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Statistical Properties of the Weibull Cumulative Exposure Model

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Abstract

This article is aimed at the investigation of some properties of a Weibull cumulative exposure model on multiple-step step-stress accelerated life test data. Although the model includes a probabilistic idea of Miner's rule in order to express the effect of cumulative damage in fatigue, it is shown that only this is not sufficient to account for degradation of specimens and the shape parameter must be larger than 1. For a random variable obeying the model, its average and standard deviation are investigated on a various sets of parameter values. In addition, an example of the maximum likelihood estimation is given on an actual data set.

1 Introduction

In many industrial fields it is requested for lots of products to operate for a long period of time. Accompanied with that, it is very important to give reliability in relation to the lifetime of products. In such cases, however, life testing under a normal stress can lead to a lengthy procedure with expensive cost. As a means to cope with these problems, the study of accelerated life test (ALT) has been developed. The test makes it possible to quickly obtain information on the life distribution of products by inducing early failure with stronger stress than normal.

One important way in ALT is step-stress accelerated life test (SSALT). There are mainly two types of SSALTs, a simple SSALT and a multiple-step SSALT. In the simple SSALT there is a single change of stress during the test. Miller and Nelson [6] have shown optimum simple SSALT plans in an exponential cumulative exposure (CE) model. Xiong [11] have studied an exponential CE model with a threshold parameter in the simple SSALT. Park and Yum [10] have shown optimum modified simple SSALT plans in an exponential CE model, under the consideration that it is desirable to increase the stress at some finite rate. Lu and Rudy [4] have dealt with a Weibull step-stress model in the simple SSALT.

On the other hand, in the multiple-step SSALT there are changes of stress more than once. Yeo and Tang [12] have investigated a three-step SSALT in an exponential CE model. Khamis [3] has proposed an exponential CE model with k explanatory variables and investigated it on three-step SSALT data. McSorley, Lu and Li [5] have shown the properties of the maximum likelihood (ML) estimators of parameters in a Weibull CE model on three-step SSALT data. Nelson [8] has proposed an important idea, which gives a CE model for life as a function of constant stress from SSALT data. This is a probabilistic analog of Miner's rule [7], which is stated on a deterministic situation, and gives the basis of all models mentioned above. He has also performed the ML estimation in a Weibull CE model under the inverse power law on multiple-step SSALT data concerning time to breakdown of an electrical insulation. Hirose [1] has proposed a generalized model of this model, which has a threshold parameter.

As we have seen, there are many kinds of studies about SSALTs on the basis of the CE model and these provide significant understanding of step-stress models and SSALTs. However, the validity of such models is not necessarily clear. Then, in this article we devote ourselves to answering the following questions:

- Can the CE model really account for degradation of products?
- If so, what condition on the parameters is necessary for it?
- When a random variable obeys a Weibull CE model with a threshold parameter, how do its average and standard deviation behave under a condition?

In Section 2 we introduce a CE model with a threshold parameter, which is a generalization of the CE model provided by Nelson. In Section 3 we state about a Weibull law with an explanatory variable expressing stress. After giving a Weibull step-stress model with cumulative effect of exposure in Section 4, we analyze it in Section 5. In Section 6 we illustrate an example of the ML estimation on an actual data set and lastly give the conclusion.

2 Cumulative Exposure Model

We construct a generalized CE model with the help of the CE model proposed by Nelson [8, 9], whose model gives the distribution function of a random variable for failure time. Although the general model is obtained in a similar way to Nelson's, it differs in having a threshold parameter that decides whether a specimen is influenced by stress or not. Also in [1], a similar formulation has been given, provided that the step stress at the present time is not lower than that at the past time. Note that there is not the limitation in the following.

The assumptions to obtain the CE model have been given by Nelson as follows:

i) The remaining life of specimens depends only on the current cumulative fraction accumulated.

ii) If held at the current stress, survivors will fail according to the distribution function for that stress but starting at the previously accumulated fraction failed.

Using a distribution function F of a non-negative random variable with an explanatory variable V and a threshold V_{th}, we construct the distribution function G of a random variable T for failure time in a sequential way. Denote by V_i a stress that a specimen is subjected to in an interval $(t_{i-1}, t_i]$ (i=1,2,...).

First of all, we define G by

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_0; V_1) & (V_1 > V_{th}), \\ 0 & (V_1 \leq V_{th}) \end{cases}$$

for $t_0 \leq t \leq t_1$.

Next, for $t_1 < t \leq t_2$ we define

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_1 + s_1; V_2) & (V_2 > V_{th}), \\ F(s_1; V_2) & (V_2 \leq V_{th}). \end{cases}$$

Here, according to Assumption ii), s_1 is a positive value that satisfies $G(t_1) = F(s_1; V_2)$.

Similarly, for $t_{i-1} < t \leq t_i$ we define

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_{i-1} + s_{i-1}; V_i) & (V_i > V_{th}), \\ F(s_{i-1}; V_i) & (V_i \leq V_{th}), \end{cases}$$

where s_{i-1} is a positive value that satisfies $G(t_{i-1}) = F(s_{i-1}; V_i)$.

3 A Weibull Law with an Explanatory Variable

We are related with the Weibull law in that the underlying distribution is the two parameter Weibull and the scale parameter is replaced with the function of an explanatory variable. For a random variable subjected to the law, we show the relationship between the function and the average of the natural logarithm of the random variable.

Let V be an explanatory variable and a function of it $\phi(V)$. When the scale parameter is replaced with $\phi(V)$ in the Weibull distribution function, the distribution is given by

$$W(t; V) = 1 - \exp\left[-\left(\frac{t}{\phi(V)} \right)^{\beta} \right].$$

Assume that a random variable T obeys the distribution and define that $U \stackrel{\text{def}}{=} \ln T$. Then, the probability that $U \leq u$ for a value u is

$$P(U \leq u) = 1 - \exp\left[-\left(\frac{\mathrm{e}^u}{\phi(V)}\right)^\beta\right].$$

By denoting the right-hand by $H(u)$ and differentiating this twice, we obtain

$$\frac{\mathrm{d}^2 H}{\mathrm{d}u^2} = \beta\left\{1 - \left(\frac{\mathrm{e}^u}{\phi(V)}\right)^\beta\right\}\frac{\mathrm{d}H}{\mathrm{d}u}.$$

This shows that $\dfrac{\mathrm{d}^2 H}{\mathrm{d}u^2} = 0$ holds when $u = \ln\phi(V)$. Thus, we can see that observations for U will be scattered around $u = \ln\phi(V)$ at the highest frequency. That is, $\ln\phi(V)$ is the mode. On the other hand, the expectation $E[U]$ of U is expressed by

$$E[U] = -\frac{\gamma}{\beta} + \ln\phi(V),$$

where $\gamma(= 0.57721\cdots)$ is Euler's constant. The equation above indicates that we can regard $\ln\phi(V)$ nearly as the average when $\gamma/\beta \ll |\ln\phi(V)|$.

When an explanatory variable V means stress, ϕ is usually a monotone decreasing function of V because there is a tendency that life time becomes shorter as stress becomes stronger in general. In the sequel we assume the inverse power law in ϕ for $V > V_{th}$:

$$\phi(V) = \frac{K}{(V - V_{th})^n}, \tag{3. 1}$$

where K and n are positive parameters and V_{th} is a non-negative parameter.

4 Step-stress Model

We deal with a multiple-step SSALT under the condition that specimens were subjected to a normal level of stress and did not fail before the test. As we will see in the next section, this setting has a possibility of throwing light on new aspects concerning step-stress models. In this section, first we introduce the multiple-step SSALT and secondly we give a step-stress model under the condition.

4.1 Multiple-step SSALT

During the multiple-step SSALT, specimens are subjected to successively higher levels of stress as follows. After a specimen was used at a normal level of stress, it is subjected to an initial level of stress for a predetermined time interval at the first stage in the test. If it does not fail, it is subjected to a higher level of stress for a predetermined time interval at the next stage. In analogy, it is repeatedly subjected to a higher level of stress until it fails. The other specimens are tested similarly. The pattern of stress levels and time intervals is the same for all specimens.

4.2 A Weibull Step-stress Model with Cumulative Effect of Exposure

We construct a step-stress model by combining the CE model in Section 2 and the Weibull law in Section 3. In addition, we give the log likelihood function in the case when the step-stress data are given under the condition mentioned above.

Denote by V_s and T_s a normal level of stress and the length of the time interval during that a specimen is used before the test, respectively. In addition, denote by V_i the stress that a specimen is subjected to at the $(i-1)$-st stage in the test, and let be t_{i-1} the start time of the stage $(i = 2, 3, 4, \ldots)$. Because the level of stress becomes higher as the stage proceeds to a higher level in the test, the relationship $V_i < V_j$ holds when $2 \leq i < j$.

If we consider a case including the specimens that fail before the test, in the $(i-1)$-st stage the cumulative distribution function G in the step-stress model is given as follows: if $V_s > V_{th}$ and

$$V_2 < V_3 < \cdots < V_{k-1} \leq V_{th} < V_k < \cdots < V_{i-1} < V_i,$$

$$G(t) = 1 - \exp\left[-\varepsilon^{\beta}(t)\right], \qquad t_{i-1} < t \leq t_i, \tag{4. 1}$$

where

$$\varepsilon(t) \stackrel{\text{def}}{=} \frac{T_s}{\phi(V_s)} + \frac{t_k - t_{k-1}}{\phi(V_k)} + \cdots + \frac{t_{i-1} - t_{i-2}}{\phi(V_{i-1})} + \frac{t - t_{i-1}}{\phi(V_i)}. \tag{4. 2}$$

Here, note that t_1 is the start time of the test and t_0 is set at $t_1 - T_s$.

From (4. 1) we obtain the cumulative distribution function under the condition that a specimen was subjected to a normal level of stress and did not fail before the test:

$$G(t|t > t_1) = \frac{\{G(t) - G(t_1)\} I_{\{t>t_1\}}(t)}{1 - G(t_1)}, \tag{4. 3}$$

where

$$I_{\{t>t_1\}}(t) \stackrel{\text{def}}{=} \begin{cases} 1 & (t > t_1), \\ 0 & (t \leq t_1). \end{cases}$$

Thus, the log likelihood function $\ln L$ under the condition in the model is expressed by the following: if we denote by N and l the sample size and the level of the stage at which a specimen fails, and use superscript (j) to show that a variable is related to the j-th specimen,

$$\ln L = \sum_{j=1}^{N} \ln\left\{\exp\left(-\varepsilon^{\beta}(t_{l-1}^{(j)}; T_s^{(j)})\right) - \exp\left(-\varepsilon^{\beta}(t_l^{(j)}; T_s^{(j)})\right)\right\} + \sum_{j=1}^{N} \varepsilon^{\beta}(t_1^{(j)}; T_s^{(j)}), \tag{4. 4}$$

where we express $\varepsilon(t)$ by $\varepsilon(t; T_s)$ in order to show clearly that each specimen has each T_s.

5 Statistical Properties

We consider the statistical properties of the model under the condition mentioned in the last section. First we state the role of the shape parameter β in the distribution function (4. 3) and secondly we investigate the relationship between statistical quantities and the values of parameters after we simplify the model without loss of generality. In the sequel we express $G(t|t > t_1)$ by $G(t|t > t_1; T_s)$ when it is necessary to show clearly the length of the time interval during that a specimen is used before the test. In analogy, we express $G(t|t > t_1)$ by $G(t|t > t_1; V_s)$ when it is necessary to show clearly the normal stress that a specimen is subjected to before the test. Because we are interested in elapsed time from the start time of test, in the sequel we suppose that t_1 is the base point in time. That is, we may consider t_1 equal to 0.

Depending on the magnitude of β, the distribution function has a different aspect as follows.

Theorem 5.1 *Assume that $T_{s1} < T_{s2}$. Then, the following holds for $t > t_1$.*

i) If $0 < \beta < 1$,

$$G(t|t > t_1; T_{s1}) > G(t|t > t_1; T_{s2}).$$

ii) If $\beta = 1$,

$$G(t|t > t_1; T_{s1}) = G(t|t > t_1; T_{s2}).$$

iii) If $\beta > 1$,

$$G(t|t > t_1; T_{s1}) < G(t|t > t_1; T_{s2}).$$

■

Proof. The substitutions of (4. 1), $\varepsilon(t_1) = T_s/\phi(V_s)$ and (4. 2) into (4. 3) yield

$$G(t|t > t_1; T_s) = 1 - \exp\left[\left\{\frac{T_s}{\phi(V_s)}\right\}^\beta - \left\{\frac{T_s}{\phi(V_s)} + \sum_{l=k}^{i-1}\frac{t_l - t_{l-1}}{\phi(V_l)} + \frac{t - t_{i-1}}{\phi(V_i)}\right\}^\beta\right].$$

By differentiating this with respect to T_s and arranging it, we find

$$\frac{\partial G(t|t > t_1; T_s)}{\partial T_s} = \frac{\beta}{\phi(V_s)}\left[\varepsilon^{\beta-1}(t) - \varepsilon^{\beta-1}(t_1)\right]\exp\left[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)\right].$$

Noting $\varepsilon(t) > \varepsilon(t_1)$, we can see

i) if $0 < \beta < 1$,
$G(t|t > t_1; T_s)$ is a strictly decreasing function of T_s since $\partial G(t|t > t_1; T_s)/\partial T_s < 0$,

ii) if $\beta = 1$,
$G(t|t > t_1; T_s)$ does not depend on T_s since $\partial G(t|t > t_1; T_s)/\partial T_s = 0$,

iii) if $\beta > 1$,
$G(t|t > t_1; T_s)$ is a strictly increasing function of T_s since $\partial G(t|t > t_1; T_s)/\partial T_s > 0$.

This completes the proof. □

The statement i) in the theorem means that specimens become more durable as they are used longer before the test. This is clearly irrational. Thus, in this sense every value in $(0, 1)$ is inadmissible for β. The statement ii) deals with a situation when the underlying distribution is exponential. It indicates that the step-stress model inherits the memoryless property from the exponential distribution. The statement iii) expresses the most realistic situation, in which the durability of specimens decreases as the duration of their use becomes longer before the test. Note that these things hold even in different ϕ's from (3. 1).

In a similar fashion, we can obtain the following theorem.

Theorem 5.2 *Assume that* $(V_{th} <)V_{s1} < V_{s2}$*. Then, the following holds for* $t > t_1$*.*

i) If $0 < \beta < 1$*,*

$$G(t|t > t_1; V_{s1}) > G(t|t > t_1; V_{s2}).$$

ii) If $\beta = 1$*,*

$$G(t|t > t_1; V_{s1}) = G(t|t > t_1; V_{s2}).$$

iii) If $\beta > 1$*,*

$$G(t|t > t_1; V_{s1}) < G(t|t > t_1; V_{s2}).$$

■

From this theorem, we can know a similar fact to Theorem 5.1. Especially, note that the statement iii) expresses the most realistic situation, in which the durability of specimens decreases as the normal stress imposed before the test becomes higher.

Next we assume that the length of the time interval and the breadth of upsurge of stress are constant in the test. That is, we set

$$\Delta t \overset{\text{def}}{=} t_i - t_{i-1}, \quad \Delta V \overset{\text{def}}{=} V_{i+1} - V_i \quad (i = 2, 3, \ldots) \quad \text{and} \quad V_2 = \Delta V.$$

Let us simplify (4. 2) and seek the expectation and second moment of a random variable obeying (4. 3).

By using the above constants and rewriting (3. 1) and (4. 2), we can obtain

$$\varepsilon(t) = \frac{\Delta t}{\phi(V_s)}\tilde{T}_s + \frac{\Delta t}{\phi(V_k)} + \cdots + \frac{\Delta t}{\phi(V_{i-1})} + \frac{\Delta t}{\phi(V_i)}\frac{t - t_{i-1}}{\Delta t}, \tag{5. 1}$$

$$\frac{\Delta t}{\phi(V_s)} = \frac{(1 - \tilde{V}_{th})^n}{\tilde{K}}, \quad \frac{\Delta t}{\phi(V_m)} = \frac{((m-1)\Delta\tilde{V} - \tilde{V}_{th})^n}{\tilde{K}} \quad (m = k, k+1, \ldots, i), \tag{5. 2}$$

where

$$\tilde{T}_s \overset{\text{def}}{=} \frac{T_s}{\Delta t}, \quad \tilde{K} \overset{\text{def}}{=} \frac{K}{(\Delta t V_s)^n}, \quad \Delta\tilde{V} \overset{\text{def}}{=} \frac{\Delta V}{V_s}, \quad \tilde{V}_{th} \overset{\text{def}}{=} \frac{V_{th}}{V_s}$$

and $(k-2)\Delta\tilde{V} \leq \tilde{V}_{th} < (k-1)\Delta\tilde{V}$ holds. These expressions indicate that we can take Δt and V_s as a unit of time and a unit of stress, respectively. In addition, we can suppose that $0 \leq \tilde{V}_{th} < 1$ when we deal with the case that $V_s > V_{th}$.

By means of a similar procedure and the arrangement of expressions, we can obtain another $\varepsilon(t)$ for a different stress V_s', say $\varepsilon'(t)$, in the following form:

$$\varepsilon'(t) = \frac{\Delta t}{\phi(V_s')}\tilde{T}_s + \frac{\Delta t}{\phi(V_k)} + \cdots + \frac{\Delta t}{\phi(V_{i-1})} + \frac{\Delta t}{\phi(V_i)}\frac{t - t_{i-1}}{\Delta t},$$

where

$$\frac{\Delta t}{\phi(V_s')} = \frac{((V_s'/V_s) - \tilde{V}_{th})^n}{\tilde{K}}.$$

Note that only the first terms in the right-hand sides differ in the expressions of $\varepsilon(t)$ and $\varepsilon'(t)$. Thus, once we obtain the values of parameters, we can decide the distribution function in the case of another stress V_s' ($> V_s$) by replacing only the first term in the right-hand side of (5. 1).

Let us seek the expectation of a random variable T obeying (4. 3). We first seek the following conditional expectation as preliminaries: for $n > k$,

$$
\begin{aligned}
E[T|T \le t_n] &= \sum_{i=k}^{n} \int_{t_{i-1}}^{t_i} t \frac{\partial}{\partial t} G(t|t > t_1)\mathrm{d}t \\
&= \sum_{i=k}^{n} \left[\Big[tG(t|t > t_1) \Big]_{t_{i-1}}^{t_i} - \int_{t_{i-1}}^{t_i} G(t|t > t_1)\mathrm{d}t \right] \\
&= -t_n \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t_n)] + t_{k-1} + \sum_{i=k}^{n} \int_{t_{i-1}}^{t_i} \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)]\mathrm{d}t.
\end{aligned}
$$

In the last equation the relationship $\varepsilon(t_1) = \varepsilon(t_{k-1})$ is used.

When we denote by m a positive integer such that $1/\phi(V_i) < 1$ holds for any $i > m$, we can see that

$$
\begin{aligned}
t_n \exp[-\varepsilon^\beta(t_n)] &= (n-1)\Delta t \exp\left[-\left\{ \frac{\Delta t}{\phi(V_s)} \tilde{T}_s + \sum_{i=k}^{n} \frac{\Delta t}{\phi(V_i)} \right\}^\beta \right] \\
&< (n-1)\Delta t \exp\left[-\left\{ \frac{\Delta t}{\phi(V_s)} \tilde{T}_s + \sum_{i=k}^{m} \frac{\Delta t}{\phi(V_i)} + (n-m)\Delta t \right\}^\beta \right]
\end{aligned}
$$

and the right-hand side converges to 0 as $n \to \infty$.

From the things above and $E[T] = \lim_{n\to\infty} E[T|T \le t_n]$, we obtain

$$
\begin{aligned}
\frac{E[T]}{\Delta t} &= \frac{t_{k-1}}{\Delta t} + \frac{1}{\Delta t} \sum_{i=k}^{\infty} \int_{t_{i-1}}^{t_i} \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)]\mathrm{d}t \\
&= \frac{t_{k-1}}{\Delta t} + \frac{1}{\beta} \sum_{i=k}^{\infty} \frac{\phi(V_l)}{\Delta t} \Big\{ -A(t_i) + A(t_{i-1}) \Big\} \qquad (5.\ 3)
\end{aligned}
$$

as the expectation in the case that Δt is used as a unit of time. Here,

$$
A(t) \stackrel{\mathrm{def}}{=} \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)] \int_0^\infty \Big\{ u + \varepsilon^\beta(t) \Big\}^{1/\beta - 1} \mathrm{e}^{-u}\mathrm{d}u.
$$

Eq. (5. 3) is useful for stable numerical calculations when T_s takes a large value.

In a similar fashion we obtain

$$
\begin{aligned}
\frac{E[T^2]}{(\Delta t)^2} &= \left(\frac{t_{k-1}}{\Delta t} \right)^2 + \frac{2}{(\Delta t)^2} \sum_{i=k}^{\infty} \int_{t_{i-1}}^{t_i} t \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)]\mathrm{d}t \\
&= \left(\frac{t_{k-1}}{\Delta t} \right)^2 + \frac{2}{\beta} \sum_{i=k}^{\infty} \left(\frac{\phi(V_l)}{\Delta t} \right)^2 \Big\{ -B_i(t_i) + B_i(t_{i-1}) \Big\} \qquad (5.\ 4)
\end{aligned}
$$

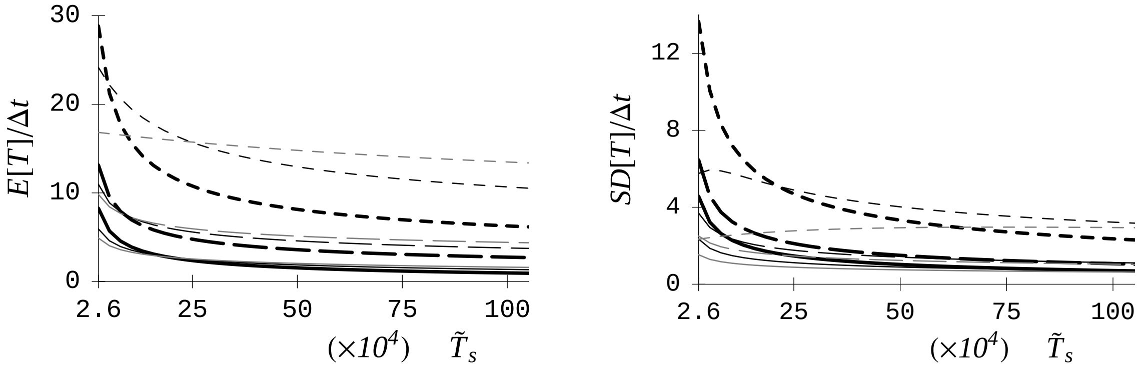
as the second moment in the case that Δt is used as a unit of time. Here,

$$
\begin{aligned}
B_i(t) \stackrel{\mathrm{def}}{=}\ & \exp[\varepsilon^\beta(t_1) - \varepsilon^\beta(t)] \\
& \times \int_0^\infty \left\{ \Big(u + \varepsilon^\beta(t) \Big)^{1/\beta} - \left(\varepsilon(t_{i-1}) - \frac{t_{i-1} - t_1}{\phi(V_i)} \right) \right\} \Big(u + \varepsilon^\beta(t) \Big)^{1/\beta - 1} \mathrm{e}^{-u}\mathrm{d}u.
\end{aligned}
$$

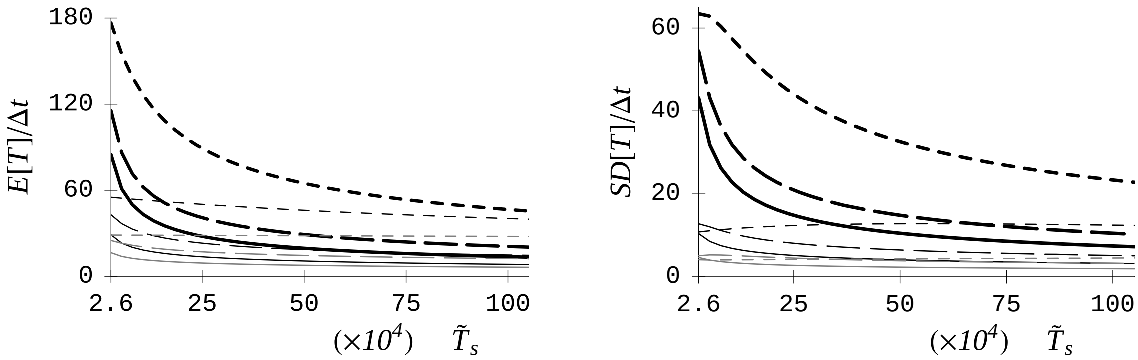
Using (5. 3) and (5. 4), we can calculate the mean and standard deviation of $T/\Delta t$ for the parameter values in Table 1. The results are shown on Fig. 1 and 2. In these figures the solid, rough dotted and fine dotted lines correspond to the cases of $\tilde{V}_{th} = 0$, 0.5 and 0.9, respectively. At the same time the bold, normal and gray ones correspond to the cases of $n = 1$, 2 and 3, respectively.

Table 1: Parameter values

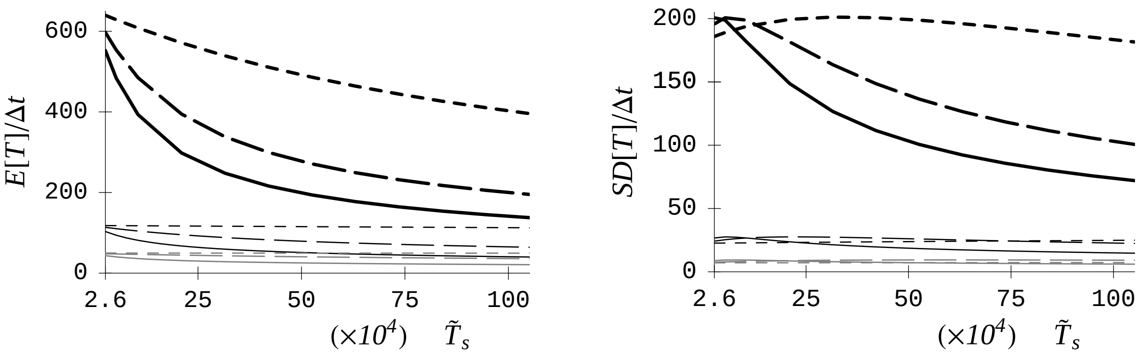
$\tilde{K}$	$\Delta\tilde{V}$	$\tilde{V}_{th}$	β	n
10^3, 10^4, 10^5	0.39	0, 0.5, 0.9	2, 3	1, 2, 3



The case of $\tilde{K} = 10^3$

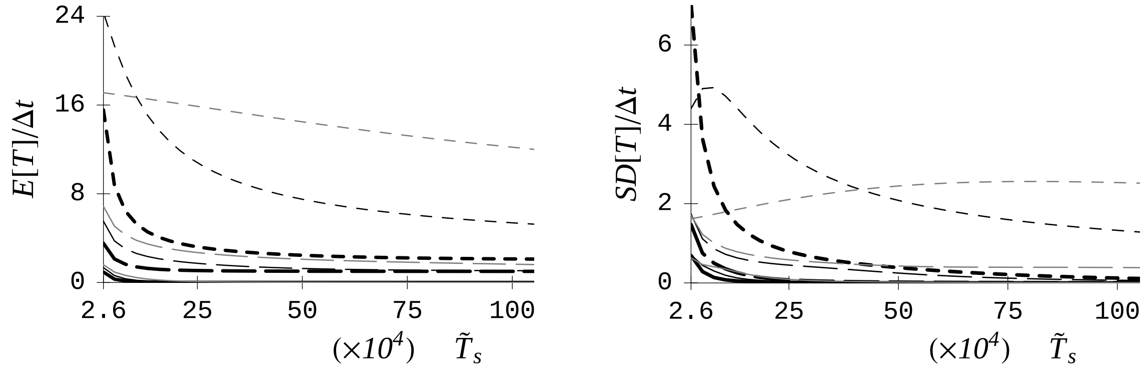


The case of $\tilde{K} = 10^4$

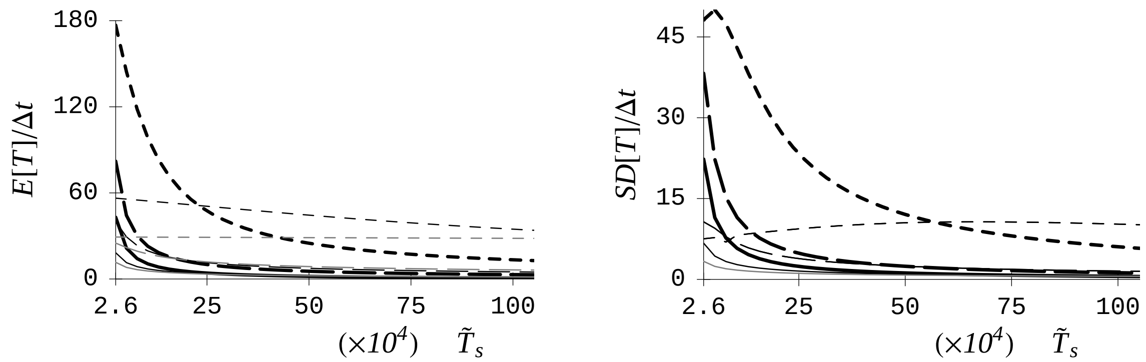


The case of $\tilde{K} = 10^5$

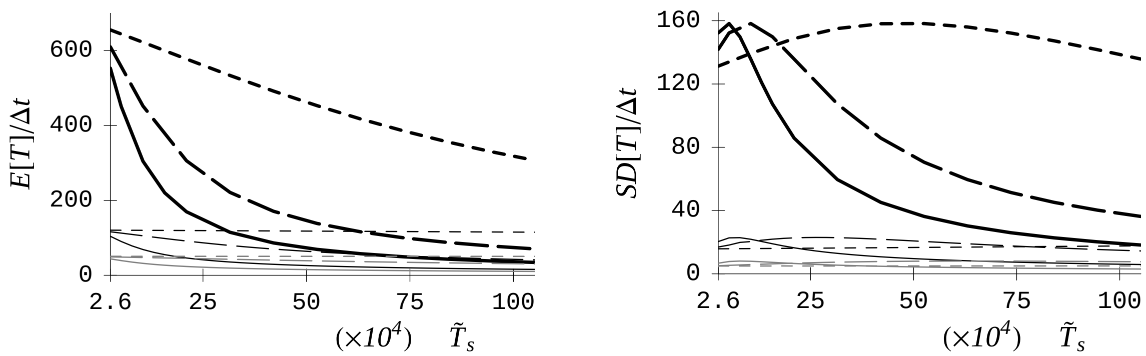
Figure 1: The mean and standard deviation of $T/\Delta t$ when $\beta = 2$



The case of $\tilde{K} = 10^3$



The case of $\tilde{K} = 10^4$



The case of $\tilde{K} = 10^5$

Figure 2: The mean and standard deviation of $T/\Delta t$ when $\beta = 3$

6 Example

On the basis of the results obtained in the previous section, we show an example of the ML estimation in (4. 3) on a actual data set. The data set is one for the breakdown voltage of cross-linked polyethylene-insulated cables. These data are part of breakdown voltage data introduced in [2], for which the insulation class is 22kV (in 3-phase) and the years in use are up to 22 years.

6.1 Data set

We show the data set in Table 2. For each specimen j, the first column indicates the length of the time interval during that the specimen is used before the test, and the second column indicates elapsed time from the start time of test by the start time of the stage on which a specimen fails. The unit of time is ten minutes. The third column indicates the number of data in each row. The fourth and fifth columns indicate the average and standard deviation of data, respectively, in the case that an outrageous datum is not taken into account.

Table 2: Step-stress data

$\tilde{T}_s^{(j)}$	$t_{l-1}^{(j)}/\Delta t$	Num.	Ave.	SD
157680	54, 56, 59, 64	4	58.3	4.3
473040	16	1	16	$*$
578160	16, 16, 18, 19, 22, 24, 46, 22, 48, 50	9	28.8	14.7
630720	24, 24, 28, 30, 32	5	27.6	3.6
735840	19, 20, 23, 35, 39, 42	6	30.0	10.2
788400	12, 12, 12, 13, 14, 14, 14, 17, 17	9	13.9	2.0
840960	12, 14, 14, 23, 24, 74†	6	17.4	5.6
893520	16, 18, 20, 20, 26, 35	6	22.5	7.0
946080	16, 16, 16, 17, 18, 18, 20, 22, 26	9	18.8	3.4
998640	8, 10, 11, 12, 12, 12, 13, 13, 14, 15, 22	11	12.9	3.6
1051200	11, 12, 12	3	11.7	0.6
1156320	11, 12, 12, 13, 14, 14	6	12.7	1.2

The marts † and $*$ mean outrageous and incomputable, respectively.

6.2 Maximum likelihood estimation

We seek the ML estimates of the parameters in (4. 3), (5. 1) and (5. 2). Because only the parameter $\tilde{K}$ possibly has an estimate whose absolute value is much larger than those of the other parameters, instead of $\tilde{K}$ we use a new parameter ζ defined by $\tilde{K}/\tilde{K}_0$ for a constant $\tilde{K}_0$.

By differentiating (4. 4) with respect to each parameter and arranging each equation, we can obtain the likelihood equation in a simplified form as follows:

$$\sum_{j=1}^{N} \frac{\lambda_\theta^{(j)}}{d_j} + \sum_{j=1}^{N} \delta_\theta(t_1^{(j)}; \tilde{T}_s^{(j)}) = 0, \quad \theta \in \{\beta, n, \zeta, \tilde{V}_{th}\}, \tag{6. 1}$$

where

$$
\begin{aligned}
d_j &\stackrel{\text{def}}{=} \exp\left(-\varepsilon^\beta(t_{l-1}^{(j)};\tilde{T}_s^{(j)})\right) - \exp\left(-\varepsilon^\beta(t_l^{(j)};\tilde{T}_s^{(j)})\right), \\
\lambda_\theta^{(j)} &\stackrel{\text{def}}{=} -\delta_\theta(t_{l-1}^{(j)};\tilde{T}_s^{(j)})\exp\left(-\varepsilon^\beta(t_{l-1}^{(j)};\tilde{T}_s^{(j)})\right) + \delta_\theta(t_l^{(j)};\tilde{T}_s^{(j)})\exp\left(-\varepsilon^\beta(t_l^{(j)};\tilde{T}_s^{(j)})\right), \\
\delta_\beta(t_i;\tilde{T}_s) &\stackrel{\text{def}}{=} \varepsilon^\beta(t_i;\tilde{T}_s)\ln\varepsilon(t_i;\tilde{T}_s), \\
\delta_n(t_i;\tilde{T}_s) &\stackrel{\text{def}}{=} \frac{1}{\tilde{K}_0}\varepsilon^{\beta-1}(t_i;\tilde{T}_s)\left\{\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^n\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\}, \\
\delta_\zeta(t_i;\tilde{T}_s) &\stackrel{\text{def}}{=} \varepsilon^\beta(t_i;\tilde{T}_s), \quad \delta_{\tilde{V}_{th}}(t_i;\tilde{T}_s) \stackrel{\text{def}}{=} \varepsilon^{\beta-1}(t_i;\tilde{T}_s)\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}.
\end{aligned}
$$

In the expressions above, note that $\varepsilon(t)$ in (5. 1) is expressed by $\varepsilon(t;\tilde{T}_s)$ as usual.

We seek the zero of (6. 1) by means of Newton's method. We set $\tilde{K}_0 = 10^4$ and use as a vector of initial guesses $(\beta, n, \zeta, \tilde{V}_{th}) = (2.0, 2, 1, 0.5)$. These are guessed from the comparison between Fig. 1 or 2 and the averages or standard deviations in Table 2. Note that $\Delta\tilde{V} = 0.39$ on the data set because that $V_s = 22$kV (in 3-phase) $= 22/\sqrt{3}$kV (in single phase) and $\Delta V = 5$kV (in single phase) [2].

In this model, the calculation for ML estimates is so sensitive that, depending on initial guesses, a sequence of approximates by Newton's iteration can converge to a vector of estimates in which the estimate of β is less 1 even if the value of the likelihood function is not a maximum value. According to our observation, when this phenomenon occurs, the estimate of $\tilde{V}_{th}$ always becomes 0. Thus, including calculations in the next subsection, we always adopt the strategy below.

1. We seek roughly the profile of the expression in the left-hand side of (6. 1) with respect to $\tilde{V}_{th}$. That is, while changing the value of $\tilde{V}_{th}$ from a value α_0 to another value α_1 in incremental steps, we seek the estimates of the other parameters in each step. We set $\alpha_0 = 0.5$ or $\alpha_0 = 0.85$ in the calculation for ML estimates or the simulation, respectively. On the other hand, we set $\alpha_1 = 0.99$ and the increment of $\tilde{V}_{th}$ at 0.01 in each step.

2. In analogy, we seek more precisely the part of the profile by narrowing the interval (α_0, α_1).

3. Among the points on the part of the profile, we select the point at that the profile achieves its maximum, and then seek the ML estimates of all parameters simultaneously by using the point as a vector of initial values and performing Newton's iteration without fixing $\tilde{V}_{th}$.

The derivatives of the expressions in the left-hand side of (6. 1) are given in Appendix B.

Ultimately, we can obtain the following ML estimates:

$$\beta = 5.016812, \quad n = 1.603875, \quad \zeta = 0.548237, \quad \tilde{V}_{th} = 0.944054. \tag{6. 2}$$

Then, the value of the likelihood function (4. 4) is -244.4626. The mean and some statistical quantities of T and the scatter plot of data are given on Fig. 3. In the figure the solid line indicates the mean and the upper or under dotted line indicates the mean plus or minus the standard deviation, respectively. Each dot indicates $(t_{l-1}^{(j)} + t_l^{(j)})/2$ for each sample j.

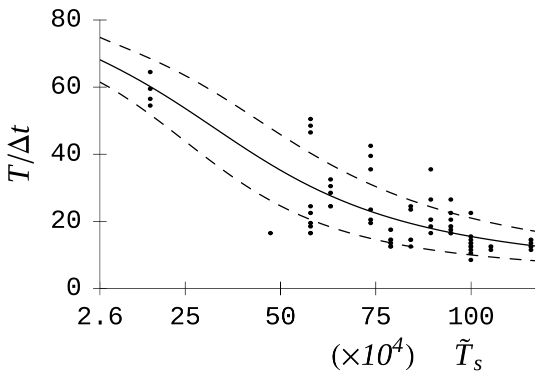


Figure 3: The mean and the scatter plot

6.3 Bias and variance

We suppose that the foregoing ML estimates are the true values of parameters and investigate the bias and variance of the ML estimator by means of Monte Carlo simulation.

The simulation conditions are as follows. The prespecified values of $\tilde{T}_s$'s, $\Delta\tilde{V}$ and $\tilde{K}_0$ and the unit of time are the same as those for the data set in Table 2. Besides, each sample number for each $\tilde{T}_s$ is the same as the data set except the outrageous data. Thus, the sample size is 74. The values in (6. 2) is used as the true values of β, n, ζ and $\tilde{V}_{th}$.

Next we explain the way of generating random samples. By setting $u \stackrel{\text{def}}{=} G(t|t > T_s)$ for $(\infty >)\ t > T_s$, we can obtain from (4. 1), (4. 3), (5. 1) and (5. 2)

$$\begin{aligned} \varepsilon(t) &= \Big[-\ln\{(1-u)(1-G(T_s))\}\Big]^{1/\beta} \\ &= \left[-\ln(1-u) + \left(\frac{(1-\tilde{T}_{th})^n}{\tilde{K}_0\zeta}\tilde{T}_s\right)^{\beta}\right]^{1/\beta}. \end{aligned} \qquad (6.\ 3)$$

Let us denote by q the last expression. Then, (5. 1) and (5. 2) yield

$$\sum_{m=k}^{i-1} \frac{((m-1)\Delta\tilde{V} - \tilde{V}_{th})^n}{\tilde{K}_0\zeta} < q - \frac{(1-\tilde{T}_{th})^n}{\tilde{K}_0\zeta}\tilde{T}_s \le \sum_{m=k}^{i} \frac{((m-1)\Delta\tilde{V} - \tilde{V}_{th})^n}{\tilde{K}_0\zeta}. \qquad (6.\ 4)$$

Note that $u \in (0, 1)$ and $i \ge 2$ since $T_s < t < \infty$. Consequently, in order to generate a pseudo-random sample, all things we have to do is generate a uniform random number, seek q by (6. 3) and find i which satisfies (6. 4). Then, this i means the stress level l in which failure occurs. For pseudo-random samples, repeat the procedure above.

In the simulation, 1000 sets of independent pseudo-random samples are considered, except sample sets where the ML estimates could not be obtained. The result is given in Table 3.

Table 3: Simulation result

	β	n	ζ	$\tilde{V}_{th}$
Bias	-0.114307	0.048888	-0.017306	0.000984
Variance	0.460027	0.048186	0.181128	0.002013

7 Conclusion

We have considered the two-parameter Weibull CE model with the threshold parameter in the multiple-step SSALT under the condition that specimens were subjected to a normal level of stress and did not fail before the test. This consideration has revealed that the shape parameter β must be larger than 1 for the model to fit the realistic situation in which the durability of specimens decreases as they are used longer or with higher stress. Thus, the exponential CE model can not fit such a situation.

After simplifying the model without loss of generality, for a various sets of parameter values we have shown the average and standard deviation of failure time versus the duration $\tilde{T}_s$ of the specimen's use before the test. We have utilized them on the first stage of calculation to obtain the ML estimates in Section 6.

In the section we have illustrated the example of the ML estimation, in which we have sought the vector of the ML estimates and shown the biases and variances of the ML estimators by means of Monte Carlo simulation.

We did not perform a goodness of fit test in the example because the number of samples is too small to use an asymptotic distribution for the test. In the case when there are enough samples in each $\tilde{T}_s$, however, we can perform the chi-square goodness of fit test on such grouped data as the example in order to see whether the CE model is consistent with these data.

Appendix

A How to construct the CE model

We explain the way of constructing the CE model in Section 2 by probabilistic analogy with that for Miner's rule. In addition to the symbols introduced in the section, we denote by d_{i-1} and $\Delta d_{i-1}(\Delta t)$ damage accumulated up to t_{i-1} and damage accumulated during Δt from time t_{i-1}, respectively.

[Assumptions]

i) When damage D is accumulated in a specimen, it fails with probability 1.

ii) If a specimen does not fail in an interval $(t_{i-1}, t_i]$, the following holds:

$$d_i = d_{i-1} + \Delta d_{i-1}(t_i - t_{i-1}).$$

iii) For a positive value s_{i-1}, the following relationship between damage and failure probability holds:

$$\frac{d_{i-1}}{D} + \frac{\Delta d_{i-1}(\Delta t)}{D} = \begin{cases} F(\Delta t + s_{i-1}; V_i) & (V_i > V_{th}), \\ F(s_{i-1}; V_i) & (V_i \leq V_{th}). \end{cases} \tag{A. 1}$$

iv) Damage accumulated for a moment is 0. That is,

$$\lim_{\Delta t \to 0} \Delta d_{i-1}(\Delta t) = 0. \tag{A. 2}$$

Assumption iii) indicates that the probability that failure occurs by the time when relative damage in the left-hand side of (A. 1) is accumulated, is given by the expression in the right-hand side.

When we set $d_{i-1} = 0$ in (A. 1), we can obtain the relationship between damage and failure probability

$$\frac{\Delta d_{i-1}(\Delta t)}{D} = \begin{cases} F(\Delta t; V_i) & (V_i > V_{th}), \\ 0 & (V_i \leq V_{th}) \end{cases} \tag{A. 3}$$

for a constant stress V_i since $s_{i-1} = 0$ from (A. 2).

Using the things above, let us construct G.

When we set $d_0 = 0$, (A. 3) gives

$$\frac{\Delta d_0(\Delta t)}{D} = \begin{cases} F(\Delta t; V_1) & (V_1 > V_{th}), \\ 0 & (V_1 \leq V_{th}). \end{cases}$$

From this, we define

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_0; V_1) & (V_1 > V_{th}), \\ 0 & (V_1 \leq V_{th}) \end{cases}$$

for $t_0 \leq t \leq t_1$.

If failure does not occur by t_1, (A. 1) gives

$$\frac{d_1}{D} + \frac{\Delta d_1(\Delta t)}{D} = \begin{cases} F(\Delta t + s_1; V_2) & (V_2 > V_{th}), \\ F(s_1; V_2) & (V_2 \leq V_{th}). \end{cases}$$

By using the above equations and letting Δt tend to 0, we obtain

$$G(t_1) = F(s_1; V_2). \tag{A. 4}$$

From this, we define

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_1 + s_1; V_2) & (V_2 > V_{th}), \\ F(s_1; V_2) & (V_2 \leq V_{th}) \end{cases}$$

for $t_1 < t \leq t_2$. Here, s_1 is a positive value that satisfies (A. 4).

Similarly, for $t_{i-1} < t \leq t_i$, we define

$$G(t) \stackrel{\text{def}}{=} \begin{cases} F(t - t_{i-1} + s_{i-1}; V_i) & (V_i > V_{th}), \\ F(s_{i-1}; V_i) & (V_i \leq V_{th}), \end{cases}$$

where s_{i-1} is a positive value such that $G(t_{i-1}) = F(s_{i-1}; V_i)$ holds.

[Example] In order to look at the relationship with Miner's rule, assume a uniform distribution as $F(t;V)$:

$$F(t;V) = \frac{t}{V}.$$

If $V_1, V_2, \ldots > V_{th}$, from (A. 1),

$$\begin{aligned}
\sum_{j=1}^{i-1} \frac{\Delta d_{j-1}(t_j - t_{j-1})}{D} + \frac{\Delta d_{i-1}(t - t_{i-1})}{D} &= \frac{s_{i-1}}{V_i} + \frac{t - t_{i-1}}{V_i} \\
&= s_{i-1} + \frac{t_i - t_{i-1}}{V_{i-1}} + \frac{t - t_{i-1}}{V_i} \\
&\quad \vdots \qquad\qquad \vdots \\
&= \sum_{j=1}^{i-1} \frac{t_j - t_{j-1}}{V_j} + \frac{t - t_{i-1}}{V_i}
\end{aligned}$$

for $t_{i-1} < t \leq t_i$. In addition, when we set $d_{i-1} = 0$, (A. 3) gives

$$\frac{\Delta d_{i-1}(\Delta t)}{D} = \frac{\Delta t}{V_i}.$$

These two equations are the counterparts of Miner's rule, usually stated deterministically.

B Derivatives needed for the ML estimation

For $\theta_1,\ \theta_2 \in \{\beta, n, \zeta, \tilde{V}_{th}\}$, the derivatives of the expressions in the left-hand side of (6. 1) are given as follows:

$$\begin{aligned}
&\frac{\partial}{\partial \theta_2} \left\{ \sum_{j=1}^{N} \frac{\lambda_{\theta_1}^{(j)}}{d_j} + \sum_{j=1}^{N} \delta_{\theta_1}(t_1^{(j)}; \tilde{T}_s^{(j)}) \right\} \\
&\quad = \sum_{j=1}^{N} \left\{ -\left(\frac{1}{d_j} \frac{\partial d_j}{\partial \theta_2} \right) \frac{\lambda_{\theta_1}^{(j)}}{d_j} + \frac{1}{d_j} \frac{\partial \lambda_{\theta_1}^{(j)}}{\partial \theta_2} + \frac{\partial}{\partial \theta_2} \delta_{\theta_1}(t_1^{(j)}; \tilde{T}_s^{(j)}) \right\} \\
&\quad = \sum_{j=1}^{N} \left\{ -C_{\theta_2} \frac{\lambda_{\theta_1}^{(j)}}{d_j} \frac{\lambda_{\theta_2}^{(j)}}{d_j} + \frac{1}{d_j} \frac{\partial \lambda_{\theta_1}^{(j)}}{\partial \theta_2} + \frac{\partial}{\partial \theta_2} \delta_{\theta_1}(t_1^{(j)}; \tilde{T}_s^{(j)}) \right\},
\end{aligned}$$

where

$$C_\theta \overset{\text{def}}{=} \begin{cases} 1 & (\theta = \beta), \\ \frac{\beta}{\zeta} & (\theta = n), \\ -\frac{n}{\zeta} & (\theta = \zeta), \\ -\frac{\beta n}{\zeta} & (\theta = \tilde{V}_{th}), \end{cases} \qquad E_j \overset{\text{def}}{=} \exp\left(-\varepsilon^\beta(t_l^{(j)}; \tilde{T}_s^{(j)}) + \varepsilon^\beta(t_{l-1}^{(j)}; \tilde{T}_s^{(j)}) \right),$$

$$\begin{aligned}
\frac{\lambda_\theta^{(j)}}{d_j} &= \frac{-\delta_\theta(t_{l-1}^{(j)}; \tilde{T}_s^{(j)}) + \delta_\theta(t_l^{(j)}; \tilde{T}_s^{(j)}) E_j}{1 - E_j}, \\
\frac{1}{d_j} \frac{\partial \lambda_{\theta_1}^{(j)}}{\partial \theta_2} &= \left[-\left\{ \frac{\partial}{\partial \theta_2} \delta_{\theta_1}(t_{l-1}^{(j)}; \tilde{T}_s^{(j)}) - C_{\theta_2} \delta_{\theta_1}(t_{l-1}^{(j)}; \tilde{T}_s^{(j)}) \delta_{\theta_2}(t_{l-1}^{(j)}; \tilde{T}_s^{(j)}) \right\} \right. \\
&\quad \left. + \left\{ \frac{\partial}{\partial \theta_2} \delta_{\theta_1}(t_l^{(j)}; \tilde{T}_s^{(j)}) - C_{\theta_2} \delta_{\theta_1}(t_l^{(j)}; \tilde{T}_s^{(j)}) \delta_{\theta_2}(t_l^{(j)}; \tilde{T}_s^{(j)}) \right\} E_j \right] \Big/ (1 - E_j).
\end{aligned}$$

In the above, note that the following equation does not necessarily hold because (6. 1) is a simplified equation not the original likelihood equation:

$$\frac{\partial}{\partial \theta_2}\left\{\sum_{j=1}^{N}\frac{\lambda_{\theta_1}^{(j)}}{d_j}+\sum_{j=1}^{N}\delta_{\theta_1}(t_1^{(j)};\tilde{T}_s^{(j)})\right\}=\frac{\partial}{\partial \theta_1}\left\{\sum_{j=1}^{N}\frac{\lambda_{\theta_2}^{(j)}}{d_j}+\sum_{j=1}^{N}\delta_{\theta_2}(t_1^{(j)};\tilde{T}_s^{(j)})\right\}.$$

Each derivative of δ_θ ($\theta \in \{\beta, n\zeta, \tilde{V}_{th}\}$) is given in the following. The arguments in some derivatives are omitted as far as it does not cause a confusion.

$$\frac{\partial \delta_\beta}{\partial \beta}=\delta_\beta \ln\varepsilon,\quad \frac{\partial \delta_\beta}{\partial n}=\left(C_n\ln\varepsilon+\frac{1}{\zeta}\right)\delta_n,\quad \frac{\partial \delta_\beta}{\partial \zeta}=\left(C_\zeta\ln\varepsilon-\frac{1}{\zeta}\right)\delta_\zeta,$$

$$\frac{\partial \delta_\beta}{\partial \tilde{V}_{th}}=\left(C_{\tilde{V}_{th}}\ln\varepsilon-\frac{n}{\zeta}\right)\delta_{\tilde{V}_{th}},\quad \frac{\partial \delta_n}{\partial \beta}=\delta_n\ln\varepsilon,$$

$$\begin{aligned}&\frac{\partial}{\partial n}\delta_n(t_i;\tilde{T}_s)\\&\quad=\left(C_n-\frac{1}{\zeta}\right)\varepsilon^{\beta-2}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^n\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\}^2\\&\qquad+\varepsilon^{\beta-1}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^n\Big(\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\Big)^2\right\},\end{aligned}$$

$$\frac{\partial \delta_n}{\partial \zeta}=\left(C_\zeta+\frac{1}{\zeta}\right)\delta_n,$$

$$\begin{aligned}\frac{\partial}{\partial \tilde{V}_{th}}\delta_n(t_i;\tilde{T}_s)&=\left(C_{\tilde{V}_{th}}+\frac{n}{\zeta}\right)\varepsilon^{\beta-2}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}\right\}\\&\quad\times\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^n\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\}-\delta_{\tilde{V}_{th}}(t_i;\tilde{T}_s)\\&\quad-n\varepsilon^{\beta-1}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\},\end{aligned}$$

$$\frac{\partial \delta_\zeta}{\partial \beta}=\delta_\beta,\quad \frac{\partial \delta_\zeta}{\partial n}=C_n\delta_n,\quad \frac{\partial \delta_\zeta}{\partial \zeta}=C_\zeta\delta_\zeta,\quad \frac{\partial \delta_\zeta}{\partial \tilde{V}_{th}}=C_{\tilde{V}_{th}}\delta_{\tilde{V}_{th}},\quad \frac{\partial \delta_{\tilde{V}_{th}}}{\partial \beta}=\delta_{\tilde{V}_{th}}\ln\varepsilon,$$

$$\begin{aligned}&\frac{\partial}{\partial n}\delta_{\tilde{V}_{th}}(t_i;\tilde{T}_s)\\&\quad=\left(C_n-\frac{1}{\zeta}\right)\varepsilon^{\beta-2}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^n\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\}\\&\qquad\times\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}\right\}\\&\qquad+\varepsilon^{\beta-1}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}\ln((m-1)\Delta\tilde{V}-\tilde{V}_{th})\right\},\end{aligned}$$

$$\frac{\partial \delta_{\tilde{V}_{th}}}{\partial \zeta}=\left(C_\zeta+\frac{1}{\zeta}\right)\delta_{\tilde{V}_{th}},$$

$$\frac{\partial}{\partial \tilde{V}_{th}}\delta_{\tilde{V}_{th}}(t_i;\tilde{T}_s)=\left(C_{\tilde{V}_{th}}+\frac{n}{\zeta}\right)\varepsilon^{\beta-2}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-1}\right\}^2$$

$$- (n-1)\varepsilon^{\beta-1}(t_i;\tilde{T}_s)\left\{\frac{1}{\tilde{K}_0}\sum_{m=k}^{i}((m-1)\Delta\tilde{V}-\tilde{V}_{th})^{n-2}\right\}.$$

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