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Supplement: Efficient Weak Second-order Stochastic Runge-Kutta Methods for Non-commutative Stochastic Differential Equations

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Abstract

This paper gives a modification of a class of stochastic Runge-Kutta methods proposed in a paper by Komori (2007). The slight modification can reduce the computational costs of the methods significantly.

1 Introduction

Runge-Kutta type methods for stochastic differential equations (SDEs) have been recently developed by many researchers [1, 3]. As an example of such methods, Komori [4] derived a stochastic Runge-Kutta (SRK) scheme with weak order 2 for non-commutative Stratonovich SDEs from a framework of SRK methods. Compared with other previous schemes, the scheme had the advantage that it can reduce the number of random variables that need to be simulated. Rößler [5], however, has pointed out that for this scheme the computational costs linearly depend on the dimension of the Wiener process for each diffusion coefficient, and has proposed new schemes without this drawback. But this requires 55 order conditions to be solved in order to construct weak second-order methods.

In the present paper, we show that the drawback can be also removed in Komori's framework of SRK methods and only 38 order conditions need to be solved. The paper is organized as follows. In Section 2, we will briefly introduce this new class of SRK methods and the expression of their order conditions with rooted trees. In Section 3 we will concretely seek the order conditions under a modified setting on parameters and random variables. Lastly, we will give a brief discussion.

2 Preliminary

As preparation for the following sections, we give a brief introduction to a framework of our SRK methods and expressions for the order conditions in order to attain weak order two. Consider a d-dimensional Stratonovich stochastic differential equation

$$\mathrm{d}\boldsymbol{y}(\tau) = \boldsymbol{g}_0(\boldsymbol{y}(\tau))\mathrm{d}\tau + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(\tau)) \circ \mathrm{d}W_j(\tau), \quad 0 \le \tau \le T_{end}, \quad \boldsymbol{y}(0) = \boldsymbol{x}_0,$$

where $W_j(\tau)$ is a scalar Wiener process. Let τ_n be an equidistant grid point nh ($n = 0, 1, \ldots, M$) with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and $\boldsymbol{y}_n$ a discrete approximation to the solution $\boldsymbol{y}(\tau_n)$. In addition, suppose that the initial approximate random variable $\boldsymbol{y}_0$ has the same probability law with all moments finite as that of $\boldsymbol{x}_0$, and define weak order in a usual way [1, 4]. As numerical methods for weak approximations, our SRK methods are given by

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{i=1}^{s} \sum_{j_a, j_b=0}^{m} c_i^{(j_a,j_b)} \boldsymbol{Y}_i^{(j_a,j_b)}, \\ \boldsymbol{Y}_{i_a}^{(j_a,j_b)} &= \tilde{\eta}_{i_a}^{(j_a,j_b)} \boldsymbol{g}_{j_b}(\boldsymbol{y}_n + \sum_{i_b=1}^{s} \sum_{j_c,j_d=0}^{m} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)} \boldsymbol{Y}_{i_b}^{(j_c,j_d)}) \end{aligned} \qquad (2.\ 1)$$

($1 \le i_a \le s$, $0 \le j_a, j_b \le m$), where $c_i^{(j_a,j_b)}$ and $\alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)}$ are constant parameters and where each $\tilde{\eta}_{i_a}^{(j_a,j_b)}$ is a random variable independent of $\boldsymbol{y}_n$ and its $2k$th moment is supposed to be equal to $K_1 h^{2k}$ if $j_b = 0$, or $K_2 h^k$ otherwise for constants K_1, K_2 and $k = 1, 2, \ldots$. We can express the weak order conditions by multi-colored rooted trees (MRTs) [4].

Definition 2.1 (MRT) *An MRT with a root* ⓙ *(colored with a label j from 0 to m) is a tree recursively defined in the following manner: i) $\tau^{(j)}$ is the primitive tree having only a vertex* ⓙ *, ii) If $t_1, \ldots, t_k$ are MRTs, then $[t_1, \ldots, t_k]^{(j)}$ is also an MRT with the root* ⓙ. *The totality of MRTs is denoted by T.*

Definition 2.2 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows:*

$$\Phi(\tau^{(j)}; s_0) = \int_{\tau_n}^{s_0} \circ \mathrm{d}W_j(s_1), \quad \Phi(t; s_0) = \int_{\tau_n}^{s_0} \prod_{i=1}^{k} \Phi(t_i; s_1) \circ \mathrm{d}W_j(s_1) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)},$$

where $\circ \mathrm{d}W_0(s_1) \stackrel{\text{def}}{=} \mathrm{d}s_1$. *Especially, $\Phi(t; \tau_{n+1})$ is denoted by $\Phi(t)$ for ease of notation.*

Definition 2.3 (Elementary numerical weight $\tilde{\Phi}(t)$ on T) *Let s be the stage number of* (2.1) *and m the maximum value of the range of values of the index j_a or j_b in* (2.1). *An elementary numerical weight of $t \in T$ is given as follows:*

i) Trace the vertices of t in the direction from the root to upper vertices. Then, for the root vertex, prepare indices i_1 and j'_1 and set $\Theta = c_{i_1}^{(j'_1,j)} \tilde{\eta}_{i_1}^{(j'_1,j)}$ if the color is j. For each vertex except the root vertex, prepare new indices i_{k+1} and j'_{k+1}, multiply Θ by $\alpha_{i_k i_{k+1}}^{(j'_k,j,j'_{k+1},l)} \tilde{\eta}_{i_{k+1}}^{(j'_{k+1},l)}$ if the color is l, and reset Θ by it, where i_k and j'_k mean the indices for the parent vertex and where j means the color of the parent vertex.

ii) Define $\tilde{\Phi}(t)$ by the summation of Θ over $i_\cdot$ from 1 to s and over $j'_\cdot$ from 0 to m.

The weak order conditions are given as follows. Let $\rho(t)$ be the number of vertices of $t \in T$ and $r(t)$ the number of vertices of t with the color 0, and suppose that any component of $\boldsymbol{g}_j$ is sufficiently smooth and the regularity of the time discrete approximation is satisfied. If the following conditions are satisfied, the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ with weak (global) order q as $h \to 0$:

$$E\left[\prod_{j=1}^{L} \tilde{\Phi}(t_j)\right] = E\left[\prod_{j=1}^{L} \Phi(t_j)\right] \tag{2. 2}$$

for any $t_1, \ldots, t_L \in T$ $(1 \leq L \leq 2q)$ satisfying $\sum_{j=1}^{L}(\rho(t_j) + r(t_j)) \leq 2q$ and

$$E\left[\tilde{\Phi}(t)\right] = 0 \tag{2. 3}$$

for any $t \in T$ satisfying $\rho(t) + r(t) = 2q + 1$ [4].

3 Weak second order conditions for our SRK methods

In (2. 1) we seek weak second order conditions that lead to a reduction in the number of evaluations on the diffusion coefficients. We can achieve this by slightly changing the parameter settings considered in [4]. Taking generality into account, we will leave implicitness in parameters as much as possible.

We use the same simplifying assumptions as those in [4], which are given by seven equalities for $\tilde{\Phi}(t)$. Four of them, for examples, are as follows: for $\tau^{(j)}$, $[\tau^{(0)}]^{(j)}$, $[\tau^{(l)}]^{(j)}$

and $[\tau^{(j)}]^{(l)}$ $(0 < j < l)$

$$
\begin{aligned}
&\sum_{i_a=1}^{s}\sum_{j_a=0}^{m} c_{i_a}^{(j_a,j)}\tilde{\eta}_{i_a}^{(j_a,j)} = \triangle W_j,\\
&\sum_{i_a,i_b=1}^{s}\sum_{j_a,j_b=0}^{m} c_{i_a}^{(j_a,j)}\tilde{\eta}_{i_a}^{(j_a,j)}\alpha_{i_a i_b}^{(j_a,j,j_b,l)}\tilde{\eta}_{i_b}^{(j_b,l)} = \frac{\triangle W_j(\triangle W_l + \triangle \tilde{W}_l)}{2},\\
&\sum_{i_a,i_b=1}^{s}\sum_{j_a,j_b=0}^{m} c_{i_a}^{(j_a,j)}\tilde{\eta}_{i_a}^{(j_a,j)}\alpha_{i_a i_b}^{(j_a,j,j_b,0)}\tilde{\eta}_{i_b}^{(j_b,0)} = \frac{h\triangle W_j}{2},\\
&\sum_{i_a,i_b=1}^{s}\sum_{j_a,j_b=0}^{m} c_{i_a}^{(j_a,l)}\tilde{\eta}_{i_a}^{(j_a,l)}\alpha_{i_a i_b}^{(j_a,l,j_b,j)}\tilde{\eta}_{i_b}^{(j_b',j)} = \frac{\triangle W_j(\triangle W_l - \triangle \tilde{W}_l)}{2},
\end{aligned}
\tag{3. 1}
$$

where $\triangle W_j$ $(j = 1, \ldots, m)$ and $\triangle \tilde{W}_l$ $(l = 2, \ldots, m)$ are mutually independent random variables satisfying

$$
E\left[(\triangle W_j)^k\right] = \begin{cases} 0 & (k = 1, 3, 5),\\ (k-1)h^{k/2} & (k = 2, 4),\\ O(h^3) & (k \geq 6), \end{cases} \qquad E\left[(\triangle \tilde{W}_l)^k\right] = \begin{cases} 0 & (k = 1, 3),\\ h & (k = 2),\\ O(h^2) & (k \geq 4). \end{cases}
$$

Further, as in the same as [4] we set

$$
\begin{aligned}
&\tilde{\eta}_i^{(0,0)} = h, \qquad \tilde{\eta}_i^{(j,j)} = \triangle W_j \quad (j > 0), \qquad c_i^{(j_a,0)} = c_i^{(0,j_b)} = 0 \quad (j_a, j_b \neq 0),\\
&\alpha_{i_a i_b}^{(j_a,0,j_c,0)} = 0 \quad (j_a \neq 0 \text{ or } j_c \neq 0), \qquad \alpha_{i_a i_b}^{(j_a,0,j_c,j)} = 0 \quad (j_a \neq 0 \text{ or } j_c \neq j).
\end{aligned}
$$

On the other hand, for each $(1 \leq)$ j $(\leq m)$, chose a value in $\{1, 2, \ldots, j-1, j+1, \ldots, m\}$, say $k(j)$, and assume

$$
\begin{aligned}
&c_i^{(j_a,j)} = 0 \;\; (j_a \neq j, k(j)), \quad c_i^{(k(j),j)} = 0 \;\; (i \leq s-3),\\
&\alpha_{i_a i_b}^{(j_a,j,j_c,j_d)} = 0 \;\; (j_a \neq j, k(j) \text{ and } (j_c \neq 0 \text{ or } j_d \neq 0)),\\
&\alpha_{i_a i_b}^{(j,j,j_c,j_d)} = 0 \;\; (j_c \neq j_d), \quad \alpha_{i_a i_b}^{(k(j),j,j_c,j_d)} = 0 \;\; (j_c \neq 0, j),\\
&\alpha_{i_a i_b}^{(k(j),j,0,j_d)} = 0 \;\; (j_d \neq 0), \quad \alpha_{i_a i_b}^{(k(j),j,j,j_d)} = 0 \;\; (j_d = 0, j)
\end{aligned}
\tag{3. 2}
$$

for $j > 0$ and

$$
\begin{aligned}
&\tilde{\eta}_{s-2}^{(j,l)} = \begin{cases} \triangle W_j \triangle \tilde{W}_l/\sqrt{h} & (l > j),\\ -\triangle \tilde{W}_j \triangle W_l/\sqrt{h} & (j > l), \end{cases} \qquad \tilde{\eta}_i^{(j,l)} = \sqrt{h} \;\; (i > s-2),\\
&\alpha_{i_a i_b}^{(k(j),j,j,l)} = 0 \;\; (i_a, i_b \leq s-3 \text{ or } i_a \leq i_b)
\end{aligned}
\tag{3. 3}
$$

for $j \neq l$ and $j, l > 0$ (we always assume the restrictions for j, l in the sequel). Note that $\tilde{\eta}_i^{(j,0)}$, $\tilde{\eta}_i^{(0,j)}$ and $\tilde{\eta}_i^{(j,l)}$ $(i \leq s-3)$ do not need to be set since they are not used below.

From the first equations in (3. 1) and (3. 2) we obtain

$$
\sum_{i_a=1}^{s} c_{i_a}^{(j,j)} = 1, \qquad c_{s-2}^{(k(j),j)} = c_{s-1}^{(k(j),j)} + c_s^{(k(j),j)} = 0.
\tag{3. 4}
$$

Similarly, from this and the other three equations in (3. 1) we have

$$
\begin{aligned}
&\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,0,0)} = \frac{1}{2}, \qquad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,l,l)} = \frac{1}{2}, \\
&\sum_{i_a=s-1}^{s} \sum_{i_b=1}^{s} c_{i_a}^{(k(j),j)} \alpha_{i_a i_b}^{(k(j),j,0,0)} = 0, \\
&c_{s-1}^{(k(j),j)} \alpha_{s-1,s-2}^{(k(j),j,j,l)} + c_{s}^{(k(j),j)} \alpha_{s,s-2}^{(k(j),j,j,l)} = \tfrac{1}{2}, \qquad c_{s}^{(k(j),j)} \alpha_{s,s-1}^{(k(j),j,j,l)} = 0.
\end{aligned}
\tag{3. 5}
$$

Here, note that the last equation in (3. 4) and the last two equations in (3. 5) yield

$$
\alpha_{s,s-1}^{(k(j),j,j,l)} = 0. \tag{3. 6}
$$

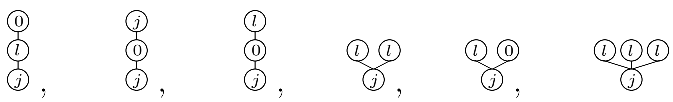
Remark 3.1 *We also have the following.*

i) For the trees in which a node and its child node are colored with the same color, their elementary numerical weights do not have $\tilde{\eta}_i^{(j,l)}$ because $\alpha_{i_a i_b}^{(j_a,j,j_c,j)} = 0$ ($j_a \neq j$ or $j_c \neq j$) and $\alpha_{i_a i_b}^{(j_a,0,j_c,0)} = 0$ ($j_a \neq 0$ or $j_c \neq 0$).

ii) For the trees in which the root is colored with 0, their elementary numerical weights do not have $\tilde{\eta}_i^{(j,l)}$ because $\alpha_{i_a i_b}^{(j_a,0,j_c,j_d)} = 0$ ($j_a \neq 0$ or $j_c \neq j_d$).

iii) For the trees in which the root is colored with j and has a child node colored with l which has a child node colored with k ($\neq 0, l$), their elementary numerical weights do not have $\tilde{\eta}_i^{(j,l)}$ or $\tilde{\eta}_i^{(l,k)}$ because $\alpha_{i_a i_b}^{(j_a,j,j_b,l)} \alpha_{i_b i_c}^{(j_b,l,j_c,k)} = 0$ ($j_a \neq j$ or $j_b \neq l$ or $j_c \neq k$) from (3. 2), (3. 3) and (3. 6).

Consequently, concerning weak order 2, all the trees whose elementary numerical weights have $\tilde{\eta}_i^{(j,l)}$ are



as well as $[\tau^{(0)}]^{(j)}$ and $[\tau^{(l)}]^{(j)}$ dealt with in (3. 5). Let us seek the order conditions concerning the above MRTs. For the MRTs except the second and fourth ones, (2. 2) holds automatically. In order to satisfy (2. 2) for the others and (2. 3) for the second one, we obtain

$$
\begin{aligned}
&\sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,0,0)} \alpha_{i_b i_c}^{(0,0,j,j)} = 0, \qquad \sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j,l,l)} \alpha_{i_a i_c}^{(j,j,l,l)} = \frac{1}{2}, \\
&\sum_{i_a=s-1}^{s} c_{i_a}^{(k(j),j)} \left(\alpha_{i_a,s-2}^{(k(j),j,j,l)} \right)^2 = 0.
\end{aligned}
\tag{3. 7}
$$

Incidentally, in the same way as that in [4] we can obtain the order conditions concerning the trees whose elementary numerical weights do not have $\tilde{\eta}_i^{(j,l)}$. Summarizing all mentioned up to here, we have all 38 order conditions for weak order 2:

$$\sum_{i_a=1}^{s} c_{i_a}^{(0)} = \sum_{i_a=1}^{s} c_{i_a}^{(j)} = 1, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,j)} = \sum_{i_a=1}^{s} c_{i_a}^{(0)} A_{i_a}^{(0,j)} = \sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,0)} = \sum_{i_a=1}^{s} c_{i_a}^{(0)} A_{i_a}^{(0,0)} = \frac{1}{2},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} A_{i_b}^{(j,0)} = \frac{1}{4}, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(0)} \alpha_{i_a i_b}^{(0,j)} A_{i_b}^{(j,j)} = \frac{1}{4}, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,0)} A_{i_b}^{(0,j)} = 0,$$

$$\sum_{i_a=1}^{s} c_{i_a}^{(0)} \left(A_{i_a}^{(0,j)}\right)^2 = \frac{1}{2}, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,0)} A_{i_a}^{(j,j)} = \frac{1}{4}, \quad \sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} \alpha_{i_b i_c}^{(j,j)} A_{i_c}^{(j,j)} = \frac{1}{24},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} \left(A_{i_b}^{(j,j)}\right)^2 = \frac{1}{12}, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,j)} A_{i_b}^{(j,j)} = \frac{1}{8}, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} \left(A_{i_a}^{(j,j)}\right)^3 = \frac{1}{4},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} A_{i_b}^{(j,j)} = \frac{1}{6}, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} \left(A_{i_a}^{(j,j)}\right)^2 = \frac{1}{3}, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,l)} = \frac{1}{2},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,l)} \alpha_{i_a i_b}^{(j,l)} A_{i_b}^{(l,j)} = 0, \quad \sum_{i_a=1}^{s} c_{i_a}^{(j)} \left(A_{i_a}^{(j,l)}\right)^2 = \frac{1}{2}, \quad \sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} \alpha_{i_b i_c}^{(j,l)} A_{i_c}^{(l,l)} = \frac{1}{8},$$

$$\sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,l)} \alpha_{i_b i_c}^{(l,j)} A_{i_c}^{(j,l)} = \sum_{i_a,i_b,i_c=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,l)} \alpha_{i_b i_c}^{(l,l)} A_{i_c}^{(l,j)} = 0, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} \left(A_{i_b}^{(j,l)}\right)^2 = \frac{1}{4},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,l)} A_{i_b}^{(l,l)} A_{i_b}^{(l,j)} = 0, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,j)} \alpha_{i_a i_b}^{(j,l)} A_{i_b}^{(l,l)} = \frac{1}{8},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,l)} \alpha_{i_a i_b}^{(j,j)} A_{i_b}^{(j,l)} = \sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,j)} \left(A_{i_a}^{(j,l)}\right)^2 = \frac{1}{4},$$

$$\sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,j)} A_{i_b}^{(j,l)} = \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,l)} A_{i_b}^{(l,l)} = \frac{1}{4}, \quad \sum_{i_a,i_b=1}^{s} c_{i_a}^{(j)} \alpha_{i_a i_b}^{(j,l)} A_{i_b}^{(l,j)} = 0,$$

$$\sum_{i_a=1}^{s} c_{i_a}^{(j)} A_{i_a}^{(j,j)} A_{i_a}^{(j,l)} = \frac{1}{4}, \quad c_{s-2}^{(k(j),j)} = 0,$$

$$c_{s-1}^{(k(j),j)} + c_{s}^{(k(j),j)} = \alpha_{s,s-1}^{(k(j),j,j,l)} = \sum_{i_a=s-1}^{s} c_{i_a}^{(k(j),j)} A_{i_a}^{(k(j),j,0,0)} = \sum_{i_a=s-1}^{s} c_{i_a}^{(k(j),j)} \left(\alpha_{i_a,s-2}^{(k(j),j,j,l)}\right)^2 = 0,$$

$$\sum_{i_a=s-1}^{s} c_{i_a}^{(k(j),j)} \alpha_{i_a,s-2}^{(k(j),j,j,l)} = \frac{1}{2},$$

where the following are used for ease of notation:

$$c_{i_a}^{(j_a)} \stackrel{\text{def}}{=} c_{i_a}^{(j_a,j_a)}, \quad \alpha_{i_a i_b}^{(j_a,j_b)} \stackrel{\text{def}}{=} \alpha_{i_a i_b}^{(j_a,j_a,j_b,j_b)}, \quad A_{i_a}^{(j_a,j_b)} \stackrel{\text{def}}{=} \textstyle\sum_{i_b=1}^{s} \alpha_{i_a i_b}^{(j_a,j_b)},$$
$$A_{i_a}^{(j_a,j_b,j_c,j_d)} \stackrel{\text{def}}{=} \textstyle\sum_{i_b=1}^{s} \alpha_{i_a i_b}^{(j_a,j_b,j_c,j_d)}.$$

4 Discussion

The difference in the order conditions between the present paper and [4] is only the last six equality relationships. For example, let us set $s = 4$ and use the same values for $c_{i_a}^{(j_a)}$ and $\alpha_{i_a i_b}^{(j_a,j_b)}$ as those in [4], which means that the first 32 order conditions are satisfied. Further, if we set $\alpha_{i_a i_b}^{(j,l,0,0)} = 0$ and $c_3^{(k(j),j)} = \gamma$ (a nonzero constant), then we obtain

$$c_2^{(k(j),j)} = \alpha_{43}^{(k(j),j,j,l)} = 0, \quad c_4^{(k(j),j)} = -\gamma, \quad \alpha_{32}^{(k(j),j,j,l)} = -\alpha_{42}^{(k(j),j,j,l)} = \frac{1}{4\gamma}$$

from the 6 order conditions. This new method leads to

$$\boldsymbol{Y}_3^{(k(j),j)} = \sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n + \frac{1}{4\gamma}\sum_{\substack{j_d=1\\ j_d\neq j}}^{m} \tilde{\eta}_2^{(j,j_d)}\boldsymbol{g}_{j_d}(\boldsymbol{y}_n)),$$

$$\boldsymbol{Y}_4^{(k(j),j)} = \sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n - \frac{1}{4\gamma}\sum_{\substack{j_d=1\\ j_d\neq j}}^{m} \tilde{\eta}_2^{(j,j_d)}\boldsymbol{g}_{j_d}(\boldsymbol{y}_n)).$$

Now, the necessary intermediate stage values for $\boldsymbol{y}_{n+1}$ in (2. 1) are only $\boldsymbol{Y}_3^{(k(j),j)}$ and $\boldsymbol{Y}_4^{(k(j),j)}$ in addition to $\boldsymbol{Y}_i^{(0,0)}$ and $\boldsymbol{Y}_i^{(j,j)}$ $(1 \leq i \leq 4)$, whereas the $\boldsymbol{Y}_i^{(j,l)}$'s are necessary in [4].

On the basis of our SRK methods, we will propose new schemes with good stability properties and less computational costs in the near future [2].

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