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Multi-Colored Rooted Tree Analysis of the Order Conditions of Weak Schemes for Stochastic Differential Equations with a Multi-Dimensional Wiener Process

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Abstract

The aim of the present paper is to give the tractable way of seeking weak order conditions of a stochastic Runge-Kutta family for stochastic differential equations with a multi-dimensional Wiener process. This is accomplished by the extension of the rooted tree analysis for ordinary Runge-Kutta methods. It is illustrated that weak order conditions can be obtained directly from diagrams for multi-colored rooted trees in quite a transparent manner.

1 Introduction

The importance of numerical schemes for stochastic differential equations (SDEs) has increased significantly as SDEs have been used for mathematical modeling in many fields ([15]). This is because that SDEs are analytically unsolvable in most cases.

Corresponding to the meaning of approximation, there are two kinds of numerical schemes for SDEs, that is, strong schemes and weak schemes ([10]). Strong schemes give an approximate solution in the mean square sense ([2, 4, 6, 13]). Weak schemes give an approximation to the moment of an exact solution ([1, 9, 11, 16]). In either type of scheme we have to seek order conditions and solve them in order to obtain high order schemes.

Generally speaking, it is hard to derive order conditions. Fortunately, however, the rooted tree analysis, invented by Butcher ([5]), opened the way to get the order conditions of Runge-Kutta schemes for ordinary differential equations (ODEs) in a transparent manner ([7, 8]), and it was extended to be applicable to the order conditions of schemes for SDEs. In fact, Burrage and Burrage ([2]) gave the rooted tree analysis of strong schemes for SDEs with a scalar Wiener process and they ([3]) also extend it for SDEs with a multi-dimensional Wiener process. Komori, Mitsui and Sugiura ([12]) extended the rooted analysis for ODEs into that of weak schemes for SDEs with a scalar Wiener process.

The aim of the present paper is to further extend this into the analysis of weak order conditions of a stochastic Runge-Kutta family for SDEs with a multi-dimensional Wiener process.

Next, we introduce some notations and concepts dealt with in the paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\boldsymbol{W}(t) = [W_1(t), \ldots, W_m(t)]$ an m-dimensional Wiener process defined on the probability space. We mainly consider the following d-dimensional stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_0^t \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s), \qquad 0 \le t \le T_{end}, \qquad (1.\ 1)$$

where $\boldsymbol{g}_j$ $(j = 0, 1, \ldots, m)$ are d-vector valued functions and $\circ$ means the Stratonovich formulation. The equation can be expressed, in differential form, by the SDE

$$\mathrm{d}\boldsymbol{y}(t) = \boldsymbol{g}_0(\boldsymbol{y}(t))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t)) \circ \mathrm{d}W_j(t), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \quad 0 \le t \le T_{end}.$$

In addition, the solution $\boldsymbol{y}(t)$ of (1. 1) satisfies the following Itô's stochastic integral equation

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_0^t \left[\boldsymbol{g}_0(\boldsymbol{y}(s)) + \frac{1}{2}\sum_{j=1}^{m} \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}} \boldsymbol{g}_j(\boldsymbol{y}(s))\right] \mathrm{d}s + \sum_{j=1}^{m} \int_0^t \boldsymbol{g}_j(\boldsymbol{y}(s))\mathrm{d}W_j(s). \qquad (1.\ 2)$$

This equation will be referred in Section 4.

We give equidistant grid points $\tau_n \stackrel{\text{def}}{=} nh$ $(n = 0, 1, \ldots, M)$ with step size $h \stackrel{\text{def}}{=} T_{end}/M < 1$ (M is a natural number) and consider discrete approximations $\boldsymbol{y}_n$ to $\boldsymbol{y}(\tau_n)$. Let $C_P^l(\boldsymbol{R}^d, \boldsymbol{R})$ denote the totality of l times continuously differentiable $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^d$, all of whose partial derivatives of order less than or equal to l have polynomial growth. Now we can give the following definition [3].

Definition 1.1 *Suppose that discrete approximations $\boldsymbol{y}_n$ are given by a scheme. Then, we say that the scheme is of weak (global) order q if for each $G \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$, $\exists C > 0$ (independent of h) and $\delta > 0$ such that*

$$|E[\boldsymbol{G}(\boldsymbol{y}(\tau_M)] - E[\boldsymbol{G}(\boldsymbol{y}_M)]| \leq Ch^q, \quad h \in (0, \delta).$$

The organization of the present paper is as follows. In Section 2 we will first express the Stratonovich-Taylor expansion of the solution of an SDE by a function of multi-colored rooted trees, and second give a detailed expression of the function by introducing the notions of elementary integrals, differentials and weights. In Section 3 we will first express, with a function of labeled multi-colored rooted trees, the Taylor expansion of an approximate solution given by a stochastic Runge-Kutta family, and second give a detailed expression of the function by elementary differentials, numerical integrals and weights. In Section 4 we will express the order conditions of a stochastic Runge-Kutta family in the weak sense by only the expectations of elementary weights and numerical weights. In addition, we will give the tractable way of seeking the expectations with multi-colored trees. In the appendix, we will show that the Runge-Kutta family includes a weak scheme proposed by Platen.

2 The Stratonovich-Taylor expansion by multi-colored rooted trees

The goal of this section is to represent the Stratonovich-Taylor expansion of the solution $\boldsymbol{y}(t)$ of

$$\boldsymbol{y}(\tau_{n+1}) = \boldsymbol{y}_n + \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_0(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_j(\boldsymbol{y}(s)) \circ \mathrm{d}W_j(s)$$

as functions on the set of multi-colored rooted trees.

At the first step we define the integral operator J_j as follows: for any integrable function H of $\boldsymbol{y}$ and $s > \tau_n$,

$$J_0[H](s) \stackrel{\mathrm{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1))\mathrm{d}s_1, \qquad J_j[H](s) \stackrel{\mathrm{def}}{=} \int_{\tau_n}^{s} H(\boldsymbol{y}(s_1)) \circ \mathrm{d}W_j(s_1) \quad (1 \leq j \leq m).$$

Suppose that $\boldsymbol{g}_j \in C_P^{2(q+1)}(\boldsymbol{R}^d, \boldsymbol{R})$ $(0 \leq j \leq m)$. Putting $\boldsymbol{z}(\boldsymbol{y}(s)) = \boldsymbol{y}(s) - \boldsymbol{y}_n$ as the increment of $\boldsymbol{y}$ from time 0 to s, we obtain the following formal series.

$$\begin{aligned} J_j[\boldsymbol{g}_j](s) \quad &= \quad J_j[\boldsymbol{g}_j(\boldsymbol{y}_n + \boldsymbol{z})](s) \\ &= \quad J_j\Bigg[\boldsymbol{g}_j(\boldsymbol{y}_n) + \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{z}] + \cdots + \frac{1}{(2q-1)!}\boldsymbol{g}_j^{(2q-1)}(\boldsymbol{y}_n)[\boldsymbol{z}, \ldots, \boldsymbol{z}] \\ &\quad + \frac{1}{(2q)!}\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\boldsymbol{z}, \ldots, \boldsymbol{z}]\Bigg](s) \\ &= \quad J_j[\boldsymbol{g}_j(\boldsymbol{y}_n)](s) + J_j[\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)[\boldsymbol{z}]](s) + \cdots \\ &\quad + \frac{1}{(2q-1)!}J_j[\boldsymbol{g}_j^{(2q-1)}(\boldsymbol{y}_n)[\boldsymbol{z}, \ldots, \boldsymbol{z}]](s) \\ &\quad + \frac{1}{(2q)!}J_j[\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\boldsymbol{z}, \ldots, \boldsymbol{z}]](s), \end{aligned} \qquad (2.\ 1)$$

where $0 < \theta_j < 1$.

By introducing the following notations for $k = 0, 1, \ldots$

$$\mathrm{P}_j^{(k)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k](s) = \frac{1}{k!} J_j[\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k]](s)$$

and

$$\mathrm{R}_j^{(2q)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{2q}](s) = \frac{1}{(2q)!} J_j[\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n + \theta_j \boldsymbol{z})[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{2q}]](s),$$

from Eqs. (1. 1) and (2. 1) we have

$$\begin{aligned} \boldsymbol{z}(\boldsymbol{y}(s)) \quad = \quad & \sum_{j=0}^{m} \mathrm{P}_j^{(0)}(s) + \sum_{j=0}^{m} \mathrm{P}_j^{(1)}[\boldsymbol{z}](s) + \cdots + \sum_{j=0}^{m} \mathrm{P}_j^{(2q-1)}[\boldsymbol{z}, \ldots, \boldsymbol{z}](s) \\ & + \sum_{j=0}^{m} \mathrm{R}_j^{(2q)}[\boldsymbol{z}, \ldots, \boldsymbol{z}](s). \end{aligned} \tag{2. 2}$$

Repeated application of (2. 2) implies the following formal expression for the increment.

$$\begin{aligned} \boldsymbol{y}(\tau_{n+1}) - \boldsymbol{y}_n \quad = \quad & \sum_{j=0}^{m} \mathrm{P}_j^{(0)}(\tau_{n+1}) + \sum_{j=0}^{m} \mathrm{P}_j^{(1)}[\sum_{l=0}^{m} \mathrm{P}_l^{(0)}](\tau_{n+1}) + \cdots \\ & + \sum_{j=0}^{m} \mathrm{P}_j^{(2q-1)}[\sum_{l=0}^{m} \mathrm{P}_l^{(0)}, \ldots, \sum_{l=0}^{m} \mathrm{P}_l^{(0)}](\tau_{n+1}) + \cdots. \end{aligned} \tag{2. 3}$$

The multilinearity of the operator $\mathrm{P}_j^{(k)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k](\tau_{n+1})$ can readily verify the above equations.

In the right-hand side of Eq. (2. 3), τ_{n+1} only stands for the upper bound of integral interval. In the sequel we omit this symbol from the equation as far as it does not cause a confusion.

Let N be a finite set of consecutive natural numbers, $\#S$ the cardinal number of a set S, and $V(N)$ the set of all possible partitions of N. That is, if $p \in V(N)$ and $p = \{p_1, \ldots, p_{\#p}\}$ hold, then $p_1, \ldots, p_{\#p}$ are non-empty pairwise-disjoint subsets of N, the equation $N = \bigcup_i p_i$ holds, and the elements of p_i are consecutive. For example,

$$\begin{aligned} N \quad &= \quad \{1, 2, 3\}, \\ V(N) \quad &= \quad \{\{N\}, \{\{1, 2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}\}. \end{aligned}$$

Let $\mathrm{Q}^{(n)}$ be the sum of all products of n terms of $\mathrm{P}_{\cdot}^{(\cdot)}$ appearing in the right-hand side of Eq. (2. 3). Then the following Lemma readily holds from (2. 3).

Lemma 2.1 $\mathrm{Q}^{(n)}$ *can be given recursively as follows:*

$$\mathrm{Q}^{(1)} = \sum_{j=0}^{m} \mathrm{P}_j^{(0)}, \qquad \mathrm{Q}^{(\#N)} = \sum_{p \in V(N)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)}[\mathrm{Q}^{(\#p_1)}, \ldots, \mathrm{Q}^{(\#p_{\#p})}],$$

where $1 \leq \#N \leq 2q$.

For a combinatorial description of the above expansion, we are to introduce multi-colored rooted trees.

Definition 2.1 (Multi-colored rooted tree (MRT)) *A multi-colored rooted tree with roots* $\textcircled{j}$ *(colored with a label* j *from* 0 *to* m*) is a tree recursively defined in the following way.*

1. $\tau^{(j)}$ *is the primitive tree having only a vertex* $\textcircled{j}$*.*
2. *If* $t_1, \ldots, t_k$ *are multi-colored trees, then* $[t_1, \ldots, t_k]^{(j)}$ *is also a multi-colored rooted tree with the root* $\textcircled{j}$*.*

The totality of multi-colored rooted trees is denoted by T*.*

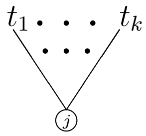


Figure 1: Generation of the tree $[t_1, \ldots, t_k]^{(j)}$

For an expression of the Stratonovich-Taylor expansion upon the multi-colored rooted trees, we introduce the following.

Definition 2.2 (Elementary integral $\mathbf{\Psi}(t)$ on T) *An elementary integral* $\mathbf{\Psi}(t)$ *for* $t \in T$ *is a function recursively given in the following.*

$$\mathbf{\Psi}(\tau^{(j)}) = \mathrm{P}_j^{(0)}, \qquad \mathbf{\Psi}(t) = \mathrm{P}_j^{(k)}\left[\mathbf{\Psi}(t_1), \ldots, \mathbf{\Psi}(t_k)\right] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

For the set N, denote by T_N the totality of MRTs with $\#N$ vertices numbered with the elements of N in the following way.

1. Along each outwardly directed arc the numbers increase.
2. Vertices of a subtree are consecutively numbered. This rule is also applied to subtrees of the subtree recursively.
3. Isomorphic trees are regarded to be identical.

Figure 2: Examples of trees in T_N

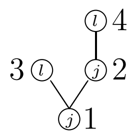


Figure 3: An example of trees not in T_N

For $u \in T_N$, $|\ u\ |$ stands for an MRT from which the numbers are removed.

Lemma 2.2 *For the set* N *satisfying* $\#N \leq 2q$ *the following equation holds.*

$$\mathrm{Q}^{(\#N)} = \sum_{u \in T_N} \mathbf{\Psi}(|\ u\ |).$$

Proof. We will prove it by a mathematical induction. When $\#N = 1$, Lemma 2.1 and Definition 2.2 imply

$$\mathrm{Q}^{(1)} = \sum_{j=0}^{m} \mathrm{P}_j^{(0)} = \sum_{j=0}^{m} \mathbf{\Psi}(\tau^{(j)}).$$

When N_0 is a finite set of consecutive natural numbers satisfying $\min(N_0) \geq 2$, suppose that the equation

$$\mathrm{Q}^{(\#N)} = \sum_{u \in T_N} \mathbf{\Psi}(|u|)$$

holds for any set N ($\subseteq N_0$) of consecutive natural numbers. Then, for $N = \{\min(N_0) - 1\} \cup N_0$, from Lemma 2.1 we can calculate as follows.

$$\begin{aligned}
\mathrm{Q}^{(\#N)} &= \mathrm{Q}^{(\#N_0+1)} \\
&= \sum_{p \in V(N_0)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)}[\mathrm{R}^{(\#p_1)}, \ldots, \mathrm{R}^{(\#p_{\#p})}] \\
&= \sum_{p \in V(N_0)} \sum_{j=0}^{m} \mathrm{P}_j^{(\#p)}[\sum_{u_1 \in T_{p_1}} \mathbf{\Psi}(|u_1|), \ldots, \sum_{u_{\#p} \in T_{p_{\#p}}} \mathbf{\Psi}(|u_{\#p}|)] \\
&= \sum_{u \in T_N} \mathbf{\Psi}(|u|),
\end{aligned}$$

where $u \in T_N$ whose root has the number $\min(N_0) - 1$, and which consists of subtrees $u_1, \ldots, u_{\#p}$. This completes the proof. □

Let $\nu(t)$ ($t \in T$) be the number of different ways of numbering on t. That is $\nu(t) = \#\{u \in T_N : |u| = t\}$. Furthermore denote by $\rho(t)$ the number of vertices of $t \in T$. From Lemma 2.2, we readily have the following.

Lemma 2.3 *The identity*

$$\mathrm{Q}^{(\#N)} = \sum_{\substack{\rho(t)=\#N \\ t \in T}} \nu(t)\mathbf{\Psi}(t) \qquad for \quad \#N \leq 2q$$

holds.

For any stochastic multiple integral x, let $\lambda(x)$ be the multiplicity of integrals with respect to a time variable or a Wiener process, and $\sigma(x)$ the multiplicity of integrals with respect to a time variable.

From Lemma 2.3, all terms x appearing in the expansion (2. 3) with $\lambda(x) \leq 2q$ can be expressed with $\mathbf{\Psi}(t)$. Actually, let $\boldsymbol{y}_{2q}(h)$ denote the truncated expansion of $\boldsymbol{y}(h)$ satisfying $\lambda(x) + \sigma(x) \leq 2q$. Then from Lemma 2.3, we have one of the main results.

Theorem 2.1 *The finitely truncated expansion has the following expression.*

$$\boldsymbol{y}_{2q}(\tau_{n+1}) = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(t)+r(t)=i \\ t \in T}} \nu(t)\mathbf{\Psi}(t),$$

where $r(t)$ is the number of vertices of t with the color 0.

Moreover, $\mathbf{\Psi}(t)$ has a more concise representation.

Definition 2.3 (Elementary weight $\Phi(t)$ on T) *An elementary weight of $t \in T$ is given recursively as follows.*

$$\Phi(\tau^{(j)}) = J_j[1], \qquad \Phi(t) = J_j\left[\prod_{i=1}^{k}\Phi(t_i)\right] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

As for the differential form of an SDE, we can introduce an elementary differential for $\boldsymbol{g}_j$ similarly to in ODEs.

Definition 2.4 (Elementary differential $\boldsymbol{F}(t)$ on T) *An elementary differential is a possibly multilinear operator recursively given as follows.*

$$\boldsymbol{F}(\tau^{(j)}) = \boldsymbol{g}_j(\boldsymbol{y}_n), \qquad \boldsymbol{F}(t) = \boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)] \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

Definition 2.5 (Elementary coefficient $\beta(t)$ on T) *The index $\beta(t)$ ($t \in T$) is defined recursively.*

$$\beta(\tau^{(j)}) = 1, \qquad \beta(t) = \frac{1}{k!}\prod_{i=1}^{k}\beta(t_i) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)}.$$

The following is the main goal of this section.

Theorem 2.2 *For any $t \in T$ we have the following identity.*

$$\boldsymbol{\Psi}(t) = \beta(t)\boldsymbol{F}(t)\Phi(t).$$

Proof. We carry out the proof by a mathematical induction. When $\rho(t) = 1$, a simple interpretation gives

$$\boldsymbol{\Psi}(\tau^{(j)}) = \mathrm{P}_j^{(0)} = J_j[\boldsymbol{g}_j(\boldsymbol{y}_n)] = \boldsymbol{g}_j(\boldsymbol{y}_n)J_j[1] = \beta(\tau^{(j)})\boldsymbol{F}(\tau^{(j)})\Phi(\tau^{(j)}).$$

Suppose that the statement is valid for all trees with $\rho(t) \le k'$. If t has a root colored with j such as $t = [t_1, \ldots, t_k]^{(j)}$ ($\rho(t) = k' + 1$), then the definition of the elementary integrals implies

$$\begin{aligned}
\boldsymbol{\Psi}(t) &= \mathrm{P}_j^{(k)}[\boldsymbol{\Psi}(t_1), \ldots, \boldsymbol{\Psi}(t_k)] \\
&= \frac{1}{k!}J_j\left[\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\beta(t_1)\boldsymbol{F}(t_1)\Phi(t_1), \ldots, \beta(t_k)\boldsymbol{F}(t_k)\Phi(t_k)]\right] \\
&= \frac{1}{k!}J_j\left[\prod_{i=1}^{k}\beta(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)]\prod_{i=1}^{k}\Phi(t_i)\right] \\
&= \frac{1}{k!}\prod_{i=1}^{k}\beta(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\boldsymbol{F}(t_1), \ldots, \boldsymbol{F}(t_k)] \cdot J_j\left[\prod_{i=1}^{k}\Phi(t_i)\right] \\
&= \beta(t)\boldsymbol{F}(t)\Phi(t).
\end{aligned}$$

Thus the statement holds in this case. □

In Theorem 2.1 and 2.2 we derived the expressions in relation to the Stratonovich-Taylor expansion upon the multi-colored rooted trees. With the use of integral forms from the begin of the derivation, we achieved this in a natural way. On the other hand, Burrage and Burrage [3] gave a slightly different expression of the expansion by modified a presentation derived on the basis of differential forms. Lastly, let us confirm both expressions of the expansion are equivalent.

For this, we need to seek an expression of $\nu(t)$ for $t \in T$. When $\rho(t) = 1$, it is trivial that $\nu(t) = 1$. Assume that $\rho(t) \geq 2$ and $t = [t_1, \ldots, t_k]^{(j)}$ and consider allocating the subsets of the set $\{2, 3, \ldots, \rho(t)\}$ to the subtrees $t_1, \ldots, t_k$ (1 is attached to the root). According to the numbering rules in T_N, we can allocate the subsets in the following way.

1. Choose a subtree t_i from among $t_1, \ldots, t_k$ and allocate the subset $\{2, 3, \ldots, \rho(t_i)+1\}$.
2. Choose another subtree $t_{i'}$ from among $t_1, \ldots, t_{i-1}, t_{i+1}, t_k$ and allocate the subset $\{\rho(t_i) + 2, \rho(t_i) + 3, \ldots, \rho(t_i) + \rho(t_{i'}) + 1\}$.
3. Repeat to allocate the rest of subsets to the rest of subtrees in a similar way.

The number of the way of allocating subsets is $k!$. When there is l subtrees that are not identical each other among $t_1, t_2, \ldots, t_k$, we express them by $t'_1, t'_2, \ldots, t'_l$ and denote by μ_i the number of the subtrees that are identical to t'_i. Then, we obtain

$$\nu(t) = \frac{k!}{\mu_1!\mu_2! \cdots \mu_l!} \prod_{i=1}^{k} \nu(t_i).$$

We can summarize the things mentioned above as a lemma.

Lemma 2.4 *The number $\nu(t)$ $(t \in T)$ is recursively given as follows.*

$$\nu(\tau^{(j)}) = 1, \qquad \nu(t) = \frac{k!}{\mu_1!\mu_2! \cdots \mu_l!} \prod_{i=1}^{k} \nu(t_i) \quad \text{for } t = [t_1, \ldots, t_k]^{(j)},$$

where l is the number of different subtrees and each μ_i $(i = 1, \ldots, l)$ is the number of identical subtrees to each different subtree.

Now we seek the coefficient of $\boldsymbol{F}(t)\Phi(t)$ in the Stratonovich-Taylor expansion. From Theorem 2.1 and 2.2 and Lemma 2.4 we can see that the coefficient by

$$\nu(t)\beta(t) = \frac{1}{\mu_1!\mu_2! \cdots \mu_l!} \prod_{i=1}^{k} \nu(t_i)\beta(t_i)$$

for $t = [t_1, \ldots, t_k]^{(j)}$. On the other hand, the counterpart in [3] is expressed by

$$\frac{\gamma(t)\alpha(t)}{\rho(t)!} = \frac{1}{\mu_1!\mu_2! \cdots \mu_l!} \prod_{i=1}^{k} \frac{\gamma(t_i)\alpha(t_i)}{\rho(t_i)!}.$$

Here $\alpha(t)$ and $\gamma(t)$ denote the number of possible different monotonic labellings of t with the root labelled first and the density of t, respectively (see [3] for details). Thus, taking into account $\nu(t)\beta(t) = \gamma(t)\alpha(t)/\rho(t)! = 1$ when $\rho(t) = 1$, we can see that

$$\nu(t)\beta(t) = \frac{\gamma(t)\alpha(t)}{\rho(t)!}$$

holds also when $\rho(t) \neq 1$. Therefore, it has been confirmed that the expressions of the expansion in the present paper and [3] are equivalent.

3 The Taylor expansion in a stochastic Runge-Kutta family

In order to obtain an approximate solution $\boldsymbol{y}_{n+1}$ of the solution $\boldsymbol{y}(t_{n+1})$ of (1. 1), we consider the stochastic Runge-Kutta family given by

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{l=1}^{s} c_l \boldsymbol{Y}_l, \\ \boldsymbol{Y}_i &= \sum_{j=0}^{m} \eta_i^{(j)} \left\{ b_i^{(j)} \boldsymbol{g}_j(\boldsymbol{y}_n + \sum_{l=1}^{s} \alpha_{il}^{(j)} \boldsymbol{Y}_l) + \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) \sum_{l=1}^{s} \gamma_{il}^{(j)} \boldsymbol{Y}_l \right\} \quad (i = 1, \ldots, s), \end{aligned} \tag{3. 1}$$

where each $\eta_i^{(j)}$ is a random variable independent of $\boldsymbol{y}_n$ and satisfies

$$E\left[\left(\eta_i^{(j)}\right)^{2k}\right] = \begin{cases} K_1 h^{2k} & (j = 0), \\ K_2 h^k & (j \neq 0) \end{cases}$$

for constants K_1, K_2 and $k = 1, 2, \ldots$. If $b_i^{(j)} \neq 0$, we can rewrite this in the following simpler form by setting $\tilde{\eta}_i^{(j)} \stackrel{\text{def}}{=} \eta_i^{(j)} b_i^{(j)}$ and $\tilde{\gamma}_{il}^{(j)} \stackrel{\text{def}}{=} \gamma_{il}^{(j)} / b_i^{(j)}$:

$$\begin{aligned} \boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + \sum_{l=1}^{s} c_l \boldsymbol{Y}_l, \\ \boldsymbol{Y}_i &= \sum_{j=0}^{m} \tilde{\eta}_i^{(j)} \left\{ \boldsymbol{g}_j(\boldsymbol{y}_n + \sum_{l=1}^{s} \alpha_{il}^{(j)} \boldsymbol{Y}_l) + \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) \sum_{l=1}^{s} \tilde{\gamma}_{il}^{(j)} \boldsymbol{Y}_l \right\} \qquad (i = 1, \ldots, s). \end{aligned} \tag{3. 2}$$

These formulations include stochastic Runge-Kutta and Rosenbrock-Wanner methods (see [3, 11] and also Appendix A).

In this section we deal with the simple formulation.

For a transparent analysis later on, we adopt nominal symbols $\boldsymbol{Y}_{s+1}$, $\tilde{\eta}_{s+1}^{(j)}$, $a_{s+1,l}^{(j)}$ and $\tilde{\gamma}_{s+1,l}^{(j)}$ and define $\alpha_{s+1,l}^{(0)}$ and $\boldsymbol{Y}_l$ by

$$\begin{aligned} \alpha_{s+1,l}^{(0)} &= c_l \qquad (l = 1, \ldots, s), \\ \boldsymbol{Y}_{s+1} &= \sum_{j=0}^{m} \tilde{\eta}_{s+1}^{(j)} \left\{ \boldsymbol{g}_j(\boldsymbol{y}_n + \sum_{l=1}^{s} \alpha_{s+1,l}^{(j)} \boldsymbol{Y}_l) + \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) \sum_{l=1}^{s} \tilde{\gamma}_{s+1,l}^{(j)} \boldsymbol{Y}_l \right\}. \end{aligned}$$

The Taylor-series expansion of $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{s+1}$ at $\boldsymbol{y}_n$ and their column-wise arrangements imply the following formal series.

$$\begin{aligned} &[\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_{s+1}] \\ &= \left[\sum_{j=0}^{m} \tilde{\eta}_1^{(j)} \boldsymbol{g}_j(\boldsymbol{y}_n), \ldots, \sum_{j=0}^{m} \tilde{\eta}_{s+1}^{(j)} \boldsymbol{g}_j(\boldsymbol{y}_n) \right] \\ &\quad + \left[\sum_{j=0}^{m} \tilde{\eta}_1^{(j)} \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) [\sum_{l=1}^{s} (\alpha_{1l} + \tilde{\gamma}_{1l}) \boldsymbol{Y}_l], \ldots, \sum_{j=0}^{m} \tilde{\eta}_{s+1}^{(j)} \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) [\sum_{l=1}^{s} (\alpha_{s+1,l} + \tilde{\gamma}_{s+1,l}) \boldsymbol{Y}_l] \right] \\ &\quad + \left[\sum_{j=0}^{m} \frac{\tilde{\eta}_1^{(j)}}{2} \boldsymbol{g}_j^{(2)}(\boldsymbol{y}_n) [\sum_{l=1}^{s} \alpha_{1l} \boldsymbol{Y}_l, \sum_{j=l}^{s} \alpha_{1l} \boldsymbol{Y}_l], \ldots, \frac{\tilde{\eta}_{s+1}^{(j)}}{2} \boldsymbol{g}_j^{(2)}(\boldsymbol{y}_n) [\sum_{l=1}^{s} \alpha_{s+1,l} \boldsymbol{Y}_l, \sum_{l=1}^{s} \alpha_{s+1,l} \boldsymbol{Y}_l] \right] \end{aligned}$$

$$+\cdots$$

$$+\left[\sum_{j=0}^{m}\frac{\tilde{\eta}_1^{(j)}}{(2q)!}\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n+\theta_j\sum_{l=1}^{s}\alpha_{1l}\boldsymbol{Y}_l)[\sum_{l=1}^{s}\alpha_{1l}\boldsymbol{Y}_l,\ldots,\sum_{l=1}^{s}\alpha_{1l}\boldsymbol{Y}_l],\ldots,\right.$$

$$\left.\sum_{j=0}^{m}\frac{\tilde{\eta}_{s+1}^{(j)}}{(2q)!}\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n+\theta_j\sum_{l=1}^{s}\alpha_{s+1,l}\boldsymbol{Y}_l)[\sum_{l=1}^{s}\alpha_{s+1,l}\boldsymbol{Y}_l,\ldots,\sum_{l=1}^{s}\alpha_{s+1,l}\boldsymbol{Y}_l]\right]. \qquad (3.\ 3)$$

For a simpler expression of Eq. (3. 3), we introduce some further symbols. First the overbar means the column-wise matrix composition like as follows. For a rectangular matrix $\boldsymbol{z}=[z_1,\ldots,z_{s+1}]$ of $(s+1)$ columns, the multilinear operator of k-th derivative of $\boldsymbol{g}_j$ induces

$$\bar{\boldsymbol{g}}_j^{(k)}[\underbrace{\boldsymbol{z},\ldots,\boldsymbol{z}}_{k\text{ times}}]=\left[\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\underbrace{z_1,\ldots,z_1}_{k\text{ times}}],\ldots,\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)[\underbrace{z_{s+1},\ldots,z_{s+1}}_{k\text{ times}}]\right],$$

and as the remainder term

$$\bar{\boldsymbol{r}}_j^{(2q)}[\underbrace{\boldsymbol{z},\ldots,\boldsymbol{z}}_{2q\text{ times}}] = \left[\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n+\theta_j\sum_{l=1}^{s}\alpha_{1l}\boldsymbol{Y}_l)[\underbrace{z_1,\ldots,z_1}_{2q\text{ times}}],\ldots,\right.$$

$$\left.\boldsymbol{g}_j^{(2q)}(\boldsymbol{y}_n+\theta_j\sum_{l=1}^{s}\alpha_{s+1,l}\boldsymbol{Y}_l)[\underbrace{z_{s+1},\ldots,z_{s+1}}_{2q\text{ times}}]\right].$$

Second, we introduce several matrices related to the formula parameters of the scheme (3. 1) as follows:

$$A^{(j)}=\begin{bmatrix}\alpha_{11}^{(j)} & \cdots & \alpha_{s+1,1}^{(j)}\\ \vdots & \ddots & \vdots\\ \alpha_{1s}^{(j)} & \cdots & \alpha_{s+1,s}^{(j)}\\ 0 & \cdots & 0\end{bmatrix},\qquad \tilde{A}^{(j)}=\begin{bmatrix}\alpha_{11}^{(j)}+\tilde{\gamma}_{11}^{(j)} & \cdots & \alpha_{s+1,1}^{(j)}+\tilde{\gamma}_{s+1,1}^{(j)}\\ \vdots & \ddots & \vdots\\ \alpha_{1s}^{(j)}+\tilde{\gamma}_{1s}^{(j)} & \cdots & \alpha_{s+1,s}^{(j)}+\tilde{\gamma}_{s+1,s}^{(j)}\\ 0 & \cdots & 0\end{bmatrix},$$

$$D^{(j)}=\begin{bmatrix}\tilde{\eta}_1^{(j)} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \tilde{\eta}_{s+1}^{(j)}\end{bmatrix}.$$

Finally let us denote the rectangular matrix composed with $\{\boldsymbol{Y}_i\}$ by

$$\bar{\boldsymbol{Y}}=[\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_{s+1}].$$

Applying these symbols, we can rewrite Eq. (3. 3) simply into the following:

$$\bar{\boldsymbol{Y}} = \sum_{j=0}^{m}\bar{\boldsymbol{g}}_j^{(0)}D^{(j)}+\sum_{j=0}^{m}\bar{\boldsymbol{g}}_j^{(1)}[\bar{\boldsymbol{Y}}\tilde{A}^{(j)}]D^{(j)}+\frac{1}{2}\sum_{j=0}^{m}\bar{\boldsymbol{g}}_j^{(2)}[\bar{\boldsymbol{Y}}A^{(j)},\bar{\boldsymbol{Y}}A^{(j)}]D^{(j)}+\cdots$$

$$+\frac{1}{(2q)!}\sum_{j=0}^{m}\bar{\boldsymbol{r}}_j^{(2q)}[\bar{\boldsymbol{Y}}A^{(j)},\ldots,\bar{\boldsymbol{Y}}A^{(j)}]D^{(j)}. \qquad (3.\ 4)$$

Assume that a matrix X stands for $A^{(\cdot)}$ or $\tilde{A}^{(\cdot)}$. Similarly to $\mathrm{P}_j^{(k)}$ and $\mathrm{R}_j^{(2q)}$ in the previous section, we adopt the following notations associated with X: for $k = 0, 1, \ldots$

$$\mathrm{P}_{X,j}^{(k)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k] = \frac{1}{k!}\bar{\boldsymbol{g}}_j^{(k)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k]D^{(j)}X$$

and

$$\mathrm{R}_{X,j}^{(2q)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{2q}] = \frac{1}{(2q)!}\bar{\boldsymbol{r}}_j^{(2q)}[\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{2q}]D^{(j)}X.$$

Eq. (3. 4) then yields

$$\begin{aligned}\bar{\boldsymbol{Y}}X &= \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(0)} + \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(1)}[\bar{\boldsymbol{Y}}\tilde{A}^{(j)}] + \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(2)}[\bar{\boldsymbol{Y}}A^{(j)}, \bar{\boldsymbol{Y}}A^{(j)}] + \cdots \\ &\quad + \sum_{j=0}^{m}\mathrm{R}_{j,X}^{(2q)}[\bar{\boldsymbol{Y}}A^{(j)}, \ldots, \bar{\boldsymbol{Y}}A^{(j)}]. \qquad (3.\ 5)\end{aligned}$$

Repeated substitution of (3. 5) into itself implies the following formal expression.

$$\begin{aligned}\bar{\boldsymbol{Y}}X &= \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(0)} + \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(1)}[\sum_{j_1=0}^{m}\mathrm{P}_{j_1,\tilde{A}^{(j)}}^{(0)}] + \cdots \\ &\quad + \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(2q-1)}[\sum_{j_1=0}^{m}\mathrm{P}_{j_1,A^{(j)}}^{(0)}, \ldots, \sum_{j_1=0}^{m}\mathrm{P}_{j_1,A^{(j)}}^{(0)}] + \cdots. \qquad (3.\ 6)\end{aligned}$$

The multilinearity of operator $\mathrm{P}_{j,X}^{(k)}[z_1, \ldots, z_k]$ can readily verifies the above equations. The right-hand side of Eq. (3. 6) has the following features.

(I) The case of $k \geq 2$. The arguments of $\mathrm{P}_{j,X}^{(k)}[\cdots]$ are always labeled by $A^{(j)}$. That is, it is formed by $\mathrm{P}_{\cdot,A^{(j)}}^{(\cdot)}[\cdots]$.

(II) The case of $k = 1$. The argument of $\mathrm{P}_{j,X}^{(1)}[\cdot]$ is always labeled by $\tilde{A}^{(j)}$. That is, it is formed by $\mathrm{P}_{\cdot,\tilde{A}^{(j)}}^{(\cdot)}[\cdots]$.

This observation suggests the introduction of rooted trees with labels $A^{(j)}$ or $\tilde{A}^{(j)}$ for a recursive construction of the series. Similarly to $\mathrm{Q}^{(n)}$ in the previous section, we denote by $\mathrm{Q}_X^{(n)}$ the sum of all products of n terms of $\mathrm{P}_{\cdot,\cdot}^{(\cdot)}$ in the right-hand side of Eq. (3. 6). Then the following Lemma readily holds from (3. 6).

Lemma 3.1 $\mathrm{Q}_X^{(n)}$ *can be given recursively as follows:*

$$\begin{aligned}\mathrm{Q}_X^{(1)} &= \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(0)}, \\ \mathrm{Q}_X^{(\#N+1)} &= \sum_{j=0}^{m}\mathrm{P}_{j,X}^{(1)}[\mathrm{Q}_{\tilde{A}^{(j)}}^{(\#N)}] + \sum_{p\in V(N)-\{N\}}\sum_{j=0}^{m}\mathrm{P}_{j,X}^{(\#p)}[\mathrm{Q}_{A^{(j)}}^{\#p_1}, \ldots, \mathrm{Q}_{A^{(j)}}^{\#p_{\#p}}]\end{aligned}$$

for $1 \leq \#N \leq 2q-1$, *where* X *stands for* $A^{(\cdot)}$ *or* $\tilde{A}^{(\cdot)}$.

Note that $\mathrm{Q}_X^{(n)}$ has the same structure as $\mathrm{Q}^{(n)}$ in Lemma 2.1 except with the labels.

Definition 3.1 (Multi-colored rooted tree with labels (MRTL)) *A multi-colored rooted tree with labels, denoted by t_X, is one attached by labels according to the following rule.*

(1) The label of the root is X.

(2) The label of other vertices is decided by the number of branches of the parent vertex:

i) When the parent vertex has more than one branches, the label is $A^{(j)}$ if the color bullet of the parent vertex is j.

ii) When the parent vertex has a unique branch, the label is $\tilde{A}^{(j)}$ if the color of the parent vertex is j.

The totality of multi-colored rooted tree with labels, whose label of the root is X, is denoted by $\mathcal{T}_X$. For example, some BRTLs are listed in Fig. 4.

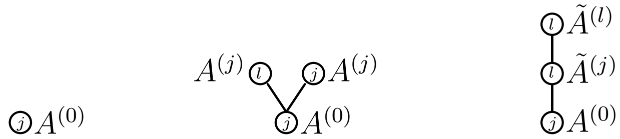


Figure 4: Examples of MRTL

Definition 3.2 (Elementary numerical integral $\bar{\boldsymbol{\Psi}}(t)$ on $\mathcal{T}_X$) *An elementary numerical integral corresponding to a MRTL is defined recursively as follows.*

$$\bar{\boldsymbol{\Psi}}(\tau_X^{(j)}) = \mathrm{P}_{j,X}^{(0)}, \qquad \bar{\boldsymbol{\Psi}}(t) = \mathrm{P}_{j,X}^{(k)}[\bar{\boldsymbol{\Psi}}(t_1), \ldots, \bar{\boldsymbol{\Psi}}(t_k)] \quad \text{for } t = [t_1, \ldots, t_k]_X^{(j)},$$

where $\tau_X^{(j)}$ and $[t_1, \ldots, t_k]_X^{(j)}$ express MRTLs whose roots are labeled by X.

From Definitions 2.2 and 3.2, we can see that both structures of $\bar{\boldsymbol{\Psi}}(t)$ and $\boldsymbol{\Psi}(t)$ are the same except with the labels as well as those of $Q_X^{(n)}$ and $Q^{(n)}$ are so. Furthermore, if we define $\hat{t}$ as an MRT obtained by removing all labels from $t \in \mathcal{T}_X$, $\Gamma : \mathcal{T}_X \ni t \mapsto \hat{t} \in T$ is a one to one correspondence from $\mathcal{T}_X$ onto T. Therefore, we can obtain the lemma bellow similar to Lemma 2.3.

Lemma 3.2 *The identity*

$$\mathrm{R}_X^{(\#N)} = \sum_{\substack{r(\hat{t})=\#N \\ t\in\mathcal{T}_X}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}(t) \qquad \text{for} \quad \#N \leq 2q$$

holds.

For any monomial x of $\tilde{\eta}_i^{(\cdot)}$, let $\bar{\lambda}(x)$ be the multiplicity of products with respect to $\tilde{\eta}_i^{(\cdot)}$, and $\bar{\sigma}(x)$ the multiplicity of products with respect to $\tilde{\eta}_i^{(\cdot)}$ except $\tilde{\eta}_i^{(0)}$.

From Lemma 3.2, all terms x appearing in the expansion of $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ with $\bar{\lambda}(x) \leq 2q$ can be expressed by $(s+1)$-st element $\bar{\boldsymbol{\Psi}}_{s+1}(t)$ of $\bar{\boldsymbol{\Psi}}(t)$.

Actually, let $\boldsymbol{y}_{n+1,2q}$ denote the truncated expansion of $\boldsymbol{y}_{n+1}$ satisfying $\bar{\lambda}(x) + \bar{\sigma}(x) \leq 2q$. One of our main results follows.

Theorem 3.1 *The finitely truncated expansion of the numerical solution by the stochastic Runge-Kutta family has the following expression.*

$$\boldsymbol{y}_{n+1,2q} = \boldsymbol{y}_n + \sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t\in\mathcal{T}_{A^{(0)}}}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}_{s+1}(t),$$

where $\bar{\boldsymbol{\Psi}}_{s+1}(t)$ *denotes the* $(s+1)$*-st element of* $\bar{\boldsymbol{\Psi}}(t)$*.*

The above theorem holds because $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ is equal to the $(s+1)$-st element of $\bar{\boldsymbol{Y}}A^{(0)}$, all terms of the expression of $\bar{\boldsymbol{Y}}A^{(0)}$ are started from $\mathrm{P}^{(\cdot)}_{A^{(0)},\cdot}$, and Lemma 3.2 holds.

Definition 3.3 (Elementary numerical weight $\bar{\Phi}(t)$ on $\mathcal{T}_X$) *An elementary weight of* $t \in \mathcal{T}_X$ *is given recursively as follows.*

$$\bar{\Phi}(\tau_X^{(j)}) = [1,\ldots,1]D^{(j)}X, \qquad \bar{\Phi}(t) = (\prod_{i=1}^{k} \bar{\Phi}(t_i))D^{(j)}X \quad \text{for } t = [t_1,\ldots,t_k]_X^{(j)},$$

where $\prod_{i=1}^{k}\bar{\Phi}(t_i)$ *means the elementwise product of row vectors* $\bar{\Phi}(t_i)$*.*

The following is the goal of this section.

Theorem 3.2 *For any* $t \in \mathcal{T}_X$ *we have the following identity.*

$$\bar{\boldsymbol{\Psi}}(t) = \beta(\hat{t})\boldsymbol{F}(\hat{t})\bar{\Phi}(t).$$

Proof. We carry out the proof by a mathematical induction as usual. When $\rho(t) = 1$, a simple interpretation gives

$$\begin{aligned}
\bar{\boldsymbol{\Psi}}(\tau_X^{(j)}) &= \mathrm{P}^{(0)}_{j,X} \\
&= \bar{\boldsymbol{g}}_j^{(0)} D^{(j)}X \\
&= [\boldsymbol{g}_j(\boldsymbol{y}_n),\ldots,\boldsymbol{g}_j(\boldsymbol{y}_n)]D^{(j)}X \\
&= \boldsymbol{g}_j(\boldsymbol{y}_n)[1,\ldots,1]D^{(j)}X \\
&= \beta(\tau^{(j)})\boldsymbol{F}(\tau^{(j)})\bar{\Phi}(\tau_X^{(j)}).
\end{aligned}$$

Suppose that the statement is valid if $r(\hat{t}) \le k'$, and let $\bar{\Phi}_i(t_j)$ denote the i-th element of $\bar{\Phi}(t_j)$. Then, for $t = [t_1,\ldots,t_k]_X^{(j)}$ $(\rho(t) = k'+1)$

$$\begin{aligned}
\bar{\boldsymbol{\Psi}}(t) &= \mathrm{P}^{(k)}_{j,X}[\bar{\boldsymbol{\Psi}}(t_1),\ldots,\bar{\boldsymbol{\Psi}}(t_k)] \\
&= \mathrm{P}^{(k)}_{j,X}\left[\beta(\hat{t}_1)\boldsymbol{F}(\hat{t}_1)\bar{\Phi}(t_1),\ldots,\beta(\hat{t}_k)\boldsymbol{F}(\hat{t}_k)\bar{\Phi}(t_k)\right] \\
&= \frac{1}{k!}\bar{\boldsymbol{g}}_j^{(k)}\left[\beta(\hat{t}_1)\boldsymbol{F}(\hat{t}_1)\bar{\Phi}(t_1),\ldots,\beta(\hat{t}_k)\boldsymbol{F}(\hat{t}_k)\bar{\Phi}(t_k)\right]D^{(j)}X \\
&= \frac{1}{k!}\left[\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)\left[\beta(\hat{t}_1)\boldsymbol{F}(\hat{t}_1)\bar{\Phi}_1(t_1),\ldots,\beta(\hat{t}_k)\boldsymbol{F}(\hat{t}_k)\bar{\Phi}_1(t_k)\right],\ldots,\right. \\
&\qquad \left.\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)\left[\beta(\hat{t}_1)\boldsymbol{F}(\hat{t}_1)\bar{\Phi}_{s+1}(t_1),\ldots,\beta(\hat{t}_k)\boldsymbol{F}(\hat{t}_k)\bar{\Phi}_{s+1}(t_k)\right]\right]D^{(j)}X \\
&= \frac{1}{k!}\left[\prod_{i=1}^{k}\beta(\hat{t}_i)\prod_{i=1}^{k}\bar{\Phi}_1(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)\left[\boldsymbol{F}(\hat{t}_1),\ldots,\boldsymbol{F}(\hat{t}_k)\right],\ldots,\right.
\end{aligned}$$

$$\prod_{i=1}^{k}\beta(\hat{t}_i)\prod_{i=1}^{k}\bar{\Phi}_{s+1}(t_i)\boldsymbol{g}_j^{(k)}(\boldsymbol{y}_n)\left[\boldsymbol{F}(\hat{t}_1),\ldots,\boldsymbol{F}(\hat{t}_k)\right]\Bigg] D^{(j)}X$$

$$= \frac{1}{k!}\prod_{i=1}^{k}\beta(\hat{t}_i)\boldsymbol{F}(\hat{t})\left[\prod_{i=1}^{k}\bar{\Phi}_1(t_i),\ldots,\prod_{i=1}^{k}\bar{\Phi}_{s+1}(t_i)\right] D^{(j)}X$$

$$= \beta(\hat{t})\boldsymbol{F}(\hat{t})\left[\prod_{i=1}^{k}\cdot\,\bar{\Phi}(t_i)\right] D^{(j)}X$$

$$= \beta(\hat{t})\boldsymbol{F}(\hat{t})\bar{\Phi}(t).$$

Therefore the statement holds, too. □

In Theorem 3.1 we obtained the Taylor expression of the approximate solution $\boldsymbol{y}_{n+1}$ in the same form as that in Theorem 2.1 except with labels. Furthermore, we showed that the elementary numerical integral $\bar{\boldsymbol{\Psi}}(t)$ can be decomposed into $\beta(t)$, $\boldsymbol{F}(t)$ and $\bar{\Phi}(t)$. This decomposition also has the same form as that in Theorem 2.2. In the next section it will be seen that these facts, that is, keeping the same form leads to an advantage to seek the order conditions of the stochastic Runge-Kutta family (3. 1).

4 Order conditions of the stochastic Runge-Kutta family

In this section we will show the transparent way of seeking the order conditions by utilizing the rooted tree analysis in Sections 2 and 3. Although the statement here is for the Runge-Kutta family (3. 1), the idea of the way can be extended for other families.

4.1 Order conditions

In relation to the weak order, Platen [10, 14] has presented an important theorem. In addition to some conditions to ensure the existence of moments of the solution and the regularity of the truncated Itô-Taylor approximation and another time discrete approximation, the theorem gives the condition that time discrete approximation should satisfy for high order convergence.

In the sequel by assuming that sufficient smoothness of g_j's and the regularity of the time discrete approximation, we devote ourselves to considering about the condition for high order convergence. The condition is written as follows: there exist constants $K < \infty$ and $r \in \{1, 2, \ldots\}$ independent of h such that for all $n = 0, \ldots, M-1$ and $(p_1, \ldots, p_l) \in \{1, \ldots, d\}^l$ $(1 \leq l \leq 2q+1)$,

$$\left| E\left[\prod_{j=1}^{l}(\boldsymbol{y}_{n+1}-\boldsymbol{y}_n)_{p_j} - \prod_{j=1}^{l}(\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1});\boldsymbol{y}_n))_{p_j} \middle| \mathcal{F}_n \right] \right| \leq K(1+\max_{0\leq k\leq n}|\boldsymbol{y}_k|^{2r})h^{q+1} \quad \text{(w.p.1).} \qquad (4.\ 1)$$

Here, $(\boldsymbol{z})_{p_j}$ and $\mathcal{F}_n$ denote, respectively, the p_j-th component of $\boldsymbol{z}$ and a non-anticipating sub-σ-algebra generated by the discretized Wiener process $\boldsymbol{W}(\tau_i)$ $(i = 0, \ldots, n)$. In addition, $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1});\boldsymbol{y}_n)$ denotes the Itô-Taylor expansion of $\boldsymbol{y}(\tau_{n+1})-\boldsymbol{y}_n$ centered at $\boldsymbol{y}_n$

and truncated up to the term x satisfying $l(x) \le q$, where $\boldsymbol{y}(\tau_{n+1})$ satisfies

$$\boldsymbol{y}(\tau_{n+1}) = \boldsymbol{y}_n + \int_{\tau_n}^{\tau_{n+1}} \left[\boldsymbol{g}_0(\boldsymbol{y}(s)) + \frac{1}{2}\sum_{j=1}^{m} \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}} \boldsymbol{g}_j(\boldsymbol{y}(s)) \right] \mathrm{d}s + \sum_{j=1}^{m} \int_{\tau_n}^{\tau_{n+1}} \boldsymbol{g}_j(\boldsymbol{y}(s))\mathrm{d}W_j(s)$$

with probability 1. If (4. 1) is satisfied, then the time discrete approximation $\boldsymbol{y}_M$ converges to the $\boldsymbol{y}(\tau_M)$ weakly with order q as $h \to 0$.

While (4. 1) is expressed by the Itô-Taylor expansion, the results in Section 2 are expressed by the Stratonovich-Taylor expansion. For this, let us rewrite (4. 1) in terms of the Stratonovich-Taylor expansion.

For $s > \tau_n$ and any natural number k, define the stochastic multiple integrals $J_{j_1 \cdots j_k}(s)$ and $I_{j_1 \cdots j_k}(s)$ by

$$J_{j_1 \cdots j_k}(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} \cdots \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{j_1}(s_1) \cdots \circ \mathrm{d}W_{j_k}(s_k)$$

and

$$I_{j_1 \cdots j_k}(s) \stackrel{\text{def}}{=} \int_{\tau_n}^{s} \cdots \int_{\tau_n}^{s_2} \mathrm{d}W_{j_1}(s_1) \cdots \mathrm{d}W_{j_k}(s_k),$$

respectively. The following relation holds between $J_{j_1 \cdots j_k}(s)$ and the Itô integral ([10], p. 173):

$$J_{j_1 \cdots j_k}(s) = \int_{\tau_n}^{s} J_{j_1 \cdots j_{k-1}}(s_k)\mathrm{d}W_{j_k}(s_k) + \frac{1}{2}\delta_{\{j_{k-1}=j_k \neq 0\}} \int_{\tau_n}^{s} J_{j_1 \cdots j_{k-2}}(s_k)\mathrm{d}s_{k-1}, \qquad (4.\ 2)$$

where $\delta_{\{j_{k-1}=j_k \neq 0\}} = 1$ if $j_{k-1} = j_k \neq 0$, or 0 otherwise. From the repeated application of this equation and the observation that $l(x) + n(x)$ is invariant for any term x in the both hand side, we can see that $J_{j_1 \cdots j_k}(s)$ is expressed by the sum of $I_{j_1 \cdots j_{k'}}(s)$ $(k' \le k)$ satisfying $k + n(J_{j_1 \cdots j_k}(s)) = k' + n(I_{j_1 \cdots j_{k'}}(s))$. This means that $J_{j_1 \cdots j_k}(s)$ satisfying $k + n(J_{j_1 \cdots j_k}(s)) > 2q$ can be expressed by the sum of $I_{j_1 \cdots j_{k'}}(s)$ with $k' > q$. From this, we can see that $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n$ includes all the term in $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$. Hence, we can replace $\boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$ in (4. 1) with $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n$ by noting that any term in $\boldsymbol{y}_{2q}(\tau_{n+1}) - \boldsymbol{y}_n - \boldsymbol{\iota}_q(\boldsymbol{y}(\tau_{n+1}); \boldsymbol{y}_n)$ does not prevent the inequality from holding.

Next, let us consider about the discrete approximate solution part in the inequality. For any term x satisfying $\lambda(x) + \nu(x) \ge 2q + 1$ in the expansion of $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$,

$$E[(\boldsymbol{x}, \boldsymbol{z})|\mathcal{F}_n] \le K_1 h^{q+1}$$

holds with probability 1, where $\boldsymbol{z}$ is any term in the expansion and K_1 is a constant. Hence, we can replace $\boldsymbol{y}_{n+1} - \boldsymbol{y}_n$ in (4. 1) with $\boldsymbol{y}_{n+1,2q} - \boldsymbol{y}_n$ by noting that any term in the expansion of $\boldsymbol{y}_{n+1} - \boldsymbol{y}_{n+1,2q}$ centered at $\boldsymbol{y}_n$ does not prevent the inequality from holding.

From the thing mentioned above and the results in Sections 2 and 3, we can rewrite the expression in the left-hand side of (4. 1) as follows.

$$\begin{aligned} & E\left[\prod_{j=1}^{l} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_A}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}_{s+1}(t)\Big)_{p_j} - \prod_{j=1}^{l} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in T}} \nu(t)\boldsymbol{\Psi}(t)\Big)_{p_j} \,\middle|\, \mathcal{F}_n \right] \\ = \; & E\left[\prod_{j=1}^{l} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_A}} \nu(\hat{t})\bar{\boldsymbol{\Psi}}_{s+1}(t)\Big)_{p_j} - \prod_{j=1}^{l} \Big(\sum_{i=1}^{2q} \sum_{\substack{\rho(\hat{t})+r(\hat{t})=i \\ t \in \mathcal{T}_A}} \nu(\hat{t})\boldsymbol{\Psi}(\hat{t})\Big)_{p_j} \,\middle|\, \mathcal{F}_n \right] \qquad \text{(w.p.1)} \end{aligned}$$

$$= \sum_{i_1=1}^{2q} \sum_{\substack{\rho(\hat{t}_1)+r(\hat{t}_1)=i_1 \\ t_1\in\mathcal{T}_A}} \cdots \sum_{i_l=1}^{2q} \sum_{\substack{\rho(\hat{t}_l)+r(\hat{t}_l)=i_l \\ t_l\in\mathcal{T}_A}} \prod_{j=1}^{l} (\nu(\hat{t}_j)\beta(\hat{t}_j)(F(\hat{t}_j))_{p_j})$$
$$\times E\left[\prod_{j=1}^{l}\bar{\Phi}_{s+1}(t_j) - \prod_{j=1}^{l}\Phi(\hat{t}_j)\middle|\mathcal{F}_n\right] \qquad \text{(w.p.1).}$$

Eventually, since $\tilde{\eta}_i^{(j)}$ is independent of $\boldsymbol{y}_n$, the inequality (4. 1) holds if the condition

$$E\left[\prod_{j=1}^{l}\bar{\Phi}_{s+1}(t_j)\right] = E\left[\prod_{j=1}^{l}\Phi(\hat{t}_j)\right] \tag{4. 3}$$

holds for any $t_1,\ldots,t_l \in \mathcal{T}_A$ satisfying $\sum_{j=1}^{l}(\rho(\hat{t}_j)+r(\hat{t}_j)) \le 2q$.

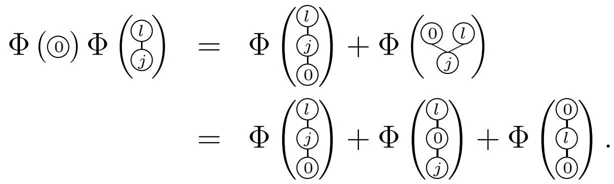
4.2 Calculation rules for elementary weights

In the next section we will give an example of seeking order conditions for weak order with the help of MRTs. As preliminaries we introduce some rules to make it easy to calculate elementary weights.

By means of the chain rule

$$J_{j_1\cdots j_k}(\tau_{n+1})J_{j_1\cdots j_{k'}}(\tau_{n+1})$$
$$= \int_{\tau_n}^{\tau_{n+1}} J_{j_1\cdots j_{k-1}}(s)J_{j_1\cdots j_{k'}}(s)\circ \mathrm{d}W_{j_k}(s) + \int_{\tau_n}^{\tau_{n+1}} J_{j_1\cdots j_k}(s)J_{j_1\cdots j_{k'-1}}(s)\circ \mathrm{d}W_{j_{k'}}(s),$$

we can express the product of elementary weights of some MRTs by the sum of elementary weights of other MRTs as the following example:



From the observation of the example we can say as follows about rewriting the product of elementary weights or the composition of subtrees in a elementary weight by the chain rule.

- The product of elementary weights of two MRTs t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to the root of t_2 and an MRT generated by grafting t_2 to the root of t_1.
- The elementary weight of an MRT having subtrees t_1, t_2 can be expressed by the sum of elementary weights of an MRT generated by grafting t_1 to the root of t_2 and an MRT generated by grafting t_2 to the root of t_1.

Hence, the product of elementary weight or the elementary weight of an MRT having a vertex with multi-branches can be expressed by the sum of MRTs whose each vertex has no more than one branch. Thus, next let us consider only this type of MRT.

For a multiple Stratonovich integral given by the elementary weight of the type of MRT, (4. 2) holds. The recursive application gives a representation of the multiple

Stratonovich integral in terms of the sum of multiple Itô integrals. Then, except the case that $j_1 = \cdots = j_k = 0$, only the second term in the right-hand side can yield a term having only integrals with respect to time. The condition for it is that the number of indices j_i's satisfying $j_i \neq 0$ is an even number and $j_i = j_{i+1}$ or $j_i = j_{i-1}$ holds for each index j_i. Finally, by noting that the expectation of any multiple Itô integral is 0 if it includes an integral with respect to a Wiener process, we can obtain the following rules for any MRT whose each vertex has no more than one branch:

- The expectation of elementary weight vanishes unless the number of vertices whose colors are different from 0 is even and each of these vertices has a parent or child vertex with the same color.
- The expectation of elementary weight of an MRT in which a vertex has a parent or child vertex with the same color is equal to a half of that of another MRT given by replacing the two vertices with one vertex with the color 0. For example,

$$E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ j \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}\right)\right].$$

4.3 The expectations of elementary weights and elementary numerical weights

First, we seek the expectations of elementary weights or the products of them for weak order 2 by utilizing the rules in the previous subsection. In the following we show only the expectations that does not vanish.

$$E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 4}$$

$$E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 5}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ j \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 6}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & \diagdown\,\diagup & \\ & 0 & \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ j \\ | \\ 0 \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 7}$$

$$E\left[\Phi\left(\begin{smallmatrix} 0 & & j \\ & \diagdown\,\diagup & \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ 0 \\ | \\ j \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 8}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ j \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{8}h^2. \tag{4. 9}$$

$$E\left[\Phi\left(\begin{smallmatrix} l \\ | \\ l \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(\begin{smallmatrix} 0 \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{8}h^2. \tag{4. 10}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & \diagdown\,\diagup & \\ & j & \\ & | & \\ & j & \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} j \\ | \\ j \\ | \\ j \\ | \\ j \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 11}$$

$$E\left[\Phi\left(\begin{smallmatrix} l & & l \\ & j & \\ & j & \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} l \\ l \\ j \\ j \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 12}$$

$$E\left[\Phi\left(\begin{smallmatrix} & & j \\ j & & j \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j \\ j \\ j \\ j \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & j & \\ & j & \end{smallmatrix}\right)\right] = \frac{3}{8}h^2. \tag{4. 13}$$

$$E\left[\Phi\left(\begin{smallmatrix} & & l \\ j & & l \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} l \\ l \\ j \\ j \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j & & l \\ & l & \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} l \\ l \\ j \\ j \end{smallmatrix}\right)\right] = \frac{1}{8}h^2. \tag{4. 14}$$

$$E\left[\Phi\left(\begin{smallmatrix} & & l \\ l & & j \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} l \\ j \\ l \\ j \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} l & & l \\ & j & \\ & j & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} l & & l \\ & j & \\ & j & \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 15}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & j & j \\ & j & \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} & & j \\ j & & j \\ & j & \end{smallmatrix}\right)\right] = \frac{3}{4}h^2. \tag{4. 16}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & l & l \\ & j & \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} & & l \\ j & & l \\ & j & \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 17}$$

$$E\left[\Phi\left(\begin{smallmatrix} 0 \\ j \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} 0 & & j \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} 0 \\ j \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 18}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ 0 \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & 0 & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j \\ 0 \\ j \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & 0 & \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 19}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ j \\ j \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} & & j \\ j & & j \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j \\ j \\ j \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 20}$$

$$E\left[\Phi\left(\begin{smallmatrix} l \\ j \\ j \end{smallmatrix}\right)\Phi\left(l\right)\right] = E\left[\Phi\left(\begin{smallmatrix} & & l \\ l & & j \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} l \\ j \\ j \\ l \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} & & l \\ l & & j \\ & j & \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 21}$$

$$E\left[\Phi\left(\begin{smallmatrix} l \\ l \\ j \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} & & l \\ j & & l \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} l \\ l \\ j \\ j \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 22}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & j & \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & j & j \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j & & j \\ & j & \\ & j & \end{smallmatrix}\right)\right] = h^2. \tag{4. 23}$$

$$E\left[\Phi\left(\begin{smallmatrix} j & & l \\ & j & \end{smallmatrix}\right)\Phi\left(l\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & l & l \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} j & & l \\ & j & \\ & l & \end{smallmatrix}\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & l & l \\ & j & \end{smallmatrix}\right)\right] = \frac{1}{4}h^2. \tag{4. 24}$$

$$E\left[\Phi\left(\begin{smallmatrix} l & & l \\ & j & \end{smallmatrix}\right)\Phi\left(j\right)\right] = E\left[\Phi\left(\begin{smallmatrix} j & l & l \\ & j & \end{smallmatrix}\right)\right] + E\left[\Phi\left(\begin{smallmatrix} l & & l \\ & j & \\ & j & \end{smallmatrix}\right)\right] = \frac{1}{2}h^2. \tag{4. 25}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ j \end{smallmatrix}\right)\right] = \frac{1}{2}E\left[\Phi\left(0\right)\right] = \frac{1}{2}h. \tag{4. 26}$$

$$E\left[\Phi\left(\begin{smallmatrix} j \\ j \end{smallmatrix}\right)\Phi\left(\begin{smallmatrix} j \\ j \end{smallmatrix}\right)\right] = 2E\left[\Phi\left(\begin{smallmatrix} & & j \\ j & & j \\ & j & \end{smallmatrix}\right)\right] = \frac{3}{4}h^2. \tag{4. 27}$$

$$E\left[\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right] = 2E\left[\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\right] = \frac{3}{2}h^2. \tag{4. 28}$$

$$E\left[\Phi\left(\substack{\textcircled{l}\\ | \\ \textcircled{j}}\right)\Phi\left(\substack{\textcircled{l}\\ | \\ \textcircled{j}}\right)\right] = 2E\left[\Phi\left(\substack{\quad\ \textcircled{l}\\ \quad\ | \\ \textcircled{l}\ \ \textcircled{j}\\ \diagdown\diagup \\ \textcircled{j}}\right)\right] = \frac{1}{2}h^2. \tag{4. 29}$$

$$E\left[\Phi\left(\substack{\textcircled{l}\\ | \\ \textcircled{j}}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{l}\right)\right] = E\left[\Phi\left(\substack{\textcircled{j}\ \ \textcircled{l}\\ \diagdown\diagup \\ \textcircled{j}}\right)\Phi\left(\textcircled{l}\right)\right] + E\left[\Phi\left(\substack{\textcircled{l}\\ | \\ \textcircled{j}\\ | \\ \textcircled{j}}\right)\Phi\left(\textcircled{l}\right)\right] = \frac{1}{2}h^2. \tag{4. 30}$$

$$E\left[\Phi\left(\textcircled{0}\right)\right] = h. \tag{4. 31}$$

$$E\left[\Phi\left(\textcircled{0}\right)\Phi\left(\textcircled{0}\right)\right] = 2E\left[\Phi\left(\substack{\textcircled{0}\\ | \\ \textcircled{0}}\right)\right] = h^2. \tag{4. 32}$$

$$E\left[\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right] = 2E\left[\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\right] = h. \tag{4. 33}$$

$$E\left[\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\Phi\left(\textcircled{j}\right)\right] = 4E\left[\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\Phi\left(\substack{\textcircled{j}\\ | \\ \textcircled{j}}\right)\right] = 3h^2. \tag{4. 34}$$

Next, we seek the expectations of the $(s+1)$-st elements of elementary numerical weights or the products of them for weak order 2. From the observation of calculations for some elementary numerical weights according to Definition 3.3, we can see that the $(s+1)$-st element of an elementary numerical weight can be obtained directly from a diagram for an MRTL by the following procedure.

- Trace vertices in the direction from the root to upper vertices.
- For the root vertex, prepare an index i and write down c_i. Then, write down $\tilde{\eta}_i^{(j)}$ if the color is j.
- For each vertex except the root, prepare a new index i_2 and write down $\alpha_{i_1 i_2}^{(j)}$ if the label is $A^{(j)}$, or $\alpha_{i_1 i_2}^{(j)} + \tilde{\gamma}_{i_1 i_2}^{(j)}$ if it is $\tilde{A}^{(j)}$, where i_1 means the index of the parent vertex. Then, write down $\tilde{\eta}_{i_2}^{(l)}$ if the color is l.
- Finally, sum over all values $(1, \ldots, s)$ of all indices.

For instance,

$$\bar{\Phi}_{s+1}\left(\substack{\textcircled{0}\tilde{A}^{(0)}\\ | \\ \textcircled{0}A^{(0)}}\right) = \sum_{i_1,i_2=1}^{s} c_{i_1}\tilde{\eta}_{i_1}^{(0)}\tilde{\alpha}_{i_1 i_2}^{(0)}\tilde{\eta}_{i_2}^{(0)}.$$

Now, let us suppose

$$E\left[\left(\tilde{\eta}_i^{(j)}\right)^3\left(\tilde{\eta}_{i'}^{(l)}\right)^k\right] = 0 \qquad (k = 0 \text{ or } 1, \quad j \neq l),$$
$$E\left[\tilde{\eta}_i^{(j)}\left(\tilde{\eta}_{i_1}^{(l_1)}\right)^{k_1}\left(\tilde{\eta}_{i_2}^{(l_2)}\right)^{k_2}\left(\tilde{\eta}_{i_3}^{(l_3)}\right)^{k_3}\right] = 0 \qquad (k_1 + k_2 + k_3 = 3, \quad j \neq l_1, l_2, l_3)$$

for $j \neq 0$. In the following we show only the expectations that does not vanish, of elementary weights or the products of them for weak order 2. For ease of notation, we use $\tilde{\alpha}_{i_1 i_2}^{(j)} \overset{\text{def}}{=} \alpha_{i_1 i_2}^{(j)} + \tilde{\gamma}_{i_1 i_2}^{(j)}$ and suppose $j, l \neq 0$ and $j \neq l$. In addition, we omit all indices and the range of values of all indices in all summations.

$$E\left[\bar{\Phi}_{s+1}\left(\substack{\textcircled{0}\tilde{A}^{(0)}\\ | \\ \textcircled{0}A^{(0)}}\right)\right] = \sum c_{i_1}\tilde{\alpha}_{i_1 i_2}^{(0)} E\left[\tilde{\eta}_{i_1}^{(0)}\tilde{\eta}_{i_2}^{(0)}\right]. \tag{4. 35}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{0}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(0)}_{i_3}\right]. \tag{4. 36}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(0)}\\ \textcircled{0}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(0)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}E\left[\tilde{\eta}^{(0)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\right]. \tag{4. 37}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(0)}\\ \textcircled{0}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(0)}_{i_2i_3}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(0)}_{i_2}\right]. \tag{4. 38}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(0)}\textcircled{j}\ \textcircled{j}A^{(0)}\\ \textcircled{0}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(0)}_{i_1i_2}\alpha^{(0)}_{i_1i_3}E\left[\tilde{\eta}^{(0)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\right]. \tag{4. 39}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{0}\ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(0)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\right]. \tag{4. 40}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 41}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}\tilde{\alpha}^{(l)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 42}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(l)}_{i_2i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 43}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(l)}_{i_2i_3}\tilde{\alpha}^{(l)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 44}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{j}\ \textcircled{j}A^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\alpha^{(j)}_{i_2i_3}\alpha^{(j)}_{i_2i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 45}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{l}\ \textcircled{l}A^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\alpha^{(j)}_{i_2i_3}\alpha^{(j)}_{i_2i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 46}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(l)}\textcircled{l}\ \textcircled{j}A^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\alpha^{(l)}_{i_2i_3}\alpha^{(l)}_{i_2i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 47}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ A^{(j)}\textcircled{j}\ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 48}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(l)}\\ A^{(j)}\textcircled{j}\ \textcircled{l}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\tilde{\alpha}^{(l)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 49}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ A^{(j)}\textcircled{l}\ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 50}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ A^{(j)}\textcircled{l}\ \textcircled{l}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\tilde{\alpha}^{(l)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 51}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\\ A^{(j)}\textcircled{j}\textcircled{j}\textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\alpha^{(j)}_{i_1i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \tag{4. 52}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\\ A^{(j)}\textcircled{j}\textcircled{l}\textcircled{l}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}\alpha^{(j)}_{i_1i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \tag{4. 53}$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{0}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(0)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\right]. \qquad (4.\ 54)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(0)}\\ \textcircled{0}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(0)}_{i_1i_2}c_{i_3}E\left[\tilde{\eta}^{(0)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\right]. \qquad (4.\ 55)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 56)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{l}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(j)}_{i_2i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \qquad (4.\ 57)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(l)}_{i_2i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 58)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{l}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}\tilde{\alpha}^{(l)}_{i_2i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \qquad (4.\ 59)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{j}\ \ \textcircled{j}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 60)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{j}\ \ \textcircled{l}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{l}A^{(0)}\right)\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \qquad (4.\ 61)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}A^{(j)}\textcircled{l}\ \ \textcircled{l}A^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\alpha^{(j)}_{i_1i_2}\alpha^{(j)}_{i_1i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 62)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\right]. \qquad (4.\ 63)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 64)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 65)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}\tilde{\alpha}^{(j)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \qquad (4.\ 66)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{j}\tilde{A}^{(l)}\\ \textcircled{l}A^{(0)}\end{pmatrix}\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}\tilde{\alpha}^{(l)}_{i_3i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(l)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 67)$$

$$E\left[\bar{\Phi}_{s+1}\begin{pmatrix}\textcircled{l}\tilde{A}^{(j)}\\ \textcircled{j}A^{(0)}\end{pmatrix}\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{l}A^{(0)}\right)\right]=\sum c_{i_1}\tilde{\alpha}^{(j)}_{i_1i_2}c_{i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(l)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(l)}_{i_4}\right]. \qquad (4.\ 68)$$

$$E\left[\bar{\Phi}_{s+1}\left(\textcircled{0}A^{(0)}\right)\right]=\sum c_{i_1}E\left[\tilde{\eta}^{(0)}_{i_1}\right]. \qquad (4.\ 69)$$

$$E\left[\bar{\Phi}_{s+1}\left(\textcircled{0}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{0}A^{(0)}\right)\right]=\sum c_{i_1}c_{i_2}E\left[\tilde{\eta}^{(0)}_{i_1}\tilde{\eta}^{(0)}_{i_2}\right]. \qquad (4.\ 70)$$

$$E\left[\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]=\sum c_{i_1}c_{i_2}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\right]. \qquad (4.\ 71)$$

$$E\left[\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\bar{\Phi}_{s+1}\left(\textcircled{j}A^{(0)}\right)\right]$$
$$=\sum c_{i_1}c_{i_2}c_{i_3}c_{i_4}E\left[\tilde{\eta}^{(j)}_{i_1}\tilde{\eta}^{(j)}_{i_2}\tilde{\eta}^{(j)}_{i_3}\tilde{\eta}^{(j)}_{i_4}\right]. \qquad (4.\ 72)$$

The expectations above are expressed with the notation in (3. 2). By substituting $\tilde{\eta}^{(j)}_i=\eta^{(j)}_i b^{(j)}_i$ and $\tilde{\gamma}^{(j)}_{il}=\gamma^{(j)}_{il}/b^{(j)}_i$, however, we can readily obtain the expressions of the expectations for (3. 1).

5 Conclusion

By extending the rooted tree analysis ([12]) for weak schemes in the scalar Wiener process case, in Subsection 4.1 we have obtained, with multi-colored rooted trees, the expression of the weak order conditions of the stochastic Runge-Kutta family in the multi-dimensional Wiener process case. In the expressions there appear the expectations of elementary weights and numerical elementary weights. Thus, we have shown the way of seeking the expectations directly from diagrams for MRTs or MRTLs in Subsections 4.2 and 4.3. As we have seen in these subsections, we can obtain weak order conditions for the stochastic Runge-Kutta family in quite a simple way by utilizing the multi-colored rooted trees.

Appendix

A An example of the stochastic Runge-Kutta method

We show an example of schemes belonging to the stochastic Runge-Kutta family (3. 1). This scheme is of weak order 2 for Stratonovich-type SDEs with a multi-dimensional Wiener process. In fact, the scheme is the counterpart of an explicit order 2 weak scheme for Itô-type SDEs, proposed by Platen ([10], p. 486).

For (1. 1), let us consider

$$\begin{aligned}
\boldsymbol{y}_{n+1} \;=\; & \boldsymbol{y}_n + \frac{2-m}{2}\sum_{j=1}^{m} \triangle W_j \boldsymbol{g}_j(\boldsymbol{y}_n) + \frac{h}{2}\boldsymbol{g}_0(\boldsymbol{y}_n) + \frac{h}{4}\sum_{j=1}^{m} \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)\boldsymbol{g}_j(\boldsymbol{y}_n) \\
& + \frac{1}{4}\sum_{j=1}^{m}\left\{\triangle W_j + ((\triangle W_j)^2 - h)\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}_+^j\right) \\
& + \frac{1}{4}\sum_{j=1}^{m}\left\{\triangle W_j - ((\triangle W_j)^2 - h)\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\bar{\boldsymbol{y}}_-^j\right) \\
& + \frac{1}{4}\sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}\left\{\triangle W_j + (\triangle W_j \triangle W_r + V_{r,j})\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\tilde{\boldsymbol{y}}_+^r\right) \\
& + \frac{1}{4}\sum_{j=1}^{m}\sum_{\substack{r=1\\ r\neq j}}^{m}\left\{\triangle W_j - (\triangle W_j \triangle W_r + V_{r,j})\Big/\sqrt{h}\right\}\boldsymbol{g}_j\left(\tilde{\boldsymbol{y}}_-^r\right) \\
& + \frac{h}{2}\boldsymbol{g}_0(\bar{\boldsymbol{y}}) + \frac{\sqrt{h}}{24}\sum_{j=1}^{m}\left[8\left\{\boldsymbol{g}_j\left(\bar{\boldsymbol{y}} + \frac{\sqrt{h}}{2}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right) - \boldsymbol{g}_j\left(\bar{\boldsymbol{y}} - \frac{\sqrt{h}}{2}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)\right\}\right. \\
& \left. - \left\{\boldsymbol{g}_j\left(\bar{\boldsymbol{y}} + \sqrt{h}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right) - \boldsymbol{g}\left(\bar{\boldsymbol{y}} - \sqrt{h}\boldsymbol{g}_j(\bar{\boldsymbol{y}})\right)\right\}\right] \qquad \text{(A. 1)}
\end{aligned}$$

with the intermediate variables

$$\begin{aligned}
\bar{\boldsymbol{y}} &= \boldsymbol{y}_n + h\Big(\boldsymbol{g}_0(\boldsymbol{y}_n) + \frac{1}{2}\sum_{j=1}^{m}\boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n)\boldsymbol{g}_j(\boldsymbol{y}_n)\Big) + \sum_{j=1}^{m}\triangle W_j \boldsymbol{g}_j(\boldsymbol{y}_n), \\
\bar{\boldsymbol{y}}_\pm^j &= \boldsymbol{y}_n + h\Big(\boldsymbol{g}_0(\boldsymbol{y}_n) + \frac{1}{2}\sum_{l=1}^{m}\boldsymbol{g}_l^{(1)}(\boldsymbol{y}_n)\boldsymbol{g}_l(\boldsymbol{y}_n)\Big) \pm \sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n), \quad \tilde{\boldsymbol{y}}_\pm^j = \boldsymbol{y}_n \pm \sqrt{h}\boldsymbol{g}_j(\boldsymbol{y}_n).
\end{aligned}$$

Here, $\triangle W_j$'s and $V_{r,j}$'s ($r \neq j$) are mutually independent random variables satisfying

$$E\left[\triangle W_j\right] = E\left[(\triangle W_j)^3\right] = E\left[(\triangle W_j)^5\right] = 0, \quad E\left[(\triangle W_j)^2\right] = h, \quad E\left[(\triangle W_j)^4\right] = 3h^2$$

and

$$E\left[V_{r,j}\right] = 0, \quad E\left[(V_{r,j})^2\right] = h^2, \quad E\left[V_{r,j}V_{j,r}\right] = -h^2,$$

respectively, and they are also independent of $\boldsymbol{y}_n$.

Because the last term in (A. 1) can be rewritten by $(h/4)\sum_{j=1}^{m}\boldsymbol{g}_j^{(1)}(\bar{\boldsymbol{y}})\boldsymbol{g}_j(\bar{\boldsymbol{y}}) + O(h^3)$, (A. 1) is equivalent to Platen's scheme for (1. 2) except the terms being of $O(h^3)$ at least. Consequently, (A. 1) is of weak order 2. (It should be noted that there exists a typographical error in the scheme in [10].)

In the rest of this paper we show that (A. 1) belongs to the stochastic Runge-Kutta family (3. 1).

For $1 \leq i \leq m+2$, we put some of random variables and parameters as follows and set the others at 0:

$$\tilde{\eta}_1^{(j)} = \triangle W_j \quad (j \neq 0), \qquad \tilde{\eta}_i^{(i-1)} = \sqrt{h} \quad (2 \leq i \leq m+1),$$
$$\tilde{\eta}_{m+2}^{(0)} = h, \qquad \eta_{m+2}^{(j)} \gamma_{m+2,j+1}^{(j)} = \frac{\sqrt{h}}{2} \quad (j \neq 0).$$

Then, since

$$\boldsymbol{Y}_1 = \sum_{j=1}^{m} \triangle W_j \boldsymbol{g}_j(\boldsymbol{y}_n), \qquad \boldsymbol{Y}_i = \sqrt{h} \boldsymbol{g}_{i-1}(\boldsymbol{y}_n) \quad (2 \leq i \leq m+1),$$
$$\boldsymbol{Y}_{m+2} = h \boldsymbol{g}_0(\boldsymbol{y}_n) + \frac{h}{2} \sum_{j=1}^{m} \boldsymbol{g}_j^{(1)}(\boldsymbol{y}_n) \boldsymbol{g}_j(\boldsymbol{y}_n),$$

we have

$$\bar{\boldsymbol{y}} = \boldsymbol{y}_n + \boldsymbol{Y}_1 + \boldsymbol{Y}_{m+2}, \quad \bar{\boldsymbol{y}}_{\pm}^{j} = \boldsymbol{y}_n + \boldsymbol{Y}_{m+2} \pm \boldsymbol{Y}_{j+1}, \quad \tilde{\boldsymbol{y}}_{\pm}^{j} = \boldsymbol{y}_n \pm \boldsymbol{Y}_{j+1}.$$

In the sequel we often utilize the equations above.

By setting

$$\tilde{\eta}_{m+3}^{(j)} = \triangle W_j + ((\triangle W_j)^2 - h) \big/ \sqrt{h}\,, \qquad \alpha_{m+3,j+1}^{(j)} = \alpha_{m+3,m+2}^{(j)} = 1,$$
$$\tilde{\eta}_{m+4}^{(j)} = \triangle W_j - ((\triangle W_j)^2 - h) \big/ \sqrt{h}\,, \qquad \alpha_{m+4,j+1}^{(j)} = -\alpha_{m+4,m+2}^{(j)} = -1$$

for $j \neq 0$, we have

$$\boldsymbol{Y}_{m+3} = \sum_{j=1}^{m} \left\{ \triangle W_j + ((\triangle W_j)^2 - h) \big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\bar{\boldsymbol{y}}_{+}^{j} \right),$$
$$\boldsymbol{Y}_{m+4} = \sum_{j=1}^{m} \left\{ \triangle W_j - ((\triangle W_j)^2 - h) \big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\bar{\boldsymbol{y}}_{-}^{j} \right).$$

Next, we set

$$\tilde{\eta}_i^{(j)} = \begin{cases} \triangle W_j + (\triangle W_j \triangle W_r + V_{r,j}) \big/ \sqrt{h} & (j \neq r), \\ \triangle W_j + (\triangle W_j \triangle W_m + V_{m,j}) \big/ \sqrt{h} & (j = r), \end{cases} \qquad \alpha_{i,m+1}^{(r)} = \alpha_{i,r+1}^{(j)} = 1 \quad (j \neq r)$$

for $i = m+4+r$ $(1 \leq r \leq m-1)$, and

$$\tilde{\eta}_i^{(j)} = \begin{cases} \triangle W_j - (\triangle W_j \triangle W_r + V_{r,j}) \big/ \sqrt{h} & (j \neq r), \\ \triangle W_j - (\triangle W_j \triangle W_m + V_{m,j}) \big/ \sqrt{h} & (j = r), \end{cases} \qquad \alpha_{i,m+1}^{(r)} = \alpha_{i,r+1}^{(j)} = -1 \quad (j \neq r)$$

for $i = 2m+3+r$ $(1 \leq r \leq m-1)$. Then, we obtain

$$\boldsymbol{Y}_i = \sum_{\substack{j=1 \\ j \neq r}}^{m} \left\{ \triangle W_j + (\triangle W_j \triangle W_r + V_{r,j}) \big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\tilde{\boldsymbol{y}}_{+}^{r} \right)$$
$$+ \left\{ \triangle W_r + (\triangle W_r \triangle W_m + V_{m,r}) \big/ \sqrt{h} \right\} \boldsymbol{g}_r \left(\tilde{\boldsymbol{y}}_{+}^{m} \right)$$

for $i = m + 4 + r$ $(1 \le r \le m-1)$, and

$$\boldsymbol{Y}_i = \sum_{\substack{j=1 \\ j \neq r}}^{m} \left\{ \triangle W_j - (\triangle W_j \triangle W_r + V_{r,j}) \Big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\tilde{\boldsymbol{y}}_-^r \right) + \left\{ \triangle W_r - (\triangle W_r \triangle W_m + V_{m,r}) \Big/ \sqrt{h} \right\} \boldsymbol{g}_r \left(\tilde{\boldsymbol{y}}_-^m \right)$$

for $i = 2m + 3 + r$ $(1 \le r \le m-1)$.

For $3m + 3 \le i \le 4m + 2$, we set

$$\tilde{\eta}_i^{(i-3m-2)} = \sqrt{h}, \qquad \alpha_{i1}^{(j)} = \alpha_{i,m+2}^{(j)} = 1,$$

and then have

$$\boldsymbol{Y}_i = \sqrt{h} \boldsymbol{g}_{i-3m-2}(\bar{\boldsymbol{y}}).$$

Next, we set

$$\tilde{\eta}_{4m+3}^{(j)} = \begin{cases} h & (j=0), \\ -\frac{\sqrt{h}}{12} & (j \neq 0), \end{cases} \qquad \begin{array}{l} \alpha_{4m+3,1}^{(0)} = \alpha_{4m+3,m+2}^{(0)} = 1, \\ \alpha_{4m+3,1}^{(j)} = \alpha_{4m+3,m+2}^{(j)} = \alpha_{4m+3,j+3m+2}^{(j)} = 1 \quad (j \neq 0) \end{array}$$

and

$$\tilde{\eta}_i^{(j)} = \sqrt{h}, \quad \alpha_{i,1}^{(j)} = \alpha_{i,m+2}^{(j)} = 1, \quad \alpha_{i,j+3m+2}^{(j)} = \begin{cases} -1 & (i = 4m+4), \\ 1/2 & (i = 4m+5), \\ -1/2 & (i = 4m+6) \end{cases}$$

for $4m + 4 \le i \le 4m + 6$ and $j \neq 0$. Then, we obtain

$$\boldsymbol{Y}_{4m+3} = h \boldsymbol{g}_0(\bar{\boldsymbol{y}}) - \frac{\sqrt{h}}{12} \sum_{j=1}^{m} \boldsymbol{g}_j(\bar{\boldsymbol{y}} + \boldsymbol{Y}_{j+3m+2}), \qquad \boldsymbol{Y}_{4m+4} = \sqrt{h} \sum_{j=1}^{m} \boldsymbol{g}_j(\bar{\boldsymbol{y}} - \boldsymbol{Y}_{j+3m+2}),$$

$$\boldsymbol{Y}_{4m+5} = \sqrt{h} \sum_{j=1}^{m} \boldsymbol{g}_j \left(\bar{\boldsymbol{y}} + \frac{1}{2} \boldsymbol{Y}_{j+3m+2} \right), \qquad \boldsymbol{Y}_{4m+6} = \sqrt{h} \sum_{j=1}^{m} \boldsymbol{g}_j \left(\bar{\boldsymbol{y}} - \frac{1}{2} \boldsymbol{Y}_{j+3m+2} \right).$$

Finally, we have (A. 1) by setting

$$c_1 = \frac{2-m}{2}, \quad c_{m+2} = \frac{1}{2}, \quad c_{m+3} = \cdots = c_{3m+2} = \frac{1}{4},$$
$$c_{4m+3} = \frac{1}{2}, \quad c_{4m+4} = \frac{1}{24}, \quad c_{4m+5} = -c_{4m+5} = \frac{1}{3}$$

and noting that $\sum_{r=1}^{m-1} \boldsymbol{Y}_{m+r+4}$ and $\sum_{r=1}^{m-1} \boldsymbol{Y}_{2m+r+3}$ can be rewritten by

$$\sum_{j=1}^{m} \sum_{\substack{r=1 \\ r \neq j}}^{m} \left\{ \triangle W_j + (\triangle W_j \triangle W_r + V_{r,j}) \Big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\tilde{\boldsymbol{y}}_+^r \right)$$

and

$$\sum_{j=1}^{m} \sum_{\substack{r=1 \\ r \neq j}}^{m} \left\{ \triangle W_j - (\triangle W_j \triangle W_r + V_{r,j}) \Big/ \sqrt{h} \right\} \boldsymbol{g}_j \left(\tilde{\boldsymbol{y}}_-^r \right),$$

respectively.

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