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Explicit stochastic Runge-Kutta methods with large stability regions

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Abstract. Our aim is to derive explicit Runge-Kutta schemes for Stratonovich stochastic differential equations with a multi-dimensional Wiener process, which are of weak order 2 and which have large stability regions. This has been achieved by the use of a technique in Chebyshev methods for ordinary differential equations. In this talk, large stability regions of our schemes will be shown. Concerning convergence order and stability properties, the schemes will be tested in numerical experiments.

INTRODUCTION

While it has been customary to treat the numerical solution of stiff ordinary differential equations (ODEs) by implicit methods, there is a class of explicit methods with extended stability regions that are well suited to solving stiff problems whose eigenvalues lie near the negative real axis. Such problems include parabolic partial differential equations when solved by the method of lines.

An original contribution was by van der Houwen and Sommeijer [1] who constructed s-stage explicit Runge-Kutta (RK) methods whose stability functions are shifted Chebyshev polynomials $T_s(1+z/s^2)$. These have stability intervals along the negative real axis $[-2s^2, 0]$. The corresponding RK methods satisfy a three term recurrence relation which make them efficient to implement, but their drawback is that they are of order 1. Lebedev [2, 3] and Medovikov [4] constructed high order methods by computing the zeros of the optimal stability polynomials for maximal stability. But, the method is sensitive to the ordering of these zeros and there is no recurrence relationship.

Abdulle and Medovikov [5] developed a new strategy to construct Chebyshev methods with nearly optimal stability domain of order two. These methods are based on a weighted orthogonal polynomial and so the numerical methods satisfy a three-term recurrence relationship. In this case the stability interval is $[-l_s, 0]$ where $l_s \approx 0.81s^2$.

One of the drawbacks with Chebyshev methods is that the stability region can collapse to $s-1$ single points on the negative real axis due to the mini-max property of Chebyshev polynomials. Accordingly, we require the modulus of the stability polynomial to be bounded by a damping factor $\eta < 1$. The stability interval shrinks slightly but a strip around the negative real axis is included in stability region. With $\eta = 0.95$, $l_s \approx 0.81s^2$ for the second order Chebyshev methods.

In the case of stochastic differential equations (SDEs) the issues are much more complex. Nevertheless, Abdulle and Cirilli [6] have developed a family of explicit Runge-Kutta Chebyshev methods with extended mean square stability regions. These methods have strong order 0.5 and weak order 1 for non-commutative Stratonovich SDEs. They reduce to the first order Chebyshev methods when there is no noise. Such an approach is important because there are very few good numerical methods for solving stiff SDEs.

In general it is difficult to construct semi-implicit or implicit methods, especially high order methods for stiff SDEs [7, 8, 9], but if the stiffness is mild, we can still hope to construct effective explicit SRK methods for such SDEs. Fortunately, in addition, Komori [10] and Rößler [11] have succeeded in deriving SRK methods of weak order 2 for non-commutative SDEs, whose structure is suitable to naturally combine RK methods for ODEs.

In this talk, we shall put all these ideas together. We will construct a family of s-stage SRK methods with weak order 2 for a multi-dimensional Wiener process and with extended mean square stability regions. The method will reduce to the second order Chebyshev methods of Abdulle and Medovikov [5] when the noise terms are set to zero.

CHEBYSHEV METHODS FOR ODES

Consider the autonomous N-dimensional ODEs given by

$$y'(t) = f(y(t)), \qquad y(t_0) = y_0. \tag{1}$$

The class of s-stage RK methods for solving (1) is

$$y_{n+1} = y_n + h\sum_{j=1}^{s} b_j f(Y^{(j)}), \qquad Y^{(i)} = y_n + h\sum_{j=1}^{s} a_{ij} f(Y^{(j)}) \tag{2}$$

$(1 \le i \le s)$. A RK method is explicit if $a_{ij} = 0$ $(i \le j)$.

When we define that $b^\top \stackrel{\text{def}}{=} [b_1\ b_2\ \cdots\ b_s]$, A is a $s \times s$ matrix (a_{ij}) and $e \stackrel{\text{def}}{=} [1\ 1\ \cdots\ 1]^\top$ and apply (2) to the linear, scalar test problem

$$y'(t) = \lambda y, \quad \Re(\lambda) \le 0, \tag{3}$$

we have

$$y_{n+1} = R(h\lambda)y_n, \qquad \text{where} \quad R(z) \stackrel{\text{def}}{=} 1 + zb^\top(I - Az)^{-1}e.$$

Here R is called the stability function and for explicit methods $R(z)$ is a polynomial of at most degree s, namely

$$R(z) = 1 + \sum_{j=1}^{s} z^j b^\top A^{j-1} e.$$

The stability region of (2) is $\{z \mid |R(z)| \le 1\}$.

Suppose now we require

$$R_s(z) = 1 + z + \frac{1}{2}z^2 + \sum_{j=3}^{s} \alpha_{js} z^j \qquad (\alpha_{js}\text{'s are constants})$$

such that

$$|R_s(z)| \le 1 \text{ for } z \in [-l_s, 0], \qquad l_s \text{ as large as possible.}$$

Riha [12] showed that for a given s such polynomials exist and are unique. Abdulle and Medovikov [5] relaxed optimal stability and constructed approximations to these optimal stability polynomials using orthogonal polynomials such that

$$R_s(x) = w(x)P_{s-2}(x),$$

where if we write

$$w(x) \stackrel{\text{def}}{=} \bar{w}(a_s + x/d_s), \qquad P_j(x) \stackrel{\text{def}}{=} \bar{P}_j(a_s + x/d_s),$$

$\bar{w}(x)$ is of degree two with complex zeros and satisfied $\bar{w}(a_s) = 1$, then the orthogonal polynomials $\bar{P}_0(x)$, $\bar{P}_1(x), \ldots, \bar{P}_{s-2}(x)$ are orthogonal with respect to the weight function $\bar{w}^2(x)/\sqrt{1-x^2}$ on $[-1,1]$, $\bar{P}_0(a_s) = \bar{P}_1(a_s) = \cdots \bar{P}_{s-2}(a_s) = 1$, and satisfy a three-term recurrence relation. This leads to the method

$$\begin{aligned}
&K_0 \stackrel{\text{def}}{=} y_n, \qquad K_1 \stackrel{\text{def}}{=} y_n + h\mu_1 f(K_0), \qquad K_j \stackrel{\text{def}}{=} h\mu_j f(K_{j-1}) + (\theta_j + 1)K_{j-1} - \theta_j K_{j-2} \quad (j = 2,3,\ldots,s-2),\\
&K_{s-1} \stackrel{\text{def}}{=} K_{s-2} + h\sigma_s f(K_{s-2}), \qquad K_s^* \stackrel{\text{def}}{=} K_{s-1} + h\sigma_s f(K_{s-1}),\\
&K_s \stackrel{\text{def}}{=} K_s^* - h\sigma_s(1 - \tau_s/\sigma_s^2)\big(f(K_{s-1}) - f(K_{s-2})\big), \qquad y_{n+1} = K_s.
\end{aligned} \tag{4}$$

The computation of K_{s-1}, K_s^* can be viewed as a finishing procedure. When (4) is applied to (3), then

$$K_j = P_j(z)y_n \quad (j = 0,\ldots,s-2), \qquad K_s = w(z)K_{s-2}, \qquad y_{n+1} = R_s(z)y_n,$$

where

$$w(z) = 1 + 2\sigma_s z + \tau_s z^2$$

and

$$P_0(z) = 1, \qquad P_1(z) = 1 + \mu_1 z, \qquad P_j(z) = (\mu_j z + \theta_j + 1)P_{j-1}(z) - \theta_j P_{j-2}(z), \quad j = 2,3,\ldots,s-2.$$

The value for l_s depends on what damping (4) has. Away from $z = 0$ it is appropriate to require

$$|R_s(z)| \le \eta < 1, \qquad z \le -\varepsilon \quad (\varepsilon: \text{ small positive parameter})$$

and a number of authors set $\eta = 0.95$. In this case $l_s \approx 0.81s^2$ (rather than $0.82s^2$ with $\eta = 1$).

METHODS FOR SDES

Consider now the autonomous N-dimensional SDEs given by

$$\mathrm{d}y = \sum_{j=0}^{d} g_j(y) \circ \mathrm{d}w_j(t) \qquad y(t_0) = y_0$$

which we will assume is in Stratonovich form. Here $w_0(t) = t$ and the $w_j(t),\ j = 1,2,\ldots,d$ are independent Wiener processes. For solving this, let us consider the following SRK framework [10, 11]:

$$\begin{aligned}
Y_i^{(0,0)} &= hg_0\left(y_n + \left(\alpha_i^{(0)}\right)^\top Y^{(0,0)} + \left(\alpha_i^{(2)}\right)^\top \sum_{j=1}^{d} Y^{(j,j)}\right),\\
Y_i^{(j,j)} &= \zeta_i^{(j,j)} g_j\left(y_n + \left(\alpha_i^{(1)}\right)^\top Y^{(0,0)} + \left(\alpha_i^{(3)}\right)^\top Y^{(j,j)} + \left(\alpha_i^{(4)}\right)^\top \sum_{\substack{l=1\\ l\neq j}}^{d} Y^{(l,l)}\right), \quad j = 1,2,\ldots,d,\\
Y_i^{(j,l)} &= \zeta_i^{(j,l)} g_l\left(y_n + \left(\alpha_i^{(5)}\right)^\top Y^{(0,0)} + \left(\alpha_i^{(6)}\right)^\top \sum_{\substack{m=1\\ m\neq l}} Y^{(l,m)}\right), \quad j \neq l, \quad j,l = 1,2,\ldots,d,\\
y_{n+1} &= y_n + b_0^\top Y^{(0,0)} + b_1^\top \sum_{j=1}^{d} Y^{(j,j)} + b_2^\top \sum_{l=1}^{d} Y^{(k(l),l)},
\end{aligned} \tag{5}$$

where $k(l)$ is a value in $\{1,2,\ldots,l-1,l+1,\ldots,m\}$.

In order to construct weak order 2 methods the $\zeta^{(j,l)}$ are chosen as follows:

$$\zeta_i^{(j,l)} = \begin{cases} \Delta w_l & (j = l),\\ \Delta w_j \Delta\tilde{w}_l/\sqrt{h} & (l > j > 0 \text{ and } i = s-2),\\ -\Delta\tilde{w}_j \Delta w_l/\sqrt{h} & (j > l > 0 \text{ and } i = s-2),\\ \sqrt{h} & (j \neq l \text{ and } i \neq s-2), \end{cases}$$

where the $\Delta\tilde{w}_l$ are independent 2 point discrete random variables with $P(\Delta\tilde{w}_j = \pm\sqrt{h}) = 1/2$ and the Δw_j are independent 3 point discrete random variables with $P(\Delta w_j = \pm\sqrt{3h}) = 1/6$ and $P(\Delta w_j = 0) = 2/3$. In addition, we suppose

$$b_{2,i} = 0 \quad (i < s-2), \qquad \alpha_{i_a i_b}^{(6)} = 0 \quad (i_a, i_b < s-2 \text{ or } i_a \leq i_b)$$

to make the number of nonzero roles concerning stochastic parts as small as possible. Since (4) is embedded in (5) when there is no noise, $\alpha_i^{(0)}$ $(1 \leq i \leq s)$ and b_0 are given by the Chebyshev formulation in (4).

MEAN SQUARE STABILITY

We now apply our method to the linear, scalar multiplicative noise problem

$$\mathrm{d}y = \lambda y \mathrm{d}t + \sum_{j=1}^{d} \lambda_j y \circ \mathrm{d}w_j(t),$$

where $\lambda_j,\ j = 0,1,\ldots,d$ are real values. Because of the structure we can easily see that

$$Y_j^{(0,0)} = P_{j-1}(h\lambda)y_0, \quad j = 1,2,\ldots,s-3.$$

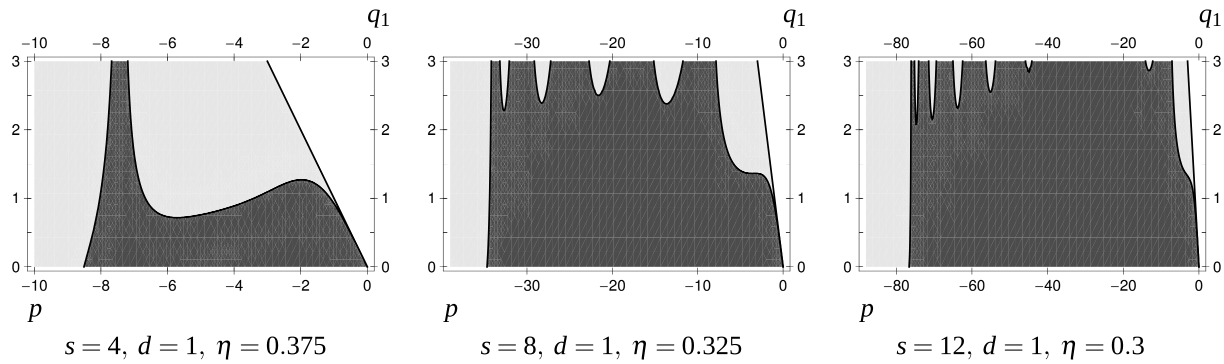

FIGURE 1. MS stability regions of SRK schemes for some s, d and η

We now compute successively $Y_i^{(0,0)}, Y_i^{(j,j)}, Y_i^{(j,l)}$ for $i = s-2, s-1, s$ and y_{n+1}, using the order conditions to try and get a simple form for these expressions. Once we have found the form

$$y_{n+1} = Ry_n,$$

the MS stability function is given by

$$\hat{R} = E[R^2].$$

$\hat{R}$ in fact will be a function of $p \overset{\text{def}}{=} h\lambda,\ q_j \overset{\text{def}}{=} h\lambda_j^2,\ \ j = 1, 2, \ldots, d$. The MS stability region of a scheme is $\{(p, q_1, q_2, \ldots, q_d) | \hat{R} \leq 1\}$. For examples, MS stability regions of our schemes are given with dark-colored parts in Figure 1, whereas the parts enclosed by the two straight lines $q_1 = -p$ and $q_1 = 0$ indicate the region in which the test SDE is stable in mean square. Concerning the importance of MS stability in weak schemes, for example, see [13].

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