{ri}(|r_i|) \Rightarrow \frac{\partial V_i}{\partial x_i} f_i \ge 0$$

hold for $i = 1, 2$. This leads to (84) and (85) for all $|r_j| \ge R_j$, $j \in \mathbf{Y}$ and $r_k = 0$, $k \in \{1, 2\} \setminus \mathbf{Y}$ since we have (94). Define $\mathbf{U}(l_1, l_2)$ as in (82). The inequalities (84) and (85) imply that trajectories starting from $\mathbf{U}(l_1, l_2)$ remain in $\mathbf{U}(l_1, l_2)$ as long as $|r_j(t)| \ge R_j$ and $r_k(t) = 0$ hold. Recall that l_1 is arbitrary in $(0, \infty)$, and independent of R_j. The trajectories for the fixed input $|r_j(t)| = R_j < \infty$ does not leave $\mathbf{U}(l_1, l_2)$ no matter how large l_1 is. This violates the ISS property[10]. Therefore, the interconnected system Σ is not ISS when (33) is violated for all pair of $\omega_i \in \mathcal{K}_\infty$ and all $\tilde{\alpha}_i \in \mathcal{K}$ fulfilling (31), $i = 1, 2$. Note that $\alpha_i \in \mathcal{P} \setminus \mathcal{K}$ can be handled as in *(i)*. In the case of $\sigma_{ri} \notin \mathcal{K}_\infty$, use $\omega_i(s) \equiv 0$ and $r_i(t) \equiv 0$. The property $\tilde{\alpha}_i(l_i) \le \sigma_i(l_{3-i}) + \sigma_{ri}(|r_i|)$ is replaced by $\tilde{\alpha}_i(l_i) \le \sigma_i(l_{3-i})$.

6.4 Proof of Lemma 2

(i) The Lipschitzness of $f_i(t, x_i, 0)$ in x_i at $x_i = 0$ uniformly in t and the Hölder's condition on V_i imply that $\partial V_i/\partial t + \partial V_i/\partial x_i \cdot f_i(t, x_i, 0) \le -\hat{\alpha}_i(|x_i|)$ requires $\hat{\alpha}_i \in \mathcal{O}(1 + a)$. Since $\hat{\alpha}_i(s) \ge \alpha_i(s)$ follows from $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$, we have $\alpha_i \in \mathcal{O}(1 + a)$.
(ii) The existence of a $\mathbf{C}^1$ function V_i and the local Lipschitzness of f_i implies that $\partial V_i/\partial x_i \cdot f_i(t, x_i, u_i)$ is locally Lipschitz with respect to (x_i, u_i) uniformly in t. From $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ it follows that

$$\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(t, x_i, u_i) \le -\alpha_i(|x_i|) + \eta_i(|x_i|)|u_i|$$

holds in a small neighborhood $\mathbf{N} := \{|u_i| < \epsilon_i\}$ of $u_i = 0$ for some $\eta_i \in \mathcal{P}$ fulfilling $b_i\epsilon_i \ge \sigma_i(\epsilon_i)$, where $b_i := \sup_{s\in\mathbb{R}_+} \eta_i(s) < \infty$. When $\sigma_i \notin \mathcal{O}(> 0)$ holds, there exists $c_i \in (0, \epsilon_i]$ such that $c_i = \min\{s \in (0, \epsilon_i] : b_i s = \sigma_i(s)\}$. Define

$$\hat{\sigma}_i(s) = \begin{cases} b_i s & \text{for } s \in [0, c_i) \\ \sigma_i(s) & \text{for } s \in [c_i, \infty) \end{cases}$$

and we arrive at $\Sigma_i \in \mathcal{S}_i(n_i, \alpha_i, \hat{\sigma}_i) \subseteq \mathcal{S}_i(n_i, \alpha_i, \sigma_i)$ and $\hat{\sigma}_i \in \mathcal{O}(1)$.

6.5 Proof of Lemma 3

The properties can be verified by merely examining the inequalities.

6.6 Proof of Theorem 6

(ii): The logical sum of *(J1)*, *(J2)*, *(J3)* is equivalent to the logical sum of

(N1) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \{\lim_{s\to\infty} \alpha_1(s) = \infty \ \vee \ \lim_{s\to\infty} \sigma_2(s) < \infty\}$

(N2) $\lim_{s\to\infty} \alpha_2(s) < \infty \ \wedge \ \lim_{s\to\infty} \sigma_1(s) < \infty$

under the assumption (46). We first prove the claim in the case of *(N1)*. For notational simplicity, we use the following notations:

$$\underline{\omega}_1 = k\hat{\omega}_1, \quad \underline{\omega}_2 = \hat{\omega}_2, \quad \tau_2 = \hat{\tau}_2, \quad \hat{\alpha}_2 = \alpha_2, \quad \hat{\sigma}_2 = \sigma_2$$

Replace σ_{ri} by $\bar{\sigma}_{ri} \in \mathcal{K}$ satisfying $\sigma_{ri}(s) \le \bar{\sigma}_{ri}(s)$ for all $s \in \mathbb{R}_+$, $i = 1, 2$. Due to (49), we can pick a class $\mathcal{K}_\infty$ function τ_{ri} fulfilling

$$\underline{\omega}_i \circ \tau_i^{-1} - \delta_i \circ \underline{\omega}_i \circ \tau_i^{-1} - \tau_{ri}^{-1} \in \mathcal{K}_\infty$$

for each $i = 1, 2$. The rest of the proof does not involve $\bar{\sigma}_{ri}$ and τ_{ri} if $\sigma_{ri}(r_i)$ is identically zero. Define

$$\begin{aligned}
\theta_i(s) &= \bar{\alpha}_i \circ \hat{\alpha}_i^{-1} \circ \tau_i \circ \hat{\sigma}_i(s), \quad s \in [0, Y_i) \\
\theta_{ri}(s) &= \bar{\alpha}_i \circ \hat{\alpha}_i^{-1} \circ \tau_{ri} \circ \bar{\sigma}_{ri}(s), \quad s \in [0, Y_{ri}) \\
Y_1 &= \lim_{s\to\infty} \hat{\sigma}_1^{-1} \circ \tau_1^{-1} \circ \hat{\alpha}_1(s) \\
Y_{r1} &= \begin{cases} \infty & , \text{ if } \lim_{s\to\infty} \hat{\alpha}_1(s) \ge \lim_{s\to\infty} \tau_{r1} \circ \bar{\sigma}_{r1} \\ \lim_{s\to\infty} \bar{\sigma}_{r1}^{-1} \circ \tau_{r1}^{-1} \circ \hat{\alpha}_1(s) & , \text{ otherwise} \end{cases} \\
Y_2 &= \infty, \quad Y_{r2} = \infty
\end{aligned} \tag{95}$$

for $i = 1, 2$. The function $\lambda_1(s)$ given by (47) satisfies $\lambda_1(s) > 0$ for all $s \in (0, \infty)$ and it is non-decreasing on $\mathbb{R}_+$ since (51) and (52). Define non-decreasing functions $\lambda_{\theta 1}, \lambda_{\theta r1} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\lambda_{\theta 1}(s) = \begin{cases} \lambda_1 \circ \theta_1(s) & , \ s \in [0, Y_1) \\ \lim_{s\to\infty} \lambda_1(s) & , \ s \in [Y_1, \infty) \end{cases} \tag{96}$$

$$\lambda_{\theta r1}(s) = \begin{cases} \lambda_1 \circ \theta_{r1}(s) & , \ s \in [0, Y_{r1}) \\ \lim_{s\to\infty} \lambda_1(s) & , \ s \in [Y_{r1}, \infty) \end{cases} \tag{97}$$

By virtue of (39), $\hat{\sigma}_1(\infty) > \tau_1^{-1} \circ \hat{\alpha}_1(\infty)$ holds if and only if $\hat{\alpha}_1(\infty) < \infty$. Thus,

$$Y_1 < \infty \vee Y_{r1} < \infty \ \Rightarrow \ \lim_{s\to\infty} \hat{\alpha}_1(s) < \infty \ \Rightarrow \ \lim_{s\to\infty} \lambda_1(s) < \infty \tag{98}$$

follows from (47). The function $\lambda_2(s)$ given by (48) is a non-decreasing function satisfying $\lambda_2(s) > 0$ for all $s \in (0, \infty)$ under (51) and (53). Define non-decreasing functions $\lambda_{\theta 2}, \lambda_{\theta r2} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\lambda_{\theta 2}(s) = \lambda_2 \circ \theta_2(s) \ , \quad \lambda_{\theta r2}(s) = \lambda_2 \circ \theta_{r2}(s) \ , \quad s \in \mathbb{R}_+$$

We obtain

$$\begin{aligned}
&\lambda_1(V_1)\{-\hat{\alpha}_1(|x_1|) + \hat{\sigma}_1(|x_2|) + \sigma_{r1}(|r_1|)\} \\
&\quad \le -\lambda_1(\underline{\alpha}_1(|x_1|)) \left[\underline{\omega}_1 \circ \tau_1^{-1} \circ \hat{\alpha}_1(|x_1|) - \tau_{r1}^{-1} \circ \hat{\alpha}_1(|x_1|)\right] + \lambda_{\theta 1}(|x_2|)\hat{\sigma}_1(|x_2|) + \lambda_{\theta r1}(|r_1|)\bar{\sigma}_{r1}(|r_1|) \qquad (99) \\
&\lambda_2(V_2)\{-\hat{\alpha}_2(|x_2|) + \hat{\sigma}_2(|x_1|) + \sigma_{r2}(|r_2|)\} \\
&\quad \le -\lambda_2(\underline{\alpha}_2(|x_2|)) \left[\underline{\omega}_2 \circ \tau_2^{-1} \circ \hat{\alpha}_2(|x_2|) - \tau_{r2}^{-1} \circ \hat{\alpha}_2(|x_2|)\right] + \lambda_{\theta 2}(|x_1|)\hat{\sigma}_2(|x_1|) + \lambda_{\theta r2}(|r_2|)\bar{\sigma}_{r2}(|r_2|) \qquad (100)
\end{aligned}$$

by combining calculations in individual cases divided by $\hat{\alpha}_i(|x_i|) \geq \tau_i \circ \hat{\sigma}_i(|x_{3-i}|)$, $\hat{\alpha}_i(|x_i|) < \tau_i \circ \hat{\sigma}_i(|x_{3-i}|)$, $\hat{\alpha}_i(|x_i|) \geq \tau_{ri} \circ \bar{\sigma}_{ri}(|r_i|)$ and $\hat{\alpha}_i(|x_i|) < \tau_{ri} \circ \bar{\sigma}_{ri}(|r_i|)$. Thus, the inequality (56) is fulfilled with

$$\alpha_{cl,i}(s) = \lambda_i(\underline{\alpha}_i(s)) \left[\underline{\omega}_i \circ \tau_i^{-1} \circ \hat{\alpha}_i - \delta_i \circ \underline{\omega}_i \circ \tau_i^{-1} \circ \hat{\alpha}_i \circ \bar{\alpha}_i^{-1} \circ \underline{\alpha}_i - \tau_{ri}^{-1} \circ \hat{\alpha}_i\right] \tag{101}$$

$$\sigma_{cl,i}(s) = \lambda_{\theta ri}(|s|)\bar{\sigma}_{ri}(|s|) \tag{102}$$

if λ_1 and λ_2 satisfy

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) \leq \lambda_2(\underline{\alpha}_2(s)) \left[\delta_2 \circ \underline{\omega}_2 \circ \tau_2^{-1} \circ \hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)\right] \tag{103}$$

$$\lambda_{\theta 2}(s)\hat{\sigma}_2(s) \leq \lambda_1(\underline{\alpha}_1(s)) \left[\delta_1 \circ \underline{\omega}_1 \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] \tag{104}$$

for all $s \in \mathbb{R}_+$. Here, $\delta_i \circ \underline{\omega}_i \in \mathcal{K}$ is used. Consider the following three conditions.

$$\begin{aligned} &\sigma_2(s) \left[\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right] \left[\lambda_{\theta 1} \circ \underline{\alpha}_2^{-1} \circ \theta_2(s)\right] \\ &\quad \leq \lambda_1(\underline{\alpha}_1(s)) \left[\delta_2 \circ \underline{\omega}_2 \circ \sigma_2(s)\right] \left[\delta_1 \circ \underline{\omega}_1 \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right], \ s \in \mathbb{R}_+ \end{aligned} \tag{105}$$

$$[\lambda_{\theta 1}(s)]\hat{\sigma}_1(s) = \lambda_2(\underline{\alpha}_2(s)) \left[\delta_2 \circ \underline{\omega}_2 \circ \tau_2^{-1} \circ \hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)\right], \ s \in [0, Y_1) \tag{106}$$

$$[\lambda_{\theta 1}(s)]\hat{\sigma}_1(s) \leq \lambda_2(\underline{\alpha}_2(s)) \left[\delta_2 \circ \underline{\omega}_2 \circ \tau_2^{-1} \circ \hat{\alpha}_2 \circ \bar{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)\right], \ s \in [Y_1, \infty) \tag{107}$$

The pair of (106) and (107) implies (103). If (37) is are satisfied, we have $\tau_1 \circ \hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq \hat{\alpha}_1(s)$. This inequality together with the definition Y_1 implies $\lim_{s\to\infty} \underline{\alpha}_2^{-1} \circ \theta_2(s) \leq Y_1$. Thus, substitution of (106) into the left hand side of (105) results in (104). Hence, the proof is completed if λ_i, $i = 1, 2$ given in (47)-(48) solve (105), (106) and (107). Combining (47) with (48), we arrive at (106). Due to (48), (98) and the definition of $\lambda_{\theta 1}$, the property (52) leads to (107). On the other hand, from (37), (54) and (55) it follows that λ_1 in (47) solves (105). In the *(N2)* case, $\hat{\sigma}_1(\infty) < \infty$ follows from (41). The property (57) guarantees $\lambda_i(\infty) < \infty$, $i = 1, 2$, in (47) and (48). Define $\sigma_{cl,i} = \lambda_i(\infty)\sigma_{ri} \in \mathcal{P}_0$, and we do not need θri.
(i): The properties (42) and (43) allow us to define θ_2 as in (95) with $Y_2 = \infty$. If $\hat{\alpha}_1 \in \mathcal{K}_\infty$ holds, θ_1 can be defined as in (95) with $Y_1 = \infty$. When $\hat{\alpha}_1 \notin \mathcal{K}_\infty$ and *(M1)* hold, the properties (37) and (39) imply $\tau_1 \circ \hat{\sigma}_1(\infty) = \hat{\sigma}_1(\infty) = \hat{\alpha}_1(\infty)$. Thus, θ_1 can be defined as in (95) with $Y_1 = \infty$. When $\hat{\alpha}_1 \notin \mathcal{K}_\infty$ and *(M2)* hold, the property (42) implies $\hat{\sigma}_1(\infty) > \tau_1^{-1} \circ \hat{\alpha}_1(\infty)$ yielding $\lambda_1(\infty) < \infty$. Although $\theta_1(s)$ defined by (95) is finite for $Y_1 < \infty$, the function $\lambda_{\theta 1}(s)$ defined by (96) is finite for all $s \in \mathbb{R}_+$.

6.7 Proof of Lemma 4

To prove the former claim, we first recall that (36) implies

$$\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} \leq 1$$

Then, the situation of $\neg$*(M3)* $\wedge$ (36) is

$$\lim_{s\to\infty} \frac{\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ \sigma_2(s)}{\alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)} < 1 \quad \vee \quad \textit{(M1)}$$

Thus, the first claim follows straightforwardly. To prove the latter claim, we first see that $\neg$*(J4)* is identical to

$$\{\lim_{s\to\infty} \alpha_1(s) = \infty \ \vee \ \lim_{s\to\infty} \sigma_2(s) < \infty\} \ \wedge \ \{\lim_{s\to\infty} \alpha_2(s) = \infty \ \vee \ \lim_{s\to\infty} \sigma_1(s) < \infty\}$$

Moreover, the above situation is equal to the logical sum of the following four cases:

$$\lim_{s\to\infty} \alpha_1(s) = \infty \ \wedge \ \lim_{s\to\infty} \alpha_2(s) = \infty \tag{108}$$
$$\lim_{s\to\infty} \sigma_2(s) < \infty \ \wedge \ \lim_{s\to\infty} \sigma_1(s) < \infty \tag{109}$$
$$\lim_{s\to\infty} \alpha_1(s) = \infty \ \wedge \ \lim_{s\to\infty} \sigma_1(s) < \infty \tag{110}$$
$$\lim_{s\to\infty} \sigma_2(s) < \infty \ \wedge \ \lim_{s\to\infty} \alpha_2(s) = \infty \tag{111}$$

Here, the situation (110) is redundant and can be removed since the case which is not covered by any one of (108), (109) and (111) is

$$\lim_{s\to\infty} \alpha_1(s) = \infty \ \wedge \ \lim_{s\to\infty} \sigma_1(s) < \infty \ \wedge \ \lim_{s\to\infty} \alpha_2(s) < \infty \ \wedge \ \lim_{s\to\infty} \sigma_2(s) = \infty \tag{112}$$

This situation, however, does not occur since (35) is assumed. Therefore, the latter claim has been proved since (108)= *(J1)*, (109)= *(J3)* and (111)= *(J2)*.

6.8 Proof of Lemma 5

First, recall again that $L \geq \lim_{s\to\infty} \tau_1^{-1} \circ \alpha(s)$ is ensured by (39) and (50). By construction, $\psi(s)$ is continuously differentiable on $(0, R)$. By definition, the following properties hold:

$$b \in \mathcal{P}, \quad \hat{\omega}_1 \in \mathcal{P}_0 \tag{113}$$
$$d, \eta, \alpha \in \mathcal{K} \tag{114}$$
$$(\mathbf{Id} + \hat{\omega}_1) \circ \eta(s) \leq \alpha(s), \ \forall s \in \mathbb{R}_+ \tag{115}$$

Here, (42) and (43) are used for (113), and the inequality (37) corresponds to (115). Assume that

$$\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) \tag{116}$$

holds. Then, the right hand side of the implication (43) holds. By virtue of (39), we obtain

$$S = L = \lim_{s\to\infty} \hat{\sigma}_1(s) = \lim_{s\to\infty} \hat{\alpha}_1(s) = R$$
$$\hat{\omega}_1(s) > 0, \quad \forall s \in (0, S)$$
$$\lim_{s\to S} s + \hat{\omega}_1(s) = S \ ,$$

Hence, the function m satisfies

$$m(0) = 0, \quad m(s) > s, \ \forall s \in (0, S) \tag{117}$$

From (63), (64) and (65), we obtain

$$\frac{\psi'(s)}{\psi(s)} = \bar{Q}(s) \geq Q(s), \quad s \in (0, R)$$

By virtue of (60), (117), $b(s) \geq 0$ and $\psi(s) \geq 0$, the following property holds.

$$[\psi(s)]\, \hat{d}(s) \leq \left[\psi(s) + (m(s) - s)\psi'(s)\right] \hat{b}(s), \ \forall s \in (0, S) \tag{118}$$

From (64) and (65) it follows that

$$\frac{d}{ds}\left(\frac{1}{\bar{Q}(s)}\right) \leq 1, \quad \forall t \in (0, R)$$

holds. This property guarantees

$$\psi''(s) = \psi(s)(\bar{Q}'(s) + \bar{Q}(s)^2) \geq 0, \quad t \in (0, R) \tag{119}$$

When (63) is satisfied by a non-decreasing function $\bar{Q}$, we have (119) again. Since the inequality (119) implies the non-decreasing property of $\psi'(s)$, the inequality (118) lead to (69) since $\eta \in \mathcal{K}$. The boundedness of ψ on $(0, S)$ follows from (60) and (117). Next, assume that (71) holds. Then, the right hand side of the implication (42) holds. From (115) and $0 < k < 1$, we obtain $S < R$, and m satisfies (117) and $\lim_{s\to S}\{m(s) - s\} > 0$. The property $\lim_{s\to\infty} \sigma_2(s) < \infty$ guaranteed by (71) implies $\lim_{s\to\infty} d(s) < \infty$. The property $\hat{\omega}_1, \hat{\omega}_2 \in \mathcal{K}$ guaranteed by (42) implies $\lim_{s\to\infty} b(s) > 0$. These two properties yields

$$\limsup_{s\to S} \frac{\hat{d}(s)}{\hat{b}(s)} < \infty \tag{120}$$

Therefore, the definition (66)-(67) and (120) ensure the boundedness of ψ on $(0, \infty)$. The rest is the same as the case of (116).

6.9 Proof of Lemma 6

First, suppose that

$$\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) = \infty \tag{121}$$

holds. Then, the assumption *(J1)* $\vee$ *(J2)* $\vee$ *(J3)* implies $\lim_{s\to\infty} \alpha_1(s) = \infty$. Hence, $L = \infty$ follows from $L = \lim_{s\to\infty} \hat{\sigma}_1(s)$, (39) and (40). Next, assume that

$$\lim_{s\to\infty} \alpha_2(s) = \lim_{s\to\infty} \sigma_2(s) < \infty \tag{122}$$

This property, however, does not admit the existence of a function $\omega_2 \in \mathcal{K}$ satisfying (45). Since both (121) and (122) contradict the assumptions of Lemma 6, the only case remaining under (35) is (71).

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