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Stability Analysis of Numerical Methods Using a Linear Test SDE with Delay and Non-delay in a Diffusion Term

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Abstract. A theorem was originally proposed to deal with the stochastic theta methods when they are applied to a linear test equation with delay and non-delay in a diffusion term. We extend the theorem to a proposition in a general form, and use it for stability analysis of stochastic orthogonal Runge–Kutta–Chebyshev methods when the methods are applied to the test equation.

INTRODUCTION

We are concerned with stabilized numerical methods for the strong approximation to the solution of stochastic delay differential equations (SDDEs). A class of such methods is the class of the stochastic theta methods. Comparing with numerical methods for stochastic differential equations without delay, the issues to analyse numerical methods for SDDEs are much more complicated. Nevertheless, Huang, Gan, and Wang [1] have analysed mean square (MS) stability properties of the stochastic theta methods when the methods are applied to a scalar test equation with delay and non-delay in a diffusion term.

Incidentally, Komori, Eremin and Burrage [2] have studied stabilized explicit numerical methods for SDDEs and have successfully derived stochastic orthogonal Runge–Kutta–Chebyshev (SROCK) methods. Using a scalar test equation with pure delay in a diffusion term, they have investigated stability properties of the SROCK methods. In [2], as a future work they left stability analysis of the SROCK methods for the test equation dealt with in [1]. In the present paper, we extend a theorem proposed by Huang et al. [1] to a proposition in a general form. After that, we will apply it to the SROCK methods and the explicit Euler–Maruyama (EM) method in order to analyse their stability properties.

MAIN RESULT IN A GENERAL FORM

Let us consider the scalar linear test equation

$$\begin{aligned} &\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + (\sigma_1 y(t) + \sigma_2 y(t-\tau))\mathrm{d}W(t), \qquad t \geq 0, \\ &y(0) = \Psi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $\tau > 0$ is a constant, $W(t)$ is a scalar Wiener process, and Ψ is continuous on $[-\tau, 0]$, and where $\lambda < 0$ and $\sigma_1, \sigma_2 \in \mathbb{R}$ satisfy

$$\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2 \mathrm{e}^{\lambda\tau} < -2\lambda. \tag{2}$$

Here, note that (2) is equivalent to $\lim_{t\to\infty} E[(y(t))^2] = 0$ for every Ψ in the case that $\sigma_2 \neq -\sigma_1 \mathrm{e}^{\lambda\tau}$ and $\sigma_1\sigma_2 < 0$ or in the case of $\sigma_1\sigma_2 \geq 0$ [3].

Let y_n denote a discrete approximation to the solution $y(t_n)$ of (1) for an equidistant grid point $t_n = nh$ ($n = 1, 2, \dots,$) with step size $h > 0$. Throughout the present paper, we assume that $\tau = Mh$ holds (M is a natural number), and define y_{n-M}, $n = 0, 1, \dots, M$, by $\Psi(t_n - \tau)$. In addition, let us denote $W(t_{n+1}) - W(t_n)$ by ΔW. Now, suppose that we have

$$y_{n+1} = A(p)y_n + B(p)(\sigma_1 y_n + \sigma_2 y_{n-M})\Delta W \tag{3}$$

if a numerical method is applied to (2), where $A(p)$ and $B(p)$ are functions of $p = \lambda h$. From this, as

$$E[y_{n+1}^2] = \left\{(A(p))^2 + (q_1 B(p))^2\right\} E[y_n^2] + (B(p))^2 \left\{2q_1 q_2 (A(p))^M + q_2^2\right\} E[y_{n-M}^2],$$

we obtain the characteristic equation

$$\xi = (A(p))^2 + (q_1 B(p))^2 + (B(p))^2 \left\{2q_1 q_2 (A(p))^M + q_2^2\right\} \xi^{-M}, \tag{4}$$

where $q_1 = \sigma_1 \sqrt{h}$ and $q_2 = \sigma_2 \sqrt{h}$. Thus, if we require that (3) is MS-stable ($\lim_{n\to\infty} E\left[|y_n|^2\right] = 0$) when p, q_1, q_2 and M are given, then all roots of (4) must satisfy $|\xi| < 1$. If $q_2 B(p)\{2q_1(A(p))^M + q_2\} = 0$, then we have $\xi = (A(p))^2 + (q_1 B(p))^2$. Thus, for $|\xi| < 1$ we obtain

$$(A(p))^2 + (q_1 B(p))^2 < 1. \tag{5}$$

In what follows, let us consider the case of

$$q_2 B(p) \left\{2q_1 (A(p))^M + q_2\right\} \neq 0. \tag{6}$$

If $q_1 = 0$, (4) reduces to $\xi^{M+1} = (A(p))^2 \xi^M + (q_2 B(p))^2$. In a similar way to [2], we can see that all roots of this equation satisfy $|\xi| < 1$ if and only if

$$(A(p))^2 + (q_2 B(p))^2 < 1. \tag{7}$$

Next, we suppose $q_1 \neq 0$. Noting that (5) and (7) imply $|A(p)| < 1$, let us assume it also in this case. For stability analysis, we utilize the root locus technique similarly to [1]. In order to investigate ξ with modulus 1, let us set $\xi = \mathrm{e}^{i\varphi}$ for $\varphi \in \mathbb{R}$. If $\xi = \mathrm{e}^{i\varphi}$ is a root of (4), then $\xi = \mathrm{e}^{-i\varphi}$ is also a root. Thus, we devote ourselves to the case of $\varphi \in [0, \pi]$. We can rewrite (4) as

$$\cos\varphi + i\sin\varphi = (A(p))^2 + (B(p))^2 \left\{q_1^2 + \left(2q_1 q_2 (A(p))^M + q_2^2\right)(\cos M\varphi - i \sin M\varphi)\right\}. \tag{8}$$

If $\varphi = 0$, then (8) leads to

$$1 = R(p, q_1, q_2) \stackrel{\text{def}}{=} (A(p))^2 + (B(p))^2 \left\{q_1^2 + 2q_1 q_2 (A(p))^M + q_2^2\right\}. \tag{9}$$

This has many roots (q_1, q_2) since $|A(p)| < 1$ and $B(p) \neq 0$, and the roots are described by an ellipse, say C_0, for each p. On the other hand, if $\varphi = \pi$, then (8) has no root for any even number M, but since (8) leads to

$$-q_1^2 + 2q_1 q_2 (A(p))^M + q_2^2 = \frac{1 + (A(p))^2}{(B(p))^2} \tag{10}$$

for any odd number M, this has many roots (q_1, q_2), and the roots are described by a pair of hyperbolas for each p. As $q_2 = 0$ does not satisfy (10), one hyperbola C_π^+ lies in the upper half plane, and the other hyperbola C_π^- lies in the lower half plane. On the other hand, as $q_1 = 0$ satisfies (10), C_π^+ and C_π^- intersects the q_2 axis. Next, if $\varphi = k\pi/M$, $k = 1, 2, \dots, M-1$, then (8) obviously does not hold.

Finally, if $\varphi \in (k\pi/M, (k+1)\pi/M)$, $k = 1, 2, \dots, M-1$, then we obtain

$$2q_1 q_2 (A(p))^M + q_2^2 + \frac{\sin\varphi}{\sin M\varphi} \frac{1}{(B(p))^2} = 0 \tag{11}$$

by comparing the imaginary parts in both sides of (8), and

$$q_1^2 = \frac{\sin(M+1)\varphi}{\sin M\varphi} \frac{1}{(B(p))^2} - \frac{(A(p))^2}{(B(p))^2} \tag{12}$$

by utilizing (11) and comparing the real parts. For any even number k, we have $\sin\varphi > 0$ and $\sin M\varphi > 0$. Thus, if $\sin(M+1)\varphi \le 0$, then (12) has no root, and even if $\sin(M+1)\varphi > 0$, (11) has no root since its discriminant is negative. For any odd number k, we have $\sin M\varphi < 0$ and $\sin(M+1)\frac{k\pi}{M} < 0$. In addition, $\frac{\sin(M+1)\varphi}{\sin M\varphi}$ is strictly monotonically decreasing in the interval $(k\pi/M, (k+1)\pi/M)$. Thus, there exists a unique $\varphi_k \in (0,\pi)$ such that

$$\frac{\sin(M+1)\frac{k\pi+\varphi_k}{M}}{\sin M\frac{k\pi+\varphi_k}{M}} = (A(p))^2.$$

If $\varphi \in (\frac{k\pi+\varphi_k}{M}, \frac{(k+1)\pi}{M})$, then (12) has no root. On the other hand, if $\varphi \in (\frac{k\pi}{M}, \frac{k\pi+\varphi_k}{M}]$, (12) and (11) have many roots (q_1, q_2). Consequently, the roots are described by the following curves

$$C_k^{\pm} = \left\{(q_1,q_2) \mid q_1^2 = \frac{\sin(M+1)\varphi}{\sin M\varphi}\frac{1}{(B(p))^2} - \frac{(A(p))^2}{(B(p))^2}, \quad q_2 = -q_1(A(p))^M \pm \sqrt{q_1^2(A(p))^{2M} - \frac{\sin\varphi}{\sin M\varphi}\frac{1}{(B(p))^2}}\right\}$$

(double sign in order), where $k = 1, 3, \ldots, l_M$ and l_M denotes the largest odd number satisfying $l_M \le M$. The curves C_k^+, $k = 1,3,\ldots,l_M$, lie in the upper half plane, and the curves C_k^- lie in the lower half plane. All C_k^+ and C_k^- intersect the q_2 axis.

Therefore, we have obtained the curves $C_0, C_\pi^{\pm}$ and $C_k^{\pm}$ each of that shows each set of (q_1, q_2) corresponding to a root $\xi = e^{i\varphi}$ of (4) for $\varphi \in [0,\pi]$. In a similar way to [1], we have the following:

i) All curves $C_0, C_\pi^{\pm}, C_k^{\pm}$ do not intersect each other in the q_1-q_2 plane for a given p.
ii) The ellipse C_0 intersects the q_1 axis twice.
iii) When M is an odd number, C_π^+ lies above all C_k^+, $k = 1,3,\ldots,l_M$, and C_π^- lies below all C_k^-.

From ii) and (9), the origin $(q_1,q_2) = (0,0)$ lies in the interior of C_0. As the roots of (4) continuously depend on the coefficients, for (q_1,q_2) in a neighborhood of the origin, all roots of (4) satisfy $|\xi| < 1$ due to $|A(p)| < 1$. Moreover, if a root $\xi = e^{\mu+i\varphi}$ has $\mu < 0$ for a pair (q_1,q_2) in the interior of C_0, then it keeps μ negative for any (q_1,q_2) in the interior of C_0. However, when a pair (q_1,q_2) moves from the interior of C_0 to the exterior, a root $\xi = e^{\mu+i\varphi}$ with $\mu < 0$ may once change to $\xi = e^{\mu+i\varphi}$ with $\mu = 0$ and $\varphi = 0$, and may change to $\xi = e^{\mu+i\varphi}$ with $\mu > 0$, continuously. In fact, if we take a q_2 in a neighborhood of $q_2 = 0$ and take a large $|q_1|$ violating (5), then (4) has a root ξ with $|\xi| > 1$. Due to iii), our next concern is what a change can be made for the roots when a pair (q_1,q_2) goes up or down across C_k^+ or C_k^-, respectively.

When we substitute $\xi = e^{\mu+i\varphi}$ with $\varphi \in (\frac{k\pi}{M}, \frac{k\pi+\varphi_k}{M}]$ for $k = 1,3,\ldots,l_M$ into (4), we obtain

$$u(q_1,q_2) \stackrel{\text{def}}{=} \left\{2q_1q_2(A(p))^M + q_2^2\right\}(B(p))^2 e^{-M\mu}\sin M\varphi + e^{\mu}\sin\varphi = 0,$$
$$v(q_1,q_2) \stackrel{\text{def}}{=} \frac{e^{\mu}\sin(M+1)\varphi}{\sin M\varphi} - (A(p))^2 - (q_1B(p))^2 = 0$$

in a similar way to (11) and (12). Here, denote by $J(q_1,q_2)$ the determinant of the Jacobian matrix of $[u(q_1,q_2)\ v(q_1,q_2)]^\top$. As

$$J(q_1,q_2) = -4(B(p))^4 q_1\left\{q_1(A(p))^M + q_2\right\}e^{-M\mu}\sin M\varphi,$$

we have $\sin M\varphi < 0$ and $q_1(A(p))^M + q_2 > 0$ or < 0 for (q_1,q_2) on C_k^+ or C_k^-, respectively. Thus, $J(q_1,q_2)q_1 > 0$ for C_k^+, whereas $J(q_1,q_2)q_1 < 0$ for C_k^-. In order to get a conclusion, we introduce the following proposition [4, p. 311].

Proposition 1 *The critical roots are in the right half plane for any parameter in the parameter region to the left of the curve $(q_1(\varphi), q_2(\varphi))$, when we follow this curve in the direction of increasing φ, whenever $J(q_1(\varphi), q_2(\varphi)) < 0$ and to the right when $J(q_1(\varphi), q_2(\varphi)) > 0$.*

Note that q_1^2 decreases when φ increases on $C_k^{\pm}$. By applying the proposition to our results, we can see that a root $\xi = e^{\mu+i\varphi}$ with $\mu < 0$ once changes to $\xi = e^{\mu+i\varphi}$ with $\mu = 0$, and changes to $\xi = e^{\mu+i\varphi}$ with $\mu > 0$, when a pair (q_1,q_2) goes up or down across C_k^+ or C_k^-, respectively. Since we do not have $C_\pi^{\pm}$ for an even number M, the parameter region for which all roots of (4) satisfy $|\xi| < 1$ is determined to be the interior of C_0 only. When M is an odd number, we have C_π^+ and C_π^- above and below the other curves, respectively. In a similar way to [1], we can see that there is no root with $|\xi| < 1$ for any parameter in the parameter regions above C_π^+ and below C_π^-. Thus, we have the same conclusion as the case of even number M. Summarising all things, we obtain the following proposition.

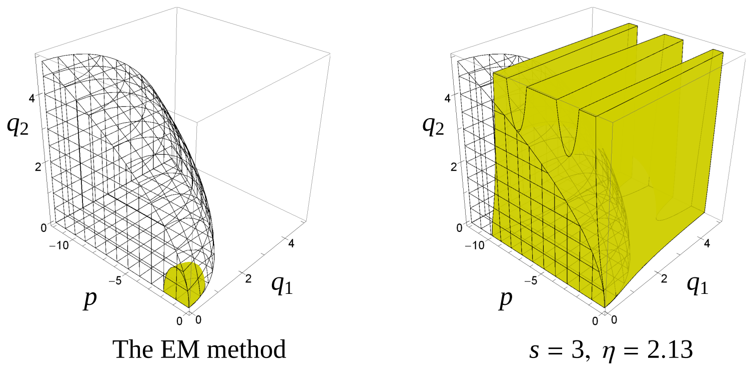

FIGURE 1. MS stability domains of the EM method and the SROCK method with $s = 3$ when $M = 1$

Proposition 2 *The numerical method (3) with a step size $h > 0$ is MS-stable for (1) if h, λ, σ_1 and σ_2 satisfy $|A(p)| < 1$ and $R(p, q_1, q_2) < 1$.*

Here, note that $R(p, q_1, q_2) < 1$ implies (5) and (7). Thus, the proposition is available even if $B(p)$ can be zero, whereas a similar theorem in [1] does not allow $B(p) = 0$.

STABILITY ANALYSIS OF SROCK METHODS

By utilizing Proposition 2, let us analyse the stability properties of the SROCK methods. When applied to (1), the method with s stages is expressed as

$$y_{n+1} = P_s(h\lambda)y_n + P_s(p)(\sigma_1 y_n + \sigma_2 y_{n-M})\Delta W,$$

where $P_s(p) = T_s(\omega_0 + \omega_1 p)/T_s(\omega_0)$, $\omega_0 = 1 + (\eta/s^2)$, $\omega_1 = T_s(\omega_0)/T'_s(\omega_0)$, and where $T_k(x)$ is the Chebyshev polynomial of degree k and $\eta \geq 0$ is a a damping parameter [2]. This yields

$$A(p) = P_s(p), \qquad R(p, q_1, q_2) = (P_s(p))^2 + (P_s(p))^2 \left\{q_1^2 + 2q_1 q_2 (P_s(p))^M + q_2^2\right\}.$$

On the other hand, due to $\tau = Mh$, (2) is rewritten as

$$q_1^2 + q_2^2 + 2q_1 q_2 \mathrm{e}^{Mp} < -2p. \tag{13}$$

Using these, we can compare the stability domain of the SROCK methods, $\{(p, q_1, q_2) \mid |P_s(p)| < 1,\ R(p, q_1, q_2) < 1\}$, with the domain that satisfies (13). In Figure 1, the stability domains of the explicit EM method and the SROCK method with $s = 3$ are indicated with the colored part. The other part enclosed by mesh indicates the domain that satisfies (13). The figure shows that the SROCK method has a much larger stability domain than the explicit EM method even when $s = 3$, but it can be further extended along the negative axis of p as s increases.

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