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New Derivation of Conservation Laws
for Optimal Control Problem
and its Application to Economic Dynamics



九州工業大学附属図書館



0010471365

Fumiyo FUJIWARA

February, 2002

Abstract

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for the derivation of conserved quantities. Though, since the theorem closely depends on the Lagrangian or the Hamiltonian structures of considering problem, there is a limit to the derivable conserved quantities. The presented paper is devoted, leaving the structures out of account, to built up a new operative procedure for discovering conserved quantities. The procedure is realized and detailed to make an effective application to the following models in optimal control problems. Various conserved quantities including non-Noether ones can be derivable in the models.

In **1**, on a direct product space of the tangent bundle of configuration manifold and time-axis, a geometric procedure for the derivation of conserved quantities is formulated within the context of exterior differential calculus. (Mimura, Ikeda and Fujiwara [31]). The procedure is carried effectively into dynamic systems without the Lagrangian or the Hamiltonian structures. Also, even for dynamic systems to which the Noether theorem is applicable, we can expect new conserved quantities including non-Noether ones. Applications for the procedure to several models are given in the following sections.

At first in **2**, we discuss an optimal control problem to maximize an integration over a period of time (finite or infinite) under constraint given by a system of functions with arbitrary degree of homogeneity (Fujiwara, Mimura and Nôno [15]). And then we deduce two independent conserved quantities, which are illustrated in a generalized model of von Neumann type which grew up from von Neumann growth model for the capital goods and the capital formations (Samuelson [40]). Finally we supply an equivalent class of utility functions in the generalized model (Fujiwara, Mimura and Nôno [27]).

In **3**, we formulate a way of constructing conserved quantities in a maximizing problem under constraint in a space with state and control variables (Fujiwara, Mimura and Nôno [11]). Particular conserved quantity is available in the case where the control variables are given by the derivative of the state variables and the integrating function is given by an arbitrary degree homogeneous one with the discount rate. The conserved quantity, which is constructed by the relating Hamiltonian in the considering problem, reduces to the usual Hamiltonian when the discount rate vanishes. In applications, we show some conserved quantities in the generalization of the neoclassical growth models including a model to which Sato [44] gave a negative answer for the existence of global conserved quantities from the Noether theorem, and also of the models in the intergenerational problem (Hotelling [23]; Hartwick [19], [20]) and in the q -theory of investment (Tobin [49]). Moreover for some models,

e.g., a model with Cobb-Douglas production function or a generalization of Hayashi model (Hayashi [21]), optimal paths are determined from the obtained conserved quantities. Every detail of Tobin's q -theory is given through the determined optimal paths in the generalized Hayashi model.

In 4, it is assumed that the derivative of the state variables can be written by first order polynomials of the state variables and the control variables. Under such constraint, we construct conserved quantities in the considering maximizing problem (Fujiwara, Mimura and Nôno [12],[14]). To make applications, we first generalize the growth model of Ramsey type (Ramsey [38]) which is characterized with the linear transformation function between consumption and capital accumulation, and then the model is extended in an external two sector version (moreover, three sector version in 5 (Fujiwara, Mimura and Nôno [13],[16])). Conserved quantities are obtained and then optimal paths are determined in the models. Typically, in this section, the utility functions in the models are not assumed to be provided with the homogeneity.

In 6, the two sector model in 4 is reformed by replacing the arbitrary utility function of the control variables with a second order polynomial of the state variable and the control variables (Fujiwara, Mimura and Nôno [17]). Similarly as in 4, optimal paths are determined through conserved quantities. The optimal paths illustrate the Pareto-efficient steady-state (Fershtman and Nitzan [8]).

In 7, we consider the n -sector differential game in which the sectors (the players) are not able to make binding commitments in advance of play on the strategies (Fujiwara, Mimura and Nôno [18]). Such strategies are called open-loop Nash strategies. In the game, each sector has his own objective functional to maximize under a constraint leaving the strategies of the other sectors out of account. This fact put difficulties for the application of, as well as the Noether theorem, the new procedure formulated in 1, so that general derivation of conserved quantities has been never discussed. Extending the original variables, we find a composite maximizing problem (an extended maximizing problem in the usual variational principle), in which the optimal paths of the original variables are those in the maximizing problem in the open-loop Nash strategies. By applying the new procedure in 1 to the composite maximizing problem in the usual variational principle, we find conserved quantities and then determined optimal paths for a model in two-sector open-loop Nash strategies. The optimal paths illustrate the level of the collective contribution of public goods at the stationary open-loop Nash equilibrium in (Fershtman and Nitzan [8]).

As final application, we consider a single particle moving under a central force directed towards a fixed center in a resisting medium, where the force is proportional to the parti-

cle's distance from the center and the medium imposes a retarding force proportional to the velocity (Fujiwara, Ikeda and Mimura [9]). By applying the new procedure in **1** to the relating Euler-Lagrange system in the problem, we find two independent conserved quantities, through which the equation of orbit of the particle is determine and analyzed in detail.

As seen in the preceding applications to various dynamic systems appeared in the considering models, the new procedure for the derivation of conserved quantities will make great contributions to a development of dynamics.

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1 A geometric derivation of new conservation laws

1.1 Introduction

The Noether theorem [37] has been extensively initiated for the derivation of conservation laws based on the symmetries in the Lagrangian or the Hamiltonian structures. However, without using the structures (which may fail to exist), Caviglia [3, 5] determined the new operative procedure for the laws via the application of a suitable version of the Noether theorem to the composite variational principle. And the procedure was analyzed by Mimura and Nôno [32] (also [33]) with various viewpoints for the derivation of the laws of a given second-order (partial) differential system.

In this paper, the local version of [32] is reformulated with a geometric notion of the calculus on manifolds (refer, e.g., to Sarlet and Cantrijn [42]). In 1.2, on a setting of a bundle $J = TM \times \mathbb{R}$ for a configuration manifold M , a set of 1-forms \mathfrak{X}_Γ^* (symmetry 1-forms of Γ) and a set of vector fields \mathfrak{X}_Γ on J or its subset \mathfrak{X}_K^ω ($\omega \in \mathfrak{X}_\Gamma^*$, $K \in \mathfrak{F}$: differentiable functions on J) are introduced associating with the equation field Γ of a given differential system in the kinematical form. In 1.3, conserved quantities of Γ , i.e., quantities C on J satisfying $\Gamma(C) = 0$, can be constructed from elements of \mathfrak{X}_Γ^* and \mathfrak{X}_K^ω , or particularly $\mathfrak{X}_K^{d\Omega}$ ($d\Omega \in \mathfrak{X}_\Gamma^*$, where Ω is a conserved quantity). In 1.4, a further derivation of constructing conserved quantities is given under a correspondence between an element of \mathfrak{X}_Γ and an element of \mathfrak{X}_Γ^* defined by a given regular Lagrangian (the regularity can be released later), or particularly an element $d\Omega \in \mathfrak{X}_\Gamma^*$ given by a conserved quantity Ω . In 1.5, such a correspondence is explicitly realized by virtue of a closed 2-form defined by the exterior derivative of Poincaré-Cartan form. The realization contributes to make the ring of all conserved quantities of Γ into an infinite dimensional Lie algebra.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

1.2 Geometric characters associated with second-order system

Adding the time-axis \mathbb{R} to the tangent bundle TM of m -dimensional configuration manifold M , let $J = TM \times \mathbb{R}$ and U be its local chart with coordinate functions $(\dot{q}, q, t) = (\dot{q}^\alpha(t), q^\alpha(t), t)$ ($\alpha = 1, \dots, m$). On the setting, we consider a given second-order system of m differential equations in the kinematical form:

$$(1.1) \quad \ddot{q}^\alpha = f^\alpha(\dot{q}, q, t),$$

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together with its equation field

$$\Gamma = f^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \frac{\partial}{\partial t}.$$

To a contact system $\{\theta^\alpha\}$, where θ^α are given in coordinates as $\theta^\alpha = dq^\alpha - \dot{q}^\alpha dt$ on U , there corresponds a contact distribution, i.e., a subset Δ of vector fields \mathfrak{X} on J (characteristic vector fields of θ^α):

$$\Delta = \{X \in \mathfrak{X} \mid i_X \theta^\alpha = 0; \alpha = 1, \dots, m\},$$

where i_X denotes the interior product (contraction) by X . Associating with the equation field Γ , the distribution is used to define

$$\mathfrak{X}_\Gamma = \{X \in \mathfrak{X} \mid [\Gamma, X] \in \Delta\}.$$

Then, in view of the relations

$$\begin{aligned} \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] &= \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial}{\partial \dot{q}^\beta} + \frac{\partial}{\partial q^\alpha}, \\ \left[\frac{\partial}{\partial q^\alpha}, \Gamma \right] &= \frac{\partial f^\beta}{\partial q^\alpha} \frac{\partial}{\partial \dot{q}^\beta}, \end{aligned}$$

a vector field $X \in \mathfrak{X}$, when expressed with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$:

$$X = \eta^\alpha \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha} + \psi \Gamma,$$

has the following commutator with Γ

$$\begin{aligned} (1.2) \quad [\Gamma, X] &= \Gamma(\eta^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \Gamma(\xi^\alpha) \frac{\partial}{\partial q^\alpha} + \Gamma(\psi) \Gamma - \eta^\alpha \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] - \xi^\alpha \left[\frac{\partial}{\partial q^\alpha}, \Gamma \right] \\ &= \left(\Gamma(\eta^\alpha) - \eta^\beta \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) \frac{\partial}{\partial \dot{q}^\alpha} + (\Gamma(\xi^\alpha) - \eta^\alpha) \frac{\partial}{\partial q^\alpha} + \Gamma(\psi) \Gamma. \end{aligned}$$

So that $i_{[\Gamma, X]} \theta^\alpha = 0$ imply $\eta^\alpha = \Gamma(\xi^\alpha)$. Therefore we have a local expression of $X \in \mathfrak{X}_\Gamma$ (cf. [42], the dynamical symmetry of Γ in Lemma 3.1; Eq. (17) and (18) in [3])

Theorem 1.1. *A vector field $X \in \mathfrak{X}_\Gamma$ is written as follows with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$:*

$$(1.3) \quad X = \Gamma(\xi^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha} + \psi \Gamma.$$

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Particularly a vector field X satisfying $[\Gamma, X] = 0$ is an element of \mathfrak{X}_Γ . In this case, (1.3) is provided with the conditions $\Gamma(\psi) = 0$ and

$$(1.4) \quad \Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} = 0.$$

Alternatively, in terms of the Lie derivative \mathcal{L}_Γ , a subset of 1-forms \mathfrak{X}^* on J (symmetry 1-forms of Γ) is defined:

$$\mathfrak{X}_\Gamma^* = \{\omega \in \mathfrak{X}^* \mid \mathcal{L}_\Gamma \omega = 0\}.$$

Here introduce 1-forms $\phi^\alpha = d\dot{q}^\alpha - f^\alpha dt$ for the basis $\{\phi^\alpha, \theta^\alpha, dt\}$ to write $\omega \in \mathfrak{X}^*$ on U as

$$\omega = \mu_\alpha \phi^\alpha + \nu_\alpha \theta^\alpha + \tau dt.$$

Then, in view of

$$\begin{aligned} \mathcal{L}_\Gamma \phi^\alpha &= \frac{\partial f^\alpha}{\partial \dot{q}^\beta} \phi^\beta + \frac{\partial f^\alpha}{\partial q^\beta} \theta^\beta, \\ \mathcal{L}_\Gamma \theta^\alpha &= \phi^\alpha, \end{aligned}$$

it follows that

$$\mathcal{L}_\Gamma \omega = \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} + \nu_\alpha \right) \phi^\alpha + \left(\Gamma(\nu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} \right) \theta^\alpha + \Gamma(\tau) dt.$$

Therefore, the condition $\mathcal{L}_\Gamma \omega = 0$ for $\omega \in \mathfrak{X}_\Gamma^*$ leads to $\Gamma(\tau) = 0$ and

$$\begin{aligned} \Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} + \nu_\alpha &= 0, \\ \Gamma(\nu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} &= 0, \end{aligned}$$

from which we have a local expression of $\omega \in \mathfrak{X}_\Gamma^*$ (cf. the adjoint symmetry in [43]; Eq. (14) in [3]):

Theorem 1.2. *A 1-form $\omega \in \mathfrak{X}_\Gamma^*$ is written as follows with respect to the basis $\{\phi^\alpha, \theta^\alpha, dt\}$:*

$$(1.5) \quad \omega = \mu_\alpha \phi^\alpha - \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) \theta^\alpha + \tau dt,$$

which is provided with the conditions $\Gamma(\tau) = 0$ and

$$(1.6) \quad \Gamma \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) - \mu_\beta \frac{\partial f^\beta}{\partial q^\alpha} = 0.$$

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Particularly a multiple $\psi\Gamma$ ($\psi \in \mathfrak{F}$: differentiable functions on J) of Γ is an element of \mathfrak{X}_Γ (see (1.3) with ξ^α). So we regard that X_1 and X_2 in \mathfrak{X}_Γ are equivalent if $X_1 - X_2$ is equal to such a multiple; and in each equivalent class of \mathfrak{X}_Γ , we can take particularly an element

$$(1.3)' \quad X = \Gamma(\xi^\alpha) \frac{\partial}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial}{\partial q^\alpha}.$$

Then, for $\omega \in \mathfrak{X}_\Gamma^*$ of the form (1.5), through (1.2) with $\psi = 0$ and $\eta^\alpha = \Gamma(\xi^\alpha)$, we have

$$(1.7) \quad i_{[\Gamma, X]} \omega = \mu_\alpha \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right).$$

For an arbitrary 1-form $\omega \in \mathfrak{X}_\Gamma^*$, associating with an element $K \in \mathfrak{F}$, we now define a subset \mathfrak{X}_K^ω of \mathfrak{X}_Γ :

$$\mathfrak{X}_K^\omega = \{X \in \mathfrak{X}_\Gamma \mid i_{[\Gamma, X]} \omega = \Gamma(K)\}.$$

Then it follows that (cf. Eq. (3.8b) in [5])

Theorem 1.3. *A vector field X in each equivalent class of \mathfrak{X}_Γ can be put as (1.3)' with respect to the basis $\{\partial/\partial \dot{q}^\alpha, \partial/\partial q^\alpha, \Gamma\}$. And then, for $K \in \mathfrak{F}$ and $\omega \in \mathfrak{X}_\Gamma^*$ of the form (1.5), $X \in \mathfrak{X}_K^\omega$ if and only if*

$$(1.8) \quad \mu_\alpha \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) = \Gamma(K).$$

Remark 1.1. For $X \in \mathfrak{X}_\Gamma$ of the form (1.3), since in view of $\Gamma(\tau) = 0$ and $i_\Gamma \phi^\alpha = i_\Gamma \theta^\alpha = 0$:

$$i_{[\Gamma, \psi\Gamma]} \omega = \Gamma(\psi) i_\Gamma \omega = \Gamma(\psi) \tau = \Gamma(\psi\tau),$$

the right hand side of (1.7) is modified by adding a term $\Gamma(\psi\tau)$. So the condition $i_{[\Gamma, X]} \omega = \Gamma(K)$ for \mathfrak{X}_K^ω differs from (1.8) essentially nothing but in that K must be replaced with $K - \psi\tau$.

1.3 Conserved quantities associated with \mathfrak{X}_Γ^* and \mathfrak{X}_K^ω

Elements of \mathfrak{X}_Γ^* and \mathfrak{X}_K^ω can be used effectively to construct conserved quantities of Γ (first integrals of the system (1.1)), i.e., quantities $C(\dot{q}, q, t)$ on J satisfying $\Gamma(C) = 0$. An element $\omega \in \mathfrak{X}_\Gamma^*$ vanishes by \mathcal{L}_Γ , i.e., $\mathcal{L}_\Gamma \omega = 0$, so that

$$i_{[\Gamma, X]} \omega = \Gamma(i_X \omega) - i_X(\mathcal{L}_\Gamma \omega) = \Gamma(i_X \omega);$$

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which implies $\Gamma(i_X\omega - K) = 0$ if $X \in \mathfrak{X}_K^\omega$. Therefore it is deduced (cf. Eq. (22) in [3]):

Theorem 1.4. *Elements $\omega \in \mathfrak{X}_\Gamma^*$ and $X \in \mathfrak{X}_K^\omega$ give rise to a conserved quantity C :*

$$(1.9) \quad C = i_X\omega - K.$$

The respective appearance (1.5) and (1.3)' of $\omega \in \mathfrak{X}_\Gamma^*$ and $X \in \mathfrak{X}_K^\omega$ give a local version of the Theorem 1.4. In fact the solutions μ_α of (1.6) and ξ^α of (1.8) yield the conserved quantity ([32], Remark 4; cf. [5], Theorem)

$$(1.10) \quad i_X\omega - K = \mu_\alpha \Gamma(\xi^\alpha) - \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right) \xi^\alpha - K.$$

Particularly, together with the solutions μ_α of (1.6), an element $X \in \mathfrak{X}_0^\omega$ ($K = 0$) satisfying $[\Gamma, X] = 0$, i.e., ξ^α of (1.3)' satisfying (1.4) yield the conserved quantity $i_X\omega$, i.e., the quantity (1.10) with $K = 0$ ([32], Theorem 2).

Whenever f^α do not depend explicitly on the time t , i.e., $f^\alpha = f^\alpha(\dot{q}, q)$, the equation field Γ supplies $[\Gamma, \Gamma - \partial/\partial t] = 0$, so that $\Gamma_0 \equiv \Gamma - \partial/\partial t \in \mathfrak{X}_0^\omega$ for arbitrary $\omega \in \mathfrak{X}_\Gamma^*$. Therefore solutions μ_α of (1.6) yield the conserved quantity ([32], Theorem 5; cf. Eq. (21) in [3])

$$(1.11) \quad i_{\Gamma_0}\omega = \mu_\alpha f^\alpha - \dot{q}^\alpha \left(\Gamma(\mu_\alpha) + \mu_\beta \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \right).$$

Remark 1.2. Similarly as in the remark 1.1, if $X \in \mathfrak{X}_K^\omega$ is not of the form (1.3)' but (1.3), K is replaced with $K - \psi\tau$ in the resulting conserved quantity (1.10).

A conserved quantity Ω satisfies $\mathcal{L}_\Gamma(d\Omega) = d\Gamma(\Omega) = 0$, i.e., $\omega \equiv d\Omega \in \mathfrak{X}_\Gamma^*$. In view of $i_X d\Omega = X(\Omega)$, it reduces the Theorem 1.4 to

Theorem 1.5. *Together with a vector field $X \in \mathfrak{X}_K^{d\Omega}$, a conserved quantity Ω gives rise to a new one C :*

$$(1.12) \quad C = X(\Omega) - K.$$

In the coordinates, since $d\Omega$ of a conserved quantity Ω is expressed as

$$(1.13) \quad d\Omega = \frac{\partial \Omega}{\partial \dot{q}^\alpha} \phi^\alpha + \frac{\partial \Omega}{\partial q^\alpha} \theta^\alpha + \Gamma(\Omega) dt = \frac{\partial \Omega}{\partial \dot{q}^\alpha} \phi^\alpha + \frac{\partial \Omega}{\partial q^\alpha} \theta^\alpha,$$

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the Theorem 1.3 implies that X of the form (1.3)' is an element of $\mathfrak{X}_K^{d\Omega}$ if and only if

$$\frac{\partial \Omega}{\partial \dot{q}^\alpha} \left(\Gamma^2(\xi^\alpha) - \Gamma(\xi^\beta) \frac{\partial f^\alpha}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\alpha}{\partial q^\beta} \right) = \Gamma(K),$$

under which the resulting conserved quantity (1.12) is written as

$$(1.14) \quad X(\Omega) - K = \Gamma(\xi^\alpha) \frac{\partial \Omega}{\partial \dot{q}^\alpha} + \xi^\alpha \frac{\partial \Omega}{\partial q^\alpha} - K.$$

Particularly an element $X \in \mathfrak{X}_0^{d\Omega}$ ($K = 0$) satisfying $[\Gamma, X] = 0$, i.e., ξ^α of (1.3)' satisfying (1.4), yields the conserved quantity (1.14) with $K = 0$ ([32], Theorem 4).

1.4 Euler-Lagrange systems

Let $L(\dot{q}, q, t)$ be a regular Lagrangian, i.e., $\det(W_{\alpha\beta}) \neq 0$ where $W_{\alpha\beta} = \partial^2 L / \partial \dot{q}^\alpha \partial \dot{q}^\beta$, and the system (1.1) have resulted from the Euler-Lagrange equations

$$(1.15) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = \ddot{q}^\gamma W_{\alpha\gamma} + \dot{q}^\gamma \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\gamma} + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial t} - \frac{\partial L}{\partial q^\alpha} = 0.$$

Then, after replacing \ddot{q}^γ with f^γ in (1.15), the equations are differentiated with respect to \dot{q}^β and q^β to obtain respectively

$$(1.16) \quad W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial \dot{q}^\beta} = -\Gamma(W_{\alpha\beta}) - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} + \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha},$$

$$(1.17) \quad W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} = -\Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} \right) + \frac{\partial^2 L}{\partial q^\alpha \partial \dot{q}^\beta};$$

where the skew-symmetric parts of (1.17) for the indices α and β lead to

$$(1.18) \quad W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} - W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} = \Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} \right).$$

Now $\mu_\alpha = W_{\alpha\beta} \xi^\beta$ are substituted for the coefficient of θ^α in (1.5), and then (1.16) is used to see

$$(1.19) \quad \begin{aligned} \nu_\alpha &\equiv \Gamma(\mu_\alpha) + \mu_\gamma \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \\ &= W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + W_{\gamma\beta} \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \right) \xi^\beta \\ &= W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial \dot{q}^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} \right) \xi^\beta; \end{aligned}$$

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and also for (1.6), together with the above ν_α , to see

$$\begin{aligned} & \Gamma \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) - W_{\gamma\beta} \xi^\beta \frac{\partial f^\gamma}{\partial q^\alpha} \\ &= W_{\alpha\beta} \Gamma^2(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \Gamma(\xi^\beta) \\ &+ \left(\Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) - W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} \right) \xi^\beta = 0, \end{aligned}$$

in which, by (1.18) and (1.17), the terms in the last parenthesis are moreover rewritten as

$$\begin{aligned} & \Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) - W_{\beta\gamma} \frac{\partial f^\gamma}{\partial q^\alpha} = -W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial q^\beta} \\ &= \Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \right) - \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta}. \end{aligned}$$

Therefore from the Theorem 1.2 it follows (cf. Eq. (3.7) in [4]; Eq. (15) in [32]):

Theorem 1.6. *A 1-form ω_L associated with the regular Lagrangian L :*

$$(1.20) \quad \omega_L = W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) \theta^\alpha + \tau dt$$

is an element of \mathfrak{X}_Γ^ if and only if $\Gamma(\tau) = 0$ and ξ^α satisfy*

$$(1.21) \quad W_{\alpha\beta} \Gamma^2(\xi^\beta) + \left(\Gamma(W_{\alpha\beta}) + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \Gamma(\xi^\beta) + \left(\Gamma \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \right) - \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta} \right) \xi^\beta = 0.$$

The left hand side of (1.21), for which (1.16) and (1.17) are substituted, is just the same as

$$W_{\alpha\gamma} \left(\Gamma^2(\xi^\gamma) - \Gamma(\xi^\beta) \frac{\partial f^\gamma}{\partial \dot{q}^\beta} - \xi^\beta \frac{\partial f^\gamma}{\partial q^\beta} \right).$$

So that, since $\det(W_{\alpha\beta}) \neq 0$, the equations (1.4) and (1.21) are to be equivalent. Therefore from the Theorem 1.1 it follows:

Theorem 1.7. *To a 1-form $\omega_L \in \mathfrak{X}_\Gamma^*$ of the form (1.20) associated with the regular Lagrangian L , there corresponds a vector field $X \in \mathfrak{X}_\Gamma$ of the form (1.3) uniquely up to a multiple of Γ satisfying $[\Gamma, X] \equiv 0 \pmod{\Gamma}$, and vice versa.*

1 A geometric derivation of new conservation laws

The correspondence in the Theorem 1.7 can be applied to the Theorem 1.4 for a derivation of conserved quantities of the Euler-Lagrange equations (1.15).

At first, in a pair of elements ω_L^1 and ω_L^2 of \mathfrak{X}_Γ^* , i.e., in those of solutions ξ_i^α ($i = 1, 2$) of (1.21), ξ_1^α define an element $X \in \mathfrak{X}_\Gamma$ of the form (1.3)' satisfying $[\Gamma, X] = 0$, while $\xi^\alpha = \xi_2^\alpha$ are left in the appearance (1.20) of ω_L^2 . Then, by the Theorem 1.4, such elements $\omega_L^2 \in \mathfrak{X}_\Gamma^*$ and $X \in \mathfrak{X}_\Gamma$ (of course $X \in \mathfrak{X}_0^\omega$ for arbitrary $\omega \in \mathfrak{X}_\Gamma^*$) yield a conserved quantity (see (1.10) with $K = 0$)

$$-i_X \omega_L^2 = \nu_\alpha \xi_1^\alpha - W_{\alpha\beta} \xi_2^\beta \Gamma(\xi_1^\alpha),$$

where ν_α are given as (1.19) in which $\xi^\alpha = \xi_2^\alpha$ and $\mu_\alpha = W_{\alpha\beta} \xi_2^\beta$. So that ([32], Theorem 6)

$$(1.22) \quad -i_X \omega_L^2 = W_{\alpha\beta} (\xi_1^\alpha \Gamma(\xi_2^\beta) - \xi_2^\beta \Gamma(\xi_1^\alpha)) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi_1^\alpha \xi_2^\beta.$$

It can be seen immediately that the equation field Γ in (1.21) and (1.22) can be replaced with the total time derivative d/dt on solutions to the relating Euler-Lagrange systems (1.15). Therefore, the regularity condition of L is released and then the derivation of conserved quantities are completed as follows:

Theorem 1.8. For the Lagrangian which may not be assumed to be regular, let ξ_i^α ($i = 1, 2$) satisfy

$$(1.23) \quad W_{\alpha\beta} \frac{d^2 \xi^\beta}{dt^2} + \left(\frac{dW_{\alpha\beta}}{dt} + \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \frac{d\xi^\beta}{dt} + \left(\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \right) - \frac{\partial^2 L}{\partial q^\alpha \partial q^\beta} \right) \xi^\beta = 0$$

on the solutions to the relating Euler-Lagrange system (1.15). Then the following conserved quantity of (1.15) is constructed:

$$(1.24) \quad \Omega = W_{\alpha\beta} \left(\xi_1^\alpha \frac{d\xi_2^\beta}{dt} - \xi_2^\beta \frac{d\xi_1^\alpha}{dt} \right) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi_1^\alpha \xi_2^\beta.$$

A further application of the Theorem 1.7 begins with a conserved quantity Ω . Since $d\Omega \in \mathfrak{X}_\Gamma^*$ as seen before, it can be written as (1.5). In fact, in view of $\Gamma(\Omega) = 0$ in

$$(1.25) \quad \begin{aligned} \left[\frac{\partial}{\partial \dot{q}^\alpha}, \Gamma \right] (\Omega) &= \left(\frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial}{\partial \dot{q}^\beta} + \frac{\partial}{\partial q^\alpha} \right) (\Omega), \\ \text{i.e.,} \quad -\Gamma \left(\frac{\partial \Omega}{\partial \dot{q}^\alpha} \right) &= \frac{\partial f^\beta}{\partial \dot{q}^\alpha} \frac{\partial \Omega}{\partial \dot{q}^\beta} + \frac{\partial \Omega}{\partial q^\alpha}, \end{aligned}$$

1 A geometric derivation of new conservation laws

$d\Omega$ of (1.13) leads to

$$(1.26) \quad d\Omega = \frac{\partial\Omega}{\partial\dot{q}^\alpha} \phi^\alpha - \left(\Gamma \left(\frac{\partial\Omega}{\partial\dot{q}^\alpha} \right) + \frac{\partial\Omega}{\partial\dot{q}^\beta} \frac{\partial f^\beta}{\partial\dot{q}^\alpha} \right) \theta^\alpha.$$

This appearance follows also from (1.20) by putting $\xi^\alpha = W^{\alpha\beta} \partial\Omega / \partial\dot{q}^\beta$ where $(W^{\alpha\beta}) = (W_{\alpha\beta})^{-1}$, i.e., $\mu_\alpha = W_{\alpha\beta} \xi^\beta = \partial\Omega / \partial\dot{q}^\alpha$; which are substituted for the first expression of ν_α in (1.19) to confirm that the coefficients of θ^α in (1.20) lead to those in the above $d\Omega$. Therefore in the Theorem 1.7, the corresponding vector field $X_\Omega \in \mathfrak{X}_\Gamma$ to $d\Omega \in \mathfrak{X}_\Gamma^*$ is

$$(1.27) \quad X_\Omega = \Gamma \left(W^{\alpha\beta} \frac{\partial\Omega}{\partial\dot{q}^\beta} \right) \frac{\partial}{\partial\dot{q}^\alpha} + W^{\alpha\beta} \frac{\partial\Omega}{\partial\dot{q}^\beta} \frac{\partial}{\partial q^\alpha},$$

which is provided with $[\Gamma, X_\Omega] = 0$. So that, together with an element $\omega \in \mathfrak{X}_\Gamma^*$ of the form (1.5), i.e., μ_α satisfying (1.6), the conserved quantity Ω , i.e., the corresponding vector field X_Ω , gives rise to a new one (see (1.9) with $K = 0$)

$$-i_{X_\Omega} \omega = -\mu_\alpha \Gamma \left(W^{\alpha\beta} \frac{\partial\Omega}{\partial\dot{q}^\beta} \right) + \left(\Gamma(\mu_\alpha) + \mu_\gamma \frac{\partial f^\gamma}{\partial\dot{q}^\alpha} \right) W^{\alpha\beta} \frac{\partial\Omega}{\partial\dot{q}^\beta},$$

in which the terms are rewritten by (1.25) as

$$\Gamma \left(W^{\alpha\beta} \frac{\partial\Omega}{\partial\dot{q}^\beta} \right) = \Gamma(W^{\alpha\beta}) \frac{\partial\Omega}{\partial\dot{q}^\beta} - W^{\alpha\beta} \left(\frac{\partial f^\gamma}{\partial\dot{q}^\beta} \frac{\partial\Omega}{\partial\dot{q}^\gamma} + \frac{\partial\Omega}{\partial q^\beta} \right);$$

so that

$$-i_{X_\Omega} \omega = \mu_\alpha \left(W^{\alpha\gamma} \frac{\partial f^\beta}{\partial\dot{q}^\gamma} + W^{\beta\gamma} \frac{\partial f^\alpha}{\partial\dot{q}^\gamma} - \Gamma(W^{\alpha\beta}) \right) \frac{\partial\Omega}{\partial\dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma(\mu_\alpha) \frac{\partial\Omega}{\partial\dot{q}^\beta} + \mu_\alpha \frac{\partial\Omega}{\partial q^\beta} \right).$$

Moreover the symmetric parts of (1.16) for the indices α and β :

$$\Gamma(W_{\alpha\beta}) = -\frac{1}{2} \left(W_{\alpha\gamma} \frac{\partial f^\gamma}{\partial\dot{q}^\beta} + W_{\beta\gamma} \frac{\partial f^\gamma}{\partial\dot{q}^\alpha} \right)$$

are substituted for the differentiation of $W_{\alpha\beta} W^{\beta\gamma} = \delta_\alpha^\gamma$ by Γ :

$$\Gamma(W_{\alpha\beta}) W^{\beta\gamma} + W_{\alpha\beta} \Gamma(W^{\beta\gamma}) = 0, \quad \text{i.e.,}$$

$$(1.28) \quad \Gamma(W^{\alpha\beta}) = -W^{\alpha\gamma} W^{\beta\sigma} \Gamma(W_{\gamma\sigma}),$$

to obtain

$$\Gamma(W^{\alpha\beta}) = \frac{1}{2} \left(W^{\alpha\gamma} \frac{\partial f^\beta}{\partial\dot{q}^\gamma} + W^{\beta\gamma} \frac{\partial f^\alpha}{\partial\dot{q}^\gamma} \right);$$

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which are used to have the final appearance ([32], Theorem 9)

$$(1.29) \quad -i_{X_\Omega} \omega = \frac{1}{2} \mu_\alpha \left(W^{\alpha\gamma} \frac{\partial f^\beta}{\partial \dot{q}^\gamma} + W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma} \right) \frac{\partial \Omega}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma(\mu_\alpha) \frac{\partial \Omega}{\partial \dot{q}^\beta} + \mu_\alpha \frac{\partial \Omega}{\partial q^\beta} \right), \quad \text{or}$$

$$(1.29)' \quad -i_{X_\Omega} \omega = \Gamma(\mu_\alpha W^{\alpha\beta}) \frac{\partial \Omega}{\partial \dot{q}^\beta} + \mu_\alpha W^{\alpha\beta} \frac{\partial \Omega}{\partial q^\beta}.$$

Particularly for conserved quantities Ω_1 and Ω_2 , ω in (1.29) is replaced with $d\Omega_1$, i.e., $\mu_\alpha = \partial\Omega_1/\partial\dot{q}^\alpha$, while $\Omega = \Omega_2$, to derive

$$-i_{X_{\Omega_2}} d\Omega_1 = \frac{1}{2} \left(W^{\alpha\gamma} \frac{\partial f^\beta}{\partial \dot{q}^\gamma} + W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} \right),$$

which, in view of (1.25), can be written as ([32], Theorem 10)

$$(1.30) \quad -i_{X_{\Omega_2}} d\Omega_1 = \frac{1}{2} W^{\alpha\gamma} \frac{\partial f^\beta}{\partial \dot{q}^\gamma} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right) + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} - \frac{\partial \Omega_1}{\partial q^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right);$$

while (1.29)' leads to

$$\begin{aligned} -i_{X_{\Omega_2}} d\Omega_1 &= \Gamma \left(W^{\alpha\beta} \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \\ &= \Gamma(W^{\alpha\beta}) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\Gamma \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \right) \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} \right) \\ &= \left(\Gamma(W^{\alpha\beta}) - W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} - \frac{\partial \Omega_1}{\partial q^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right). \end{aligned}$$

Since by (1.16) and (1.28):

$$\begin{aligned} W^{\alpha\beta} W^{\beta\sigma} \left(\frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial q^\gamma} \right) &= -W^{\alpha\gamma} W^{\beta\sigma} \left(\Gamma(W_{\gamma\sigma}) + W_{\gamma\tau} \frac{\partial f^\tau}{\partial \dot{q}^\sigma} \right) \\ &= \Gamma(W^{\alpha\beta}) - W^{\beta\gamma} \frac{\partial f^\alpha}{\partial \dot{q}^\gamma}, \end{aligned}$$

it follows, beside (1.30), the other expression

$$\begin{aligned} (1.30)' \quad -i_{X_{\Omega_2}} d\Omega_1 &= W^{\alpha\gamma} W^{\beta\sigma} \left(\frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial q^\gamma} \right) \frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} - \frac{\partial \Omega_1}{\partial q^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right) \\ &= W^{\alpha\gamma} W^{\beta\sigma} \frac{\partial^2 L}{\partial \dot{q}^\gamma \partial q^\sigma} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} - \frac{\partial \Omega_1}{\partial \dot{q}^\beta} \frac{\partial \Omega_2}{\partial \dot{q}^\alpha} \right) + W^{\alpha\beta} \left(\frac{\partial \Omega_1}{\partial \dot{q}^\alpha} \frac{\partial \Omega_2}{\partial q^\beta} - \frac{\partial \Omega_1}{\partial q^\alpha} \frac{\partial \Omega_2}{\partial \dot{q}^\beta} \right). \end{aligned}$$

1 A geometric derivation of new conservation laws

1.5 Lie algebra structure on conserved quantities

Similarly as in 1.4, let the system (1.1) have resulted from the Euler-Lagrange equations (1.15) with the regular Lagrangian L . Then, within the context of the calculus on differential forms, a procedure of constructing an infinite dimensional Lie algebra structure on the ring \mathfrak{R} of all conserved quantities of Γ , will begin with a closed 2-form Ξ which is given by the exterior derivative of Poincaré-Cartan form Θ associated with the regular Lagrangian L :

$$\Theta = \frac{\partial L}{\partial \dot{q}^\alpha} \theta^\alpha + L dt.$$

In the derivative, $d\dot{q}^\alpha$ and dq^α are replaced respectively with $\phi^\alpha + f^\alpha dt$ and $\theta^\alpha + \dot{q}^\alpha dt$ to have the appearance of the form $\Xi = d\Theta$:

$$\Xi = W_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \theta^\alpha \wedge \theta^\beta - \left(f^\beta W_{\alpha\beta} + \dot{q}^\beta \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} + \frac{\partial^2 L}{\partial q^\alpha \partial t} - \frac{\partial L}{\partial q^\alpha} \right) \theta^\alpha \wedge dt,$$

which is, by the Euler-Lagrange equations (1.15) and its equivalent form (1.1), reduced to

$$\Xi = W_{\alpha\beta} \phi^\alpha \wedge \theta^\beta - \frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} \theta^\alpha \wedge \theta^\beta.$$

For the vector field $X \in \mathfrak{X}_\Gamma$ expressed as (1.3), in view of (1.16) with an alternation of the indices α and β :

$$\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} = \Gamma(W_{\alpha\beta}) + W_{\beta\gamma} \frac{\partial f^\gamma}{\partial \dot{q}^\alpha},$$

it follows that (note: $i_\Gamma \phi^\alpha = i_\Gamma \theta^\alpha = 0$)

$$\begin{aligned} -i_X \Xi &= W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(W_{\alpha\beta} \Gamma(\xi^\beta) + \left(\frac{\partial^2 L}{\partial \dot{q}^\alpha \partial q^\beta} - \frac{\partial^2 L}{\partial \dot{q}^\beta \partial q^\alpha} \right) \xi^\beta \right) \theta^\alpha \\ (1.31) \quad &= W_{\alpha\beta} \xi^\beta \phi^\alpha - \left(\Gamma(W_{\alpha\beta} \xi^\beta) + W_{\beta\gamma} \xi^\beta \frac{\partial f^\gamma}{\partial \dot{q}^\alpha} \right) \theta^\alpha. \end{aligned}$$

Therefore, by the regularity condition of L : $\det(W_{\alpha\beta}) \neq 0$, $i_X \Xi = 0$ implies $W_{\alpha\beta} \xi^\beta = 0$, i.e., $\xi^\beta = 0$, consequently $X \equiv 0 \pmod{\Gamma}$. So that Ξ is non-degenerate on the set of equivalent classes of \mathfrak{X}_Γ .

Now, for an arbitrary element $X \in \mathfrak{X}_\Gamma$, define a 1-form ω_X by

$$(1.32) \quad \omega_X = -i_X \Xi.$$

Then, since $d\Xi = d^2\Theta = 0$ and since $i_\Gamma \Xi = 0$ by $i_\Gamma \phi^\alpha = i_\Gamma \theta^\alpha = 0$, it follows that

$$\mathcal{L}_\Gamma \Xi = i_\Gamma d\Xi + di_\Gamma \Xi = 0,$$

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and so, moreover that

$$\begin{aligned} i_{[\Gamma, X]} \Xi &= \mathcal{L}_\Gamma i_X \Xi - i_X \mathcal{L}_\Gamma \Xi \\ &= \mathcal{L}_\Gamma i_X \Xi = -\mathcal{L}_\Gamma \omega_X. \end{aligned}$$

Therefore the non-degeneracy of Ξ implies that $\omega_X \in \mathfrak{X}_\Gamma^*$, i.e., $\mathcal{L}_\Gamma \omega_X = 0$, if and only if $[\Gamma, X] \equiv 0 \pmod{\Gamma}$. Thus the correspondence in the Theorem 1.7 can be realized by Ξ .

Theorem 1.9. *Under the relation (1.32), the form Ξ defines a bijection between the set \mathfrak{X}_Γ^* and the equivalent classes of a subset of \mathfrak{X}_Γ :*

$$\mathfrak{X}_\Gamma^0 = \{X \in \mathfrak{X}_\Gamma \mid [\Gamma, X] \equiv 0 \pmod{\Gamma}\}.$$

Particularly for an element $d\Omega$ of a subset \mathfrak{X}_0^* of \mathfrak{X}_Γ^* :

$$\mathfrak{X}_0^* = \{d\Omega \mid \Omega \in \mathfrak{R}, \text{ i.e., } \Gamma(\Omega) = 0\},$$

there exists an element $X_\Omega \in \mathfrak{X}_\Gamma^0$, uniquely up to a multiple of Γ , satisfying

$$d\Omega = -i_{X_\Omega} \Xi;$$

whose appearance in coordinates follows from (1.31) by putting X as X_Ω of (1.27), i.e., ξ^α as $W^{\alpha\beta} \partial\Omega / \partial\dot{q}^\beta$, and consequently (1.27) leads to $d\Omega$ of (1.26). So that, since $\Gamma(\Omega) = 0$, it is well-defined a product $\{\Omega_1, \Omega_2\} \in \mathfrak{R}$ of elements $\Omega_i \in \mathfrak{R}$ ($i = 1, 2$):

$$(1.33) \quad \{\Omega_1, \Omega_2\} = -i_{X_{\Omega_2}} d\Omega_1 = -X_{\Omega_2}(\Omega_1),$$

which is written in coordinates as (1.30) or equivalently as (1.30)'. Then, in a familiar calculations on differential forms, we can see the anti-commutativity for $\Omega_i \in \mathfrak{R}$ ($i = 1, 2$):

$$\begin{aligned} \{\Omega_1, \Omega_2\} &= i_{X_{\Omega_2}} i_{X_{\Omega_1}} \Xi = -i_{X_{\Omega_1}} i_{X_{\Omega_2}} \Xi \\ &= -\{\Omega_2, \Omega_1\}, \end{aligned}$$

and the Leibniz identity for $\Omega_i \in \mathfrak{R}$ ($i = 1, 2, 3$):

$$\begin{aligned} \{\Omega_1 \Omega_2, \Omega_3\} &= -X_{\Omega_3}(\Omega_1 \Omega_2) = -\Omega_1 X_{\Omega_3}(\Omega_2) - \Omega_2 X_{\Omega_3}(\Omega_1) \\ &= \Omega_1 \{\Omega_2, \Omega_3\} + \Omega_2 \{\Omega_1, \Omega_3\}. \end{aligned}$$

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Moreover, in view of

$$\begin{aligned}\mathcal{L}_{X_{\Omega_1}}\Xi &= (di_{X_{\Omega_1}} + i_{X_{\Omega_1}}d)\Xi \\ &= -d^2\Omega_1 + i_{X_{\Omega_1}}d^2\Theta = 0,\end{aligned}$$

it follows that

$$\begin{aligned}i_{[X_{\Omega_1}, X_{\Omega_2}]} \Xi &= (\mathcal{L}_{X_{\Omega_1}} i_{X_{\Omega_2}} - i_{X_{\Omega_2}} \mathcal{L}_{X_{\Omega_1}}) \Xi \\ &= -\mathcal{L}_{X_{\Omega_1}} d\Omega_2 = -dX_{\Omega_1}(\Omega_2) \\ &= d\{\Omega_2, \Omega_1\} = -d\{\Omega_1, \Omega_2\};\end{aligned}$$

so, by the Theorem 1.8, that

$$X_{\{\Omega_1, \Omega_2\}} \equiv [X_{\Omega_1}, X_{\Omega_2}] \pmod{\Gamma}.$$

Therefore, since $\Gamma(\Omega_3) = 0$ for $\Omega_3 \in \mathfrak{R}$, we can see

$$\begin{aligned}\{\Omega_3, \{\Omega_1, \Omega_2\}\} &= -X_{\{\Omega_1, \Omega_2\}}(\Omega_3) = -[X_{\Omega_1}, X_{\Omega_2}](\Omega_3) \\ &= -X_{\Omega_1}(X_{\Omega_2}(\Omega_3)) + X_{\Omega_2}(X_{\Omega_1}(\Omega_3)) \\ &= X_{\Omega_1}\{\Omega_3, \Omega_2\} - X_{\Omega_2}\{\Omega_3, \Omega_1\} \\ &= -\{\{\Omega_3, \Omega_2\}, \Omega_1\} + \{\{\Omega_3, \Omega_1\}, \Omega_2\},\end{aligned}$$

which is rearranged to conclude the Jacobi identity

$$\{\{\Omega_1, \Omega_2\}, \Omega_3\} + \{\{\Omega_2, \Omega_3\}, \Omega_1\} + \{\{\Omega_3, \Omega_1\}, \Omega_2\} = 0.$$

Thus the product (1.33) gives the Poisson algebra structure on \mathfrak{R} .

Theorem 1.10. *The ring \mathfrak{R} of all conserved quantities of Γ forms an infinite dimensional Lie algebra under the Poisson product defined by (1.33).*

2 Optimal control problem in economic growths I

2.1 Introduction

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagan [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [32]) without using either Lagrangian or Hamiltonian structures. It was discussed first for a system of second-order differential equations and then the system was supposed to be given in the form of the Euler-Lagrange equations with some Lagrangian. The results were applied to various economic growth models (Mimura and Nôno [34]; Mimura, Fujiwara and Nôno [29], [30]; Fujiwara, Mimura and Nôno [11]-[14]) to discover new economic conservation laws including non-Noether ones.

Particularly in [34], there was discussed the optimal control problem to maximize an integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(2.1) \quad \int_0^T e^{-\rho t} U(\dot{q}, q) dt,$$

under constraint $F(\dot{q}, q) = 0$ given by a first degree homogeneous function $F(\dot{q}, q)$ with respect to q and \dot{q} , where $q = (q^i(t))$, $\dot{q} = (dq^i/dt)$ ($i = 1, \dots, n$) and ρ is a constant. Neamțu [35] reformed some results for constructing conserved quantities in [34] by replacing $F(\dot{q}, q) = 0$ with $F^a(\dot{q}, q) = 0$ ($a = 1, \dots, m$), where $F^a(\dot{q}, q)$ are first degree homogeneous functions.

In 2.2, we will give further generalization of discovering conservation laws for the maximizing problem of (2.1) under constraint given by a system of functions with arbitrary degree of homogeneity:

$$(2.2) \quad F^a(\dot{q}, q) = 0 \quad (a = 1, \dots, m).$$

Here recall that von Neumann [36] gave an analysis of a model which has a unique ray of balanced growth. Samuelson [39] generalized the analysis by studying transient approaches to the von Neumann balanced-growth ray, which was given now as the maximizing problem of the integration

$$\int_0^T \dot{K}^1 dt$$

under constraint $F(\dot{K}, K) = 0$ given by a first degree homogeneous function $F(\dot{K}, K)$, where $K = (K^i(t))$ and $\dot{K} = (\dot{K}^i) = (dK^i/dt)$ ($i = 1, \dots, n$) are understood respectively

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as the n capital goods and the n capital formations. This is the model called von Neumann growth model, which grew up in [34] to be the generalized growth model of von Neumann type by replacing the integrating function \dot{K}^1 with the total time derivative of first degree homogeneous function $G(K)$ with respect to K^i , while the constraint $F(\dot{K}, K) = 0$ is unchanged.

In 2.3, the model of von Neumann type in [30] is generalized moreover with respect to the constraint by replacing the first degree homogeneous function with a system of arbitrary degree homogeneous functions. In his model ($n = 2$: $i = 1, 2$), Samuelson [40] derived two conserved quantities, which conclude that the ratio of the national income and the national wealth is constant (income-wealth conservation law). In our generalized model, together with $\rho = 0$ and $n = 2$, the system of functions with arbitrary degree of homogeneity are reduced later to a homogeneous second order polynomial. Two conserved quantities are derived and then, by using of which, a class of optimal paths for the finite horizon are determined in the reduced situation. Finally, in view of the optimal paths, it is detailed the Samuelson's income-wealth conservation law in our model.

In 2.4, an equivalent class of r -th degree homogeneous Lagrangians (utility functions), which gives rise to the Samuelson's type of two conservation laws in a generalized model constructed on arbitrary number $2n$ ($n \geq 2$) of variables (capital stocks and capital formations), is determined up to total time derivatives (null Lagrangians), whose Euler-Lagrange equations are identically satisfied (Hill [22], Edelen [7]), so that the corresponding Euler-Lagrange equations in the model are invariant under an addition of the total time derivatives. Whenever $r \neq 0$, it is shown that the utility functions are such total time derivatives. In the case: $n = 2$ (just two capital stocks), this result is valid without the homogeneous condition for the utility functions. However, in the case: $n \geq 3$ (three or more capital stocks), it is important and interesting that there exists a class of non-null homogeneous utility functions of degree zero and it is determined completely up to the total time derivatives.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

2.2 Maximizing problem for optimal economic growths

Our discussion begins with an optimal control problem to maximize the integration (2.1) over a finite or an infinite period of time under the constraint (2.2) given by a system of functions $F^a(\dot{q}, q)$ ($a = 1, \dots, m$) with arbitrary degree of homogeneity with respect to q and \dot{q} . In the variational principle with the multiplier technique, we set the following

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Lagrangian as usual

$$(2.3) \quad L(\dot{\lambda}, \dot{q}, \lambda, q, t) = e^{-\rho t} U(\dot{q}, q) + \lambda_a F^a(\dot{q}, q).$$

Here put $\lambda_a = q^{n+a}$ and arrange the variables q^i and λ_a as $(q^\alpha) = (q^1, \dots, q^n, \lambda_1, \dots, \lambda_m)$. Then, according to $n+1 \leq \alpha \leq n+m$ and $1 \leq \alpha \leq n$, the Euler-Lagrange equations of (2.3):

$$(2.4) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = 0$$

separate into $F^a = 0$ ($a = 1, \dots, m$) and

$$(2.5) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0 : \quad \frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} + \lambda_a \frac{\partial F^a}{\partial \dot{q}^i} \right) = e^{-\rho t} \frac{\partial U}{\partial q^i} + \lambda_a \frac{\partial F^a}{\partial q^i}.$$

A conserved quantity (first integral) in question is a quantity $\Omega(\dot{\lambda}, \dot{q}, \lambda, q, t)$ satisfying $d\Omega/dt = 0$ (conservation law) on the optimal paths, i.e., on solutions to (2.4), or equivalently to $F^a = 0$ and (2.5). The theorem 1.8 in 1.4 is reformulated as follows ([35], Theorem 4; cf. [34], Theorem 1).

Theorem 2.1. *For the Lagrangian (2.3), let $(\xi_1^i, \eta_a^1) = (\xi_1^i(\dot{\lambda}, \dot{q}, \lambda, q, t), \eta_a^1(\dot{\lambda}, \dot{q}, \lambda, q, t))$ and $(\xi_2^i, \eta_a^2) = (\xi_2^i(\dot{\lambda}, \dot{q}, \lambda, q, t), \eta_a^2(\dot{\lambda}, \dot{q}, \lambda, q, t))$ satisfy the equations*

$$(2.6) \quad \frac{\partial F^a}{\partial \dot{q}^i} \frac{d\xi^i}{dt} + \frac{\partial F^a}{\partial q^i} \xi^i = 0,$$

$$(2.7) \quad \frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \xi^j + \frac{\partial F^a}{\partial \dot{q}^i} \eta_a \right) - \left(\frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial q^i \partial q^j} \xi^j + \frac{\partial F^a}{\partial q^i} \eta_a \right) = 0,$$

on the optimal paths for the maximizing problem of (2.1) under the constraints (2.2). Then the following conserved quantity Ω for the problem is constructed from (ξ_1^i, η_a^1) and (ξ_2^i, η_a^2) :

$$(2.8) \quad \Omega = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \left(\xi_1^i \frac{d\xi_2^j}{dt} - \xi_2^j \frac{d\xi_1^i}{dt} \right) + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} (\xi_1^i \xi_2^j - \xi_2^i \xi_1^j) + \frac{\partial F^a}{\partial \dot{q}^i} (\xi_1^i \eta_a^2 - \xi_2^i \eta_a^1).$$

In terms of $(\xi^i, \eta_a) = (\dot{q}^i, \dot{\lambda}_a + \rho \lambda_a)$ satisfying (2.6) and (2.7), the theorem 7 in [32] is also reformulated as follows ([35], Theorem 5; cf. [34], Theorem 2).

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Theorem 2.2. For the Lagrangian (2.3), let $(\xi^i, \eta_a) = (\xi^i(\dot{\lambda}, \dot{q}, \lambda, q, t), \eta_a(\dot{\lambda}, \dot{q}, \lambda, q, t))$ satisfy the equations (2.6) and (2.7) on the optimal paths for the maximizing problem of (2.1) under the constraints (2.2). Then the following conserved quantity Ω for the problem is constructed from (ξ^i, η_a) :

$$(2.9) \quad \Omega = \dot{q}^i \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \xi^j + \frac{\partial F^a}{\partial \dot{q}^i} \eta_a \right) - \left(\frac{\partial L}{\partial \dot{q}^i} + \rho \frac{\partial L}{\partial q^i} \right) \xi^i.$$

Here we impose an arbitrary degree s of homogeneity on the system of functions $F^a(\dot{q}, q)$ ($a = 1, \dots, m$) with respect to \dot{q}^j and q^j , i.e.,

$$(2.10) \quad \dot{q}^j \frac{\partial F^a}{\partial \dot{q}^j} + q^j \frac{\partial F^a}{\partial q^j} = s F^a,$$

which guarantee that $\xi^i = q^i$ satisfy the equation (2.6) whenever $F^a = 0$. So, together with $\xi^i = q^i$, the Lagrangian (2.3) is substituted for the equation (2.7). Then, by the differentiations of (2.10) with respect to \dot{q}^i and q^i :

$$(2.11) \quad \begin{aligned} \dot{q}^j \frac{\partial^2 F^a}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 F^a}{\partial \dot{q}^i \partial q^j} &= (s-1) \frac{\partial F^a}{\partial \dot{q}^i}, \\ \dot{q}^j \frac{\partial^2 F^a}{\partial \dot{q}^j \partial q^i} + q^j \frac{\partial^2 F^a}{\partial q^i \partial q^j} &= (s-1) \frac{\partial F^a}{\partial q^i}, \end{aligned}$$

the equations (2.7) are reduced to

$$\begin{aligned} \frac{d}{dt} \left(e^{-\rho t} \left(\dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 U}{\partial \dot{q}^i \partial q^j} \right) + ((s-1)\lambda_a + \eta_a) \frac{\partial F^a}{\partial \dot{q}^i} \right) \\ - \left(e^{-\rho t} \left(\dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^j \partial q^i} + q^j \frac{\partial^2 U}{\partial q^i \partial q^j} \right) + ((s-1)\lambda_a + \eta_a) \frac{\partial F^a}{\partial q^i} \right) = 0. \end{aligned}$$

Here assume also that $U(\dot{q}, q)$ is r -th degree homogeneous with respect to \dot{q}^j and q^j , i.e.,

$$(2.12) \quad \dot{q}^j \frac{\partial U}{\partial \dot{q}^j} + q^j \frac{\partial U}{\partial q^j} = rU,$$

and put $\eta_a = (r-s)\lambda_a$. Then, by the relations from (2.5):

$$\frac{d}{dt} \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} \right) - \lambda_a \frac{\partial F^a}{\partial q^i} = -\frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right) + e^{-\rho t} \frac{\partial U}{\partial q^i},$$

the equations (2.7) lead finally to

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$$\begin{aligned} \frac{d}{dt} \left(e^{-\rho t} \left(\dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 U}{\partial \dot{q}^i \partial q^j} - (r-1) \frac{\partial U}{\partial \dot{q}^i} \right) \right) \\ - e^{-\rho t} \left(\dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^j \partial q^i} + q^j \frac{\partial^2 U}{\partial q^i \partial q^j} - (r-1) \frac{\partial U}{\partial q^i} \right) = 0, \end{aligned}$$

which are satisfied identically by virtue of the following differentiations of (2.12) with respect to \dot{q}^i and q^i :

$$\begin{aligned} (2.13) \quad \dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 U}{\partial \dot{q}^i \partial q^j} &= (r-1) \frac{\partial U}{\partial \dot{q}^i}, \\ \dot{q}^j \frac{\partial^2 U}{\partial \dot{q}^j \partial q^i} + q^j \frac{\partial^2 U}{\partial q^i \partial q^j} &= (r-1) \frac{\partial U}{\partial q^i}. \end{aligned}$$

Therefore $(\xi^i, \eta_a) = (q^i, (r-s)\lambda_a)$ satisfies (2.6) and (2.7) on the optimal paths. The solution is substituted for (2.9) to obtain the conserved quantity

$$\begin{aligned} \Omega &= \dot{q}^i \left(\dot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} + (r-s)\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} \right) - q^i \left(\frac{\partial L}{\partial q^i} + \rho \frac{\partial L}{\partial \dot{q}^i} \right) \\ &= \dot{q}^i \left(\dot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \right) - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} - \rho q^i \frac{\partial L}{\partial \dot{q}^i} + (r-s)\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i}. \end{aligned}$$

Since by (2.11), (2.12) and (2.13):

$$\begin{aligned} \dot{q}^i \left(\dot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} + q^j \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \right) - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} &= (r-1)e^{-\rho t} \dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - e^{-\rho t} q^i \frac{\partial U}{\partial q^i} + (s-1)\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} \\ &= re^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) + (s-1)\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i}, \end{aligned}$$

Ω is written as

$$\Omega = re^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) - \rho q^i \frac{\partial L}{\partial \dot{q}^i} + (r-1)\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i};$$

and since by (2.10):

$$(r-1)\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} - \lambda_a q^i \frac{\partial F^a}{\partial q^i} = -r\lambda_a q^i \frac{\partial F^a}{\partial q^i} + (r-1)s\lambda_a F^a,$$

Ω is written finally as

$$\Omega = r \left(-\lambda_a q^i \frac{\partial F^a}{\partial q^i} + e^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) \right) - \rho q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right) + (r-1)s\lambda_a F^a.$$

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Thus, including the result ([35], Theorem 6), the theorem 3 in [34] is generalized as follows.

Theorem 2.3. *Let the functions $U(\dot{q}, q)$ in (2.1) and $F^a(\dot{q}, q)$ in (2.2) be r -th and s -th degree homogeneous with respect to \dot{q}^i and q^i , respectively. Then, there exists the following conserved quantity Ω for the maximizing problem of (2.1) under the constraints (2.2):*

$$(2.14) \quad \Omega = r \left(-\lambda_a q^i \frac{\partial F^a}{\partial q^i} + e^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) \right) - \rho q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right).$$

We consider now a function $\varphi(\dot{q}, q)$ which can separate the conserved quantity (2.14) into two conserved ones

$$(2.15) \quad \Omega_1 = \lambda_a q^i \frac{\partial F^a}{\partial q^i} - e^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) + \rho \varphi,$$

$$(2.16) \quad \Omega_2 = q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right) - r \varphi,$$

i.e., a function $\varphi(\dot{q}, q)$ satisfying $d\Omega_1/dt = 0$ and $d\Omega_2/dt = 0$, where $-\Omega = r\Omega_1 + \rho\Omega_2$. In $d\Omega_1/dt$, it follows the identity

$$\frac{d}{dt} \left(e^{-\rho t} \dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - e^{-\rho t} U \right) = \dot{q}^i \frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right) - e^{-\rho t} \dot{q}^i \frac{\partial U}{\partial q^i} + \rho e^{-\rho t} U,$$

and, by (2.10), the identity

$$\begin{aligned} \frac{d}{dt} \left(\lambda_a q^i \frac{\partial F^a}{\partial q^i} \right) &= \frac{d}{dt} \left(-\lambda_a \dot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} + s \lambda_a F^a \right) \\ &= -\dot{q}^i \frac{d}{dt} \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} \right) + (s-1) \lambda_a \ddot{q}^i \frac{\partial F^a}{\partial \dot{q}^i} + s \lambda_a \dot{q}^i \frac{\partial F^a}{\partial q^i} + s \dot{\lambda}_a F^a. \end{aligned}$$

So, in view of (2.3), $d\Omega_1/dt$ leads to

$$\frac{d\Omega_1}{dt} = \rho \left(\frac{d\varphi}{dt} - e^{-\rho t} U \right) - \dot{q}^i \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) + s \dot{\lambda}_a F^a,$$

also, in view of (2.10) and (2.12), $d\Omega_2/dt$ leads to

$$\frac{d\Omega_2}{dt} = -r \left(\frac{d\varphi}{dt} - e^{-\rho t} U \right) + q^i \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \right) + s \lambda_a F^a.$$

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Therefore Ω_1 and Ω_2 become conserved quantities if φ satisfies $d\varphi/dt = e^{-\rho t}U$, i.e.,

$$\varphi = \int e^{-\rho t}U dt;$$

which is substituted for (2.15) and (2.16) to deduce:

Theorem 2.4. *Let the functions $U(\dot{q}, q)$ in (2.1) and $F(\dot{q}, q)$ in (2.2) be r -th and s -th degree homogeneous with respect to \dot{q}^i and q^i , respectively. Then, there exist the following two conserved quantities Ω_1 and Ω_2 for the maximizing problem of (2.1) under the constraints (2.2):*

$$(2.17) \quad \Omega_1 = \lambda_a q^i \frac{\partial F^a}{\partial q^i} - e^{-\rho t} \left(\dot{q}^i \frac{\partial U}{\partial \dot{q}^i} - U \right) + \rho \int e^{-\rho t} U dt,$$

$$(2.18) \quad \Omega_2 = q^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{q}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{q}^i} \right) - r \int e^{-\rho t} U dt.$$

2.3 Generalized growth model of von Neumann type

The theorems established in the preceeding sections can be applied effectively for the derivation of new conservation laws in several economic growth models. Here is given an application within the n capital goods $q^i = K^i$ and n capital formations $\dot{q}^i = \dot{K}^i$. Accordingly in the Theorem 2.1 through the Theorem 2.4, $U(\dot{K}, K)$ and $F^a(\dot{K}, K)$ are regarded respectively as an r -th degree homogeneous utility function and a system of s -th degree homogeneous transformation functions with respect to \dot{K}^i and K^i , and ρ is a constant discount rate. In the situation, the conserved quantity (2.14) leads immediately to

$$-\Omega = r \left(\lambda_a K^i \frac{\partial F^a}{\partial K^i} - e^{-\rho t} \left(\dot{K}^i \frac{\partial U}{\partial \dot{K}^i} - U \right) \right) + \rho K^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{K}^i} \right).$$

Particularly for $U = c_i \dot{K}^i$ (c_i : const.; cf. [40], in which U is of the form $U = \dot{K}^1$), this quantity $-\Omega$ is reduced to

$$-\Omega = \lambda_a K^i \frac{\partial F^a}{\partial K^i} + \rho K^i \left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i} + c_i e^{-\rho t} \right).$$

Here we assume that U is a total time derivative of a first degree homogeneous function $G(K)$ with respect to K^i , i.e., $U = \dot{K}^i \partial G / \partial K^i$. Then U is first degree homogeneous with

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respect to \dot{K}^i , i.e., $\dot{K}^i \partial U / \partial \dot{K}^i = U$; and is also of degree zero with respect to K^i , i.e., $K^i \partial U / \partial K^i = 0$. Accordingly it follows that

$$\begin{aligned} \int e^{-\rho t} U dt &= \int e^{-\rho t} \dot{K}^i \frac{\partial U}{\partial \dot{K}^i} dt = e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} - \int K^i \frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial \dot{K}^i} \right) dt \\ &= e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} + \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt - \int e^{-\rho t} K^i \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{K}^i} \right) dt \\ &= e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} + \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt. \end{aligned}$$

Therefore the conserved quantities (2.17) and (2.18) with first degree ($r = 1$) homogeneous function $U = \dot{K}^i \partial G(K) / \partial K^i$ have the appearances respectively in the following theorem.

Theorem 2.5. *Let $G = G(K)$ be first degree homogeneous function with respect to the n capital goods K^i . Then for the maximizing problem of*

$$(2.1)' \quad \int_0^T e^{-\rho t} \dot{K}^i \frac{\partial G}{\partial K^i} dt,$$

under constraints of s -th degree homogeneous transformation functions

$$(2.2)' \quad F^a(\dot{K}, K) = 0 \quad (a = 1, \dots, m),$$

there exist the following two conserved quantities Ω_1 and Ω_2 :

$$(2.19) \quad \Omega_1 = \lambda_a K^i \frac{\partial F^a}{\partial K^i} + \rho \int e^{-\rho t} U dt,$$

$$(2.20) \quad \Omega_2 = \lambda_a K^i \frac{\partial F^a}{\partial \dot{K}^i} - \rho \int e^{-\rho t} K^i \frac{\partial U}{\partial \dot{K}^i} dt.$$

Remark 2.1. The conserved quantities (2.19) and (2.20) reduce respectively, if $\rho = 0$, to $\Omega_1 = \lambda_a K^i \partial F^a / \partial K^i$ and $\Omega_2 = \lambda_a K^i \partial F^a / \partial \dot{K}^i$; and moreover, if $n = 2$, $m = 1$ and $s = 1$, to the Samuelson's ones [40] $\Omega_1 = \lambda K^i \partial F / \partial K^i$ and $\Omega_2 = \lambda K^i \partial F / \partial \dot{K}^i$ in which λ , $Y \equiv K^i \partial F / \partial K^i$ and $W \equiv -K^i \partial F / \partial \dot{K}^i$ are regarded respectively as the implicit price, the national income and the national wealth.

Remark 2.2. What is important and interesting in the above theorem is that the Samuelson's conservation laws $\Omega_1 = \lambda K^i \partial F / \partial K^i$ and $\Omega_2 = \lambda K^i \partial F / \partial \dot{K}^i$ are valid for the utility of

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the form $U = dG/dt$ where $G = G(K)$ is a first degree homogeneous function with respect to K^i , and for the transformation function $F(\dot{K}, K)$ with arbitrary degree of homogeneity with respect to K^i and \dot{K}^i (in [40], $U = \dot{K}^1$ and $F(\dot{K}, K)$ is assumed to be of first degree homogeneous function).

Remark 2.3. As seen before, $(\xi^i, \eta_a) = (q^i, (r-s)\lambda_a)$ is solutions satisfying (2.6) and (2.7) on the optimal paths, which are written here respectively as $(\xi^i, \eta_a) = (K^i, (1-s)\lambda_a)$. And, in view of the equations from (2.5) with $U = \dot{K}^i \partial G(K)/\partial K^i$ and $\rho = 0$:

$$\frac{d}{dt} \left(\lambda_a \frac{\partial F^a}{\partial \dot{K}^i} \right) - \lambda_a \frac{\partial F^a}{\partial K^i} = 0,$$

we have immediately the other one $(\xi^i, \eta_a) = (0, \lambda_a)$. The above conserved quantities $-\Omega_1$ of (2.19) or Ω_2 of (2.20) can be obtained also by substituting $(\xi^i, \eta_a) = (K^i, (1-s)\lambda_a)$ for (2.9), or $(\xi_1^i, \eta_a^1) = (K^i, (1-s)\lambda_a)$ and $(\xi_2^i, \eta_a^2) = (0, \lambda_a)$ for (2.8), respectively.

In what follows, let $0 < T < \infty$, $\rho = 0$, $n = 2$ and $m = 1$, and a transformation function of two capital goods $F = F(\dot{K}^1, \dot{K}^2, K^1, K^2)$ be a homogeneous second order polynomial of the form:

$$(2.2)'' \quad F = a_1(\dot{K}^1)^2 + a_2(\dot{K}^2)^2 + \mu(a_1(K^1)^2 + a_2(K^2)^2) \quad (a_1, a_2, \mu: \text{const.}; a_1 a_2 < 0, \mu \neq 0).$$

Then (2.19) and (2.20) lead respectively to the following conserved quantities $\Xi_1 \equiv \frac{1}{2}\Omega_1/\mu$ and $\Xi_2 \equiv \frac{1}{2}\Omega_2$:

$$\Xi_1 = \lambda(a_1(K^1)^2 + a_2(K^2)^2),$$

$$\Xi_2 = \lambda(a_1 K_1 \dot{K}^1 + a_2 K_2 \dot{K}^2).$$

So, in the identity

$$\dot{\Xi}_1 = \dot{\lambda}(a_1(K^1)^2 + a_2(K^2)^2) + 2\Xi_2 = 0,$$

the following conserved quantity Ξ_3 is observed:

$$\Xi_3 = \dot{\lambda}(a_1(K^1)^2 + a_2(K^2)^2).$$

Accordingly, since $\Xi_3/\Xi_1 = \dot{\lambda}/\lambda = \alpha$ (α : const.), λ is determine as

$$\lambda = C e^{\alpha t} \quad (C: \text{const.}).$$

Here note that the Lagrangian L in the consideration is of the form

$$L = \dot{K}^1 \frac{\partial G}{\partial K^1} + \dot{K}^2 \frac{\partial G}{\partial K^2} + \lambda F,$$

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where $G = G(K^1, K^2)$ is first degree homogeneous function with respect to K^1 and K^2 . By substituting the path $\lambda = Ce^{\alpha t}$ for a part of relating Euler-Lagrange equations with the Lagrangian L :

$$\frac{d}{dt} \left(\lambda \frac{\partial F}{\partial \dot{K}^i} \right) - \lambda \frac{\partial F}{\partial K^i} = 0 \quad (i = 1, 2),$$

the following second order differential equations are obtained:

$$\ddot{K}^i + \alpha \dot{K}^i - \mu K^i = 0 \quad (i = 1, 2);$$

whose subsidiary equation has two distinct real solutions $-\frac{1}{2}(\alpha + \sqrt{D})$ and $-\frac{1}{2}(\alpha - \sqrt{D})$ if its discriminant $D \equiv \alpha^2 + 4\mu$ satisfies $D > 0$, or a coincide solution $-\frac{1}{2}\alpha$ if $D = 0$, or two complex solutions $-\frac{1}{2}\alpha \pm \frac{1}{2}i\sqrt{-D}$ if $D < 0$, which are used respectively to denote the solutions K^i . And then, together with $\lambda = Ce^{\alpha t}$, the solutions K^i are substituted for $\Xi_1 = \text{const.}$ to complete the final appearances of K^i .

Theorem 2.6. *In the maximizing problem of (2.1)' ($0 < T < \infty$, $\rho = 0$), let $G = G(K^1, K^2)$ be first degree homogeneous function with respect to the two capital goods K^1 and K^2 and the transformation function F be a homogeneous second order polynomial of the form (2.2)". Then the optimal paths λ and K^i ($i = 1, 2$) are determined as $\lambda = Ce^{\alpha t}$ (C : const.) and as follows according as $D = \alpha^2 + 4\mu$ is positive, zero and negative:*

- (i) $K^i = A^i e^{-\frac{1}{2}(\alpha + \sqrt{D})t} + B^i e^{-\frac{1}{2}(\alpha - \sqrt{D})t}$ if $D > 0$,
- (ii) $K^i = e^{-\frac{\alpha}{2}t} (A^i t + B^i)$ if $D = 0$,
- (iii) $K^i = e^{-\frac{\alpha}{2}t} (A^i \cos \frac{1}{2}\sqrt{-D}t + B^i \sin \frac{1}{2}\sqrt{-D}t)$ if $D < 0$,

where A^1, B^1 are arbitrary constants and $A^2 = \pm \sqrt{-a_1/a_2} A^1$, $B^2 = \pm \sqrt{-a_1/a_2} B^1$, in the right hand sides of which, the signs \pm correspond respectively for (ii) and (iii).

The conserved quantity Ξ_1 is always zero (so that Ξ_2 is also) for the optimal paths K^i ($i = 1, 2$) of (ii) and (iii) in the Theorem 2.6; and also for those of (i) with the constants $A^2 = \pm \sqrt{-a_1/a_2} A^1$ and $B^2 = \pm \sqrt{-a_1/a_2} B^1$ (the signs \pm correspond respectively). But nonzero conserved quantity $\Xi_1 \neq 0$ and also $\Xi_2 \neq 0$ are given respectively as $\Xi_1 = 4a_1 C A^1 B^1$ and $-\Xi_2 = 2\alpha a_1 C A^1 B^1$ by the optimal path $\lambda = Ce^{\alpha t}$ (C : const., $C \neq 0$) and the optimal paths K^i ($i = 1, 2$) of (i) in the Theorem 2.6 with the constants $A^2 = \pm \sqrt{-a_1/a_2} A^1 \neq 0$ and $B^2 = \mp \sqrt{-a_1/a_2} B^1 \neq 0$ (the signs \pm and \mp correspond respectively). So that the constant of the income-wealth (output-capital) ratio (Samuelson [40]) is $Y/W = -\mu \Xi_1 / \Xi_2 = 2\mu/\alpha$. Therefore, in view of $\alpha = \dot{\lambda}/\lambda = d(\log \lambda)/dt$, it is concluded:

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Theorem 2.7. *Together with the optimal path $\lambda = Ce^{\alpha t}$ (C : const., $C \neq 0$), the only optimal paths K^i ($i = 1, 2$) of (i) in the Theorem 2.6 with the constants $A^2 = \pm\sqrt{-a_1/a_2} A^1 \neq 0$ and $B^2 = \mp\sqrt{-a_1/a_2} B^1 \neq 0$ (the signs \pm and \mp correspond respectively) are provided with nonzero conserved quantities $\Xi_1 = 4a_1CA^1B^1$ (product of the implicit price λ and the national income $Y = K^i\partial F/\partial K^i$) and $-\Xi_2 = 2\alpha a_1CA^1B^1$ (product of the implicit price λ and the national wealth $W = -K^i\partial F/\partial \dot{K}^i$), which give the constant of the income-wealth ratio $Y/W = 2\mu/(d(\log \lambda)/dt)$:*

$$\frac{\{\text{national income}\}}{\{\text{national wealth}\}} = \frac{2\mu}{\{\text{rate of logarithm of implicit price}\}} = \text{constant}.$$

2.4 An equivalent class of utility functions

Generalizing the Samuelson's original von Neumann growth model, we set an optimal control problem to maximize the integration of a utility function $U = U(\dot{K}, K, t)$ of n capital stocks $K = (K^i(t))$ and n capital formations $\dot{K} = (\dot{K}^i) = (dK^i/dt)$ ($i = 1, \dots, n; n \geq 2$) over a period of time $[0, T]$:

$$\int_0^T U(\dot{K}, K, t) dt,$$

under a constraint:

$$F = F(\dot{K}, K) = 0,$$

for a transformation function F which is first degree homogeneous with respect to the capital stocks K^i and the capital formations \dot{K}^i , i.e., F satisfies the relation

$$(2.21) \quad \dot{K}^i \frac{\partial F}{\partial \dot{K}^i} + K^i \frac{\partial F}{\partial K^i} = F.$$

In the variational principle, as is well known, the following Lagrangian is introduced with the multiplier technique:

$$L = L(\dot{\lambda}, \dot{K}, \lambda, K, t) = U(\dot{K}, K, t) + \lambda F(\dot{K}, K),$$

whose Euler-Lagrange equations are $F = 0$ and

$$(2.22) \quad \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{K}^i} + \lambda \frac{\partial F}{\partial \dot{K}^i} \right) - \left(\frac{\partial U}{\partial K^i} + \lambda \frac{\partial F}{\partial K^i} \right) = 0.$$

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Our problem now begins with a postulation that the Samuelson's type of two conservation laws $\dot{\Omega}_1 = 0$ and $\dot{\Omega}_2 = 0$, i.e.,

$$\Omega_1 = \lambda K^i \frac{\partial F}{\partial K^i} = \text{const.},$$

$$\Omega_2 = \lambda K^i \frac{\partial F}{\partial \dot{K}^i} = \text{const.},$$

are valid on the optimal path of the Euler-Lagrange equations. We first look into the postulations by substituting the relations from (2.22):

$$\frac{d}{dt} \left(\lambda \frac{\partial F}{\partial \dot{K}^i} \right) = \lambda \frac{\partial F}{\partial K^i} - \frac{d}{dt} \frac{\partial U}{\partial \dot{K}^i} + \frac{\partial U}{\partial K^i}$$

for $\dot{\Omega}_1 = 0$ after rewriting Ω_1 by (2.21), and for $\dot{\Omega}_2 = 0$. Then, since it follows respectively that

$$\begin{aligned} \dot{\Omega}_1 &= \frac{d}{dt} \left(\lambda F - \lambda \dot{K}^i \frac{\partial F}{\partial \dot{K}^i} \right) \\ &= \dot{\lambda} F + \lambda \dot{F} - \lambda \ddot{K}^i \frac{\partial F}{\partial \dot{K}^i} - \dot{K}^i \frac{d}{dt} \left(\lambda \frac{\partial F}{\partial \dot{K}^i} \right) \\ &= \dot{\lambda} F + \lambda \dot{F} - \lambda \ddot{K}^i \frac{\partial F}{\partial \dot{K}^i} - \dot{K}^i \left(\lambda \frac{\partial F}{\partial K^i} - \frac{d}{dt} \frac{\partial U}{\partial \dot{K}^i} + \frac{\partial U}{\partial K^i} \right) \\ &= \dot{\lambda} F + \dot{K}^i \left(\frac{d}{dt} \frac{\partial U}{\partial \dot{K}^i} - \frac{\partial U}{\partial K^i} \right) = 0, \\ \dot{\Omega}_2 &= \lambda \dot{K}^i \frac{\partial F}{\partial \dot{K}^i} + K^i \frac{d}{dt} \left(\lambda \frac{\partial F}{\partial \dot{K}^i} \right) \\ &= \lambda \dot{K}^i \frac{\partial F}{\partial \dot{K}^i} + K^i \left(\lambda \frac{\partial F}{\partial K^i} - \frac{d}{dt} \frac{\partial U}{\partial \dot{K}^i} + \frac{\partial U}{\partial K^i} \right) \\ &= \lambda F - K^i \left(\frac{d}{dt} \frac{\partial U}{\partial \dot{K}^i} - \frac{\partial U}{\partial K^i} \right) = 0, \end{aligned}$$

we obtain the following equations for the utility function U on the optimal path:

$$(2.23a) \quad \dot{K}^i \left(\ddot{K}^j \frac{\partial^2 U}{\partial \dot{K}^i \partial \dot{K}^j} + \dot{K}^j \frac{\partial^2 U}{\partial \dot{K}^i \partial K^j} + \frac{\partial^2 U}{\partial \dot{K}^i \partial t} - \frac{\partial U}{\partial K^i} \right) = 0,$$

$$(2.23b) \quad K^i \left(\ddot{K}^j \frac{\partial^2 U}{\partial \dot{K}^i \partial \dot{K}^j} + \dot{K}^j \frac{\partial^2 U}{\partial \dot{K}^i \partial K^j} + \frac{\partial^2 U}{\partial \dot{K}^i \partial t} - \frac{\partial U}{\partial K^i} \right) = 0.$$

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In order that the equation (2.23a) may be satisfied for arbitrary K^i , the coefficients of \ddot{K}^j are to be zero:

$$\dot{K}^i \frac{\partial^2 U}{\partial \dot{K}^i \partial \dot{K}^j} = 0, \quad \text{i.e.,} \quad \frac{\partial}{\partial \dot{K}^j} \left(U - \dot{K}^i \frac{\partial U}{\partial \dot{K}^i} \right) = 0,$$

accordingly we can put

$$(2.24) \quad U - \dot{K}^i \frac{\partial U}{\partial \dot{K}^i} = \xi(K, t);$$

whose differentiations with respect to K^j and t , i.e.,

$$\dot{K}^i \frac{\partial^2 U}{\partial \dot{K}^i \partial K^j} = \frac{\partial U}{\partial K^j} - \frac{\partial \xi}{\partial K^j},$$

$$\dot{K}^i \frac{\partial^2 U}{\partial \dot{K}^i \partial t} = \frac{\partial U}{\partial t} - \frac{\partial \xi}{\partial t},$$

are substituted for (2.23a) to obtain

$$\frac{\partial U}{\partial t} = \dot{K}^i \frac{\partial \xi}{\partial K^i} + \frac{\partial \xi}{\partial t}.$$

Consequently U is integrated as

$$U = \dot{K}^i \int \frac{\partial \xi}{\partial K^i} dt + \xi(K, t) + g(\dot{K}, K),$$

in which, by putting

$$f(K, t) = \int \xi(K, t) dt,$$

the total time derivative \dot{f} is observed:

$$\dot{f} = \dot{K}^i \frac{\partial f}{\partial K^i} + \frac{\partial f}{\partial t} = \dot{K}^i \int \frac{\partial \xi}{\partial K^i} dt + \xi.$$

Here note that \dot{f} satisfies identically the Euler-Lagrange equations [15, 16]

$$(2.25) \quad \begin{aligned} & \frac{d}{dt} \frac{\partial \dot{f}}{\partial \dot{K}^i} - \frac{\partial \dot{f}}{\partial K^i} \\ &= \ddot{K}^j \frac{\partial^2 \dot{f}}{\partial \dot{K}^i \partial \dot{K}^j} + \dot{K}^j \frac{\partial^2 \dot{f}}{\partial \dot{K}^i \partial K^j} + \frac{\partial^2 \dot{f}}{\partial \dot{K}^i \partial t} - \frac{\partial \dot{f}}{\partial K^i} = 0, \end{aligned}$$

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i.e., it satisfies identically the equation (2.23a). Therefore, by (2.23a), U is determined as

$$(2.26) \quad U = \dot{f}(K, t) + g(\dot{K}, K),$$

which turns (2.24) into the first degree homogeneous condition of g with respect to \dot{K}^i :

$$(2.27) \quad \dot{K}^i \frac{\partial g}{\partial \dot{K}^i} = g.$$

The appearance U of (2.26) is substituted for (2.23b). Then \dot{f} 's terms are extinguished by (2.25) in the resulting equation of (2.23b). And the vanishing coefficients of \ddot{K}^j in (2.23b) lead to

$$K^i \frac{\partial^2 U}{\partial \dot{K}^i \partial \dot{K}^j} = \frac{\partial}{\partial \dot{K}^j} \left(K^i \frac{\partial g}{\partial \dot{K}^i} \right) = 0.$$

So that we can put

$$(2.28) \quad K^i \frac{\partial g}{\partial \dot{K}^i} = \varphi(K),$$

whose differentiation with respect to K^j :

$$K^i \frac{\partial^2 g}{\partial \dot{K}^i \partial K^j} = \frac{\partial \varphi}{\partial K^j} - \frac{\partial g}{\partial K^j}$$

are used in the resulting equation of (2.23b):

$$K^i \left(\dot{K}^j \frac{\partial^2 g}{\partial \dot{K}^i \partial \dot{K}^j} - \frac{\partial g}{\partial K^i} \right) = \dot{K}^j \frac{\partial \varphi}{\partial K^j} - \frac{\partial g}{\partial \dot{K}^j} - K^j \frac{\partial g}{\partial K^j} = 0,$$

to derive the final relation

$$(2.29) \quad \dot{K}^i \frac{\partial \varphi}{\partial K^i} - K^i \frac{\partial g}{\partial K^i} = \dot{K}^i \frac{\partial g}{\partial \dot{K}^i}.$$

It is a problem how to determine the function g in U of (2.26) satisfying the conditions (2.27), (2.28) and (2.29). So introduce new independent variables

$$(2.30) \quad Q^1 = \frac{\dot{K}^1}{K^1}, \quad Q^\alpha = \frac{K^1 \dot{K}^\alpha - K^\alpha \dot{K}^1}{(K^1)^2} \quad (\alpha \geq 2).$$

Here note that the index α distinguishes the ranges $\alpha = 2, \dots, n$ from $i = 1, \dots, n$. Then the following differentiations with respect to the old variables (\dot{K}, K) of a function $G(\dot{K}, K)$

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are transformed respectively into those with respect to new ones (Q, K) of $G(Q, K)$:

$$\begin{aligned} K^i \frac{\partial G(\dot{K}, K)}{\partial \dot{K}^i} &= \frac{\partial G(Q, K)}{\partial Q^1}, \\ \dot{K}^i \frac{\partial G(\dot{K}, K)}{\partial \dot{K}^i} &= Q^i \frac{\partial G(Q, K)}{\partial Q^i}, \\ K^i \frac{\partial G(\dot{K}, K)}{\partial K^i} &= -Q^i \frac{\partial G(Q, K)}{\partial Q^i} + K^i \frac{\partial G(Q, K)}{\partial K^i}, \end{aligned}$$

which are used in what follows. At first, since (2.28) leads to

$$\frac{\partial g}{\partial Q^1} = K^i \frac{\partial g}{\partial \dot{K}^i} = \varphi(K),$$

g can be put as

$$(2.31) \quad g = \varphi(K)Q^1 + \Psi(Q^2, \dots, Q^n; K);$$

which is substituted for the relation from (2.27):

$$Q^i \frac{\partial g}{\partial Q^i} = \dot{K}^i \frac{\partial g}{\partial \dot{K}^i} = g,$$

to translate the first degree homogeneous condition of g with respect to \dot{K}^i into that of Ψ with respect to Q^α :

$$(2.32) \quad Q^\alpha \frac{\partial \Psi}{\partial Q^\alpha} = \Psi.$$

Next, g of the form (2.31) is substituted for the term $K^i \partial g / \partial K^i$ in (2.29) and then (2.32) is used to derive the expression for the derivatives of new variables (2.30) and K :

$$\begin{aligned} K^i \frac{\partial g}{\partial K^i} &= -\varphi Q^1 - Q^\alpha \frac{\partial \Psi}{\partial Q^\alpha} + Q^1 K^i \frac{\partial \varphi}{\partial K^i} + K^i \frac{\partial \Psi}{\partial K^i} \\ &= -g + \frac{\dot{K}^1 K^i}{K^1} \frac{\partial \varphi}{\partial K^i} + K^i \frac{\partial \Psi}{\partial K^i}. \end{aligned}$$

So that the left hand side of (2.29) leads to

$$\dot{K}^i \frac{\partial \varphi}{\partial K^i} - K^i \frac{\partial g}{\partial K^i} = g + K^1 Q^\alpha \frac{\partial \varphi}{\partial K^\alpha} - K^i \frac{\partial \Psi}{\partial K^i},$$

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which, together with (2.27), turns the relation (2.29) into

$$(2.33) \quad K^1 Q^\alpha \frac{\partial \varphi}{\partial K^\alpha} = K^i \frac{\partial \Psi}{\partial K^i}.$$

The following situations are now in order for the determination of φ and Ψ satisfying the conditions (2.32) and (2.33).

Case 1. $n = 2$ (just two capital stocks). In this case, after arranging Ψ for

$$\Psi(Q^2; K) = \Psi(1; K) Q^2 \equiv \psi(K) Q^2,$$

the new variables $Q^1 = \dot{K}^1/K^1$ and $Q^2 = (K^1 \dot{K}^2 - K^2 \dot{K}^1)/(K^1)^2$ are substituted for g :

$$g = \varphi Q^1 + \psi Q^2 = \frac{K^1 \varphi - K^2 \psi}{(K^1)^2} \dot{K}^1 + \frac{\psi}{K^1} \dot{K}^2.$$

The condition (2.33) with $n = 2$ ($i = 1, 2$) and $\Psi = \psi Q^2$:

$$K^1 \frac{\partial \varphi}{\partial K^2} Q^2 = \left(K^1 \frac{\partial \psi}{\partial K^1} + K^2 \frac{\partial \psi}{\partial K^2} \right) Q^2,$$

$$\text{i.e.,} \quad K^1 \frac{\partial \varphi}{\partial K^2} = K^1 \frac{\partial \psi}{\partial K^1} + K^2 \frac{\partial \psi}{\partial K^2}$$

is written equivalently as

$$\frac{\partial}{\partial K^2} \frac{K^1 \varphi - K^2 \psi}{(K^1)^2} = \frac{\partial}{\partial K^1} \frac{\psi}{K^1},$$

which guarantees an existence of a function $h(K^1, K^2)$ such that

$$\frac{K^1 \varphi - K^2 \psi}{(K^1)^2} = \frac{\partial h}{\partial K^1}, \quad \frac{\psi}{K^1} = \frac{\partial h}{\partial K^2}.$$

Therefore g is determined as a total time derivative

$$g = \frac{\partial h}{\partial K^1} \dot{K}^1 + \frac{\partial h}{\partial K^2} \dot{K}^2 = \dot{h}(K^1, K^2).$$

Case 2. $n \geq 3$ (three or more capital stocks). In this case, g is assumed to be r -th degree homogeneous with respect to \dot{K}^i and K^i , i.e.,

$$\dot{K}^i \frac{\partial g}{\partial \dot{K}^i} + K^i \frac{\partial g}{\partial K^i} = r g.$$

Then, if $r \neq 0$, g becomes by (2.29) a total time derivative:

$$r g = \dot{K}^i \frac{\partial \varphi}{\partial K^i}, \quad \text{i.e.,} \quad g = \frac{d}{dt} \frac{\varphi}{r}.$$

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And if $r = 0$, (2.29) leads to

$$\dot{K}^i \frac{\partial \varphi}{\partial K^i} = 0, \quad \text{i.e.,} \quad \frac{\partial \varphi}{\partial K^i} = 0,$$

accordingly $\varphi = c$ (c : const.). So that φQ^1 in g of (2.31) is a total time derivative:

$$\varphi Q^1 = c \frac{\dot{K}^1}{K^1} = \frac{d(c \log K^1)}{dt}.$$

Moreover, (2.33) is reduced to

$$K^i \frac{\partial \Psi}{\partial K^i} = 0,$$

which implies that $\Psi(Q^2, \dots, Q^n; K)$ is homogeneous of degree zero with respect to K^i . Therefore Ψ is written as

$$\begin{aligned} \Psi(Q^2, \dots, Q^n; K^1, \dots, K^n) &= \Psi\left(Q^2, \dots, Q^n; 1, \frac{K^2}{K^1}, \dots, \frac{K^n}{K^1}\right) \\ &\equiv \Xi\left(Q^2, \dots, Q^n; \frac{K^2}{K^1}, \dots, \frac{K^n}{K^1}\right), \end{aligned}$$

in which note that the variables Q^α of (2.30) are written as $Q^\alpha = (d/dt)(K^\alpha/K^1)$. Thus, in the cases, the function g of (2.31) is determined completely up to the total time derivatives. In conclusion we have:

Theorem 2.8. *In the generalization of Samuelson's von Neumann growth model for n capital stocks $K = (K^i)$ and n capital formations $\dot{K} = (\dot{K}^i)$ ($i = 1, \dots, n$; $n \geq 2$), an equivalent class of utility functions $U(\dot{K}, K, t)$ which gives rise to the Samuelson's type of two conservation laws $\dot{\Omega}_1 = 0$ and $\dot{\Omega}_2 = 0$, where $\Omega_1 = \lambda K^i \partial F / \partial K^i$ and $\Omega_2 = \lambda K^i \partial F / \partial \dot{K}^i$, is determined completely up to total time derivatives of the form $dG(K, t)/dt$ according to the following situations.*

(i) $n = 2$ (just two capital stocks): *The utility functions are always such total time derivatives.*

(ii) $n \geq 3$ (three or more capital stocks): *Assuming that the utility functions are r -th degree homogeneous with respect to \dot{K}^i and K^i , they are also such total time derivatives if $r \neq 0$. However there exist homogeneous utility functions of degree zero with respect to \dot{K}^i and K^i , which are determined completely as follows up to the total time derivatives*

$$U = \Xi\left(Q^2, \dots, Q^n; \frac{K^2}{K^1}, \dots, \frac{K^n}{K^1}\right),$$

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where Ξ is first degree homogeneous with respect to $Q^\alpha = (d/dt)(K^\alpha/K^1)$ ($\alpha = 2, \dots, n$).

Remark 2.4. In the case $n = 2$, Samuelson's two conservation laws [32] were derived from the maximizing problem of

$$\int_0^T \dot{K}^1 dt,$$

in which \dot{K}^1 is a total time derivative of K^1 . More general example can be made by a utility function U with constant discount rate ρ , which is a total time derive of

$$G(K, t) = e^{-\rho t}(c_1 K^1 + c_2 K^2) \quad (c_1, c_2: \text{const.}),$$

so that the Samuelson's two conservation laws are derived from a maximizing problem of

$$\int_0^T e^{-\rho t}[c_1(\dot{K}_1 - \rho K_1) + c_2(\dot{K}_2 - \rho K_2)]dt.$$

3 Optimal control problem in economic growths II

3.1 Introduction

Solow [46] explored, with some models, the intergenerational problem of optimal capital accumulation by straightforward application of the max-min principle. Hartwick introduced and investigated the savings-investment rule (so-called Hartwick's rule) in the problem, first for a model with only one exhaustible resource [19] and then with many exhaustible resources [20]. He showed that Hartwick's rule (together with Hotelling's rule [23]) yields the intergenerational equity (constant consumption path). Sato and Kim [45] presented an interesting integral variational model with single exhaustible resource which gives rise to Hartwick's rule under an implicitly assumed constant consumption path. Mimura, Fujiwara and Nôno [28] developed the variational model with many exhaustible resources and discussed the intergenerational equity by generalizing the rules of Hartwick and Hotelling. They [10] made also this line of approach on the model with single exhaustible resource under implicitly assumed exponentially growing consumption. The ideas in [10], [28] and [34] were unified [29] to built up an integral variational principle for a study of optimal economic growths which can be applied effectively, e.g., to the intergenerational problem or to Tobin's q -theory of investment [53].

The purpose of this section is to look synthetically through the ideas in [10], [28], [29] and [34] with some further results. In 3.2, we introduce the extremal (maximizing or minimizing) problem of an integration under some constraints, and then present the method for constructing conserved quantities (first integrals) of the Euler-Lagrange equations which will appear in the variational approach with multiplier technique to the problem. In 3.3, we give some reductions of the extremal problem with absence of some variables which extinguishes a part of the constraints. In 3.4, we show that the results can be applied to find conservation laws in more general neoclassical growth models, the intergenerational problem for exponential consumption growth and q -theory of investment, respectively. Through the conservation laws or the relating identities with the laws, optimal paths are determined completely for these models.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

3 Optimal control problem in economic growths II

3.2 Extremal problem for optimal economic growths

Generalizing the model in [28] for the intergenerational problem, we set the following integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(3.1) \quad \int_0^T e^{-\rho t} U(x, y, u) dt,$$

where $x = (x^\mu(t))$, $y = (y^i(t))$, $u = (u^\sigma(t))$ ($\mu = 1, \dots, k$; $i = 1, \dots, n$; $\sigma = 1, \dots, \ell$) and ρ is a constant. Then our discussion of an extremal (maximizing or minimizing) problem for the integration (3.1) under constraints

$$(3.2) \quad \dot{x}^\mu = f^\mu(x, u),$$

$$(3.3) \quad \dot{y}^i = e^{-\rho t} g^i(u),$$

begins with a Lagrangian (π_μ and λ_i are the multipliers):

$$(3.4) \quad L = e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - f^\mu) + \lambda_i (\dot{y}^i - e^{-\rho t} g^i),$$

whose Euler-Lagrange equations consist of (3.2), (3.3) and

$$(3.5a) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial x^\mu} = \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu,$$

$$(3.5b) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}^i} \right) - \frac{\partial L}{\partial y^i} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial y^i} = \dot{\lambda}_i,$$

$$(3.5c) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}^\sigma} \right) - \frac{\partial L}{\partial u^\sigma} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial u^\sigma} = \frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \lambda_i.$$

A conserved quantity (first integral) for the extremal problem is a quantity Ω of the variables $(\dot{q}^\alpha) = (\dot{\pi}_\mu, \dot{\lambda}_i, \dot{x}^\mu, \dot{y}^i, \dot{u}^\sigma)$, $(q^\alpha) = (\pi_\mu, \lambda_i, x^\mu, y^i, u^\sigma)$ ($\alpha = 1, \dots, 2k + 2n + \ell$) and t whose total time derivative vanishes ($\dot{\Omega} = 0$: conservation law) on the optimal path, i.e., on solutions to the relating Euler-Lagrange equations. For a derivation of conserved quantities, the theorem 2.1 in 2.2 is now reformulated as follows.

Theorem 3.1. *For the Lagrangian (3.4), let the functions $(\xi_1^\alpha) = (\eta_\mu^1, \zeta_i^1, \varphi_1^\mu, \psi_1^i, \tau_1^\sigma)$ and $(\xi_2^\alpha) = (\eta_\mu^2, \zeta_i^2, \varphi_2^\mu, \psi_2^i, \tau_2^\sigma)$ of the variables $(\dot{q}^\alpha) = (\dot{\pi}_\mu, \dot{\lambda}_i, \dot{x}^\mu, \dot{y}^i, \dot{u}^\sigma)$, $(q^\alpha) = (\pi_\mu, \lambda_i, x^\mu, y^i, u^\sigma)$ and t satisfy the equations*

$$(3.6a) \quad \frac{d\varphi^\mu}{dt} = \frac{\partial f^\mu}{\partial x^\nu} \varphi^\nu + \frac{\partial f^\mu}{\partial u^\sigma} \tau^\sigma,$$

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$$(3.6b) \quad \frac{d\psi^i}{dt} = e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \tau^\sigma,$$

$$(3.6c) \quad \begin{aligned} \frac{d\eta_\mu}{dt} + \frac{\partial f^\nu}{\partial x^\mu} \eta_\nu + \pi_\nu \left(\frac{\partial^2 f^\nu}{\partial x^\kappa \partial x^\mu} \varphi^\kappa + \frac{\partial^2 f^\nu}{\partial u^\sigma \partial x^\mu} \tau^\sigma \right) \\ = e^{-\rho t} \left(\frac{\partial^2 U}{\partial x^\nu \partial x^\mu} \varphi^\nu + \frac{\partial^2 U}{\partial y^i \partial x^\mu} \psi^i + \frac{\partial^2 U}{\partial u^\sigma \partial x^\mu} \tau^\sigma \right), \end{aligned}$$

$$(3.6d) \quad \frac{d\zeta_i}{dt} = e^{-\rho t} \left(\frac{\partial^2 U}{\partial x^\mu \partial y^i} \varphi^\mu + \frac{\partial^2 U}{\partial y^j \partial y^i} \psi^j + \frac{\partial^2 U}{\partial u^\sigma \partial y^i} \tau^\sigma \right),$$

$$(3.6e) \quad \begin{aligned} \frac{\partial f^\mu}{\partial u^\sigma} \eta_\mu + \pi_\mu \left(\frac{\partial^2 f^\mu}{\partial x^\nu \partial u^\sigma} \varphi^\nu + \frac{\partial^2 f^\mu}{\partial u^\omega \partial u^\sigma} \tau^\omega \right) + e^{-\rho t} \left(\frac{\partial g^i}{\partial u^\sigma} \zeta_i + \lambda_i \frac{\partial^2 g^i}{\partial u^\omega \partial u^\sigma} \tau^\omega \right) \\ = e^{-\rho t} \left(\frac{\partial^2 U}{\partial x^\mu \partial u^\sigma} \varphi^\mu + \frac{\partial^2 U}{\partial y^i \partial u^\sigma} \psi^i + \frac{\partial^2 U}{\partial u^\omega \partial u^\sigma} \tau^\omega \right), \end{aligned}$$

on the optimal path for the extremal problem of (3.1) under the constraints (3.2) and (3.3). Then the following conserved quantity Ω is constructed:

$$(3.7) \quad \Omega = \eta_\mu^2 \varphi_1^\mu - \eta_\mu^1 \varphi_2^\mu + \zeta_i^2 \psi_1^i - \zeta_i^1 \psi_2^i.$$

The following conditions on U , f^μ and g^i are now imposed in order to look for solutions $(\xi^\alpha) = (\eta_\mu, \zeta_i, \varphi^\mu, \psi^i, \tau^\sigma)$ in the theorem 3.1.

1. Let the function U be of the form $U = U(x, u)$. In view of the total time derivative of (3.2) and (3.3):

$$\begin{aligned} \frac{d\dot{x}^\mu}{dt} &= \frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial f^\mu}{\partial u^\sigma} \dot{u}^\sigma, \\ \frac{d\dot{y}^i}{dt} &= -\rho \dot{y}^i + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \dot{u}^\sigma, \end{aligned}$$

we see immediately that $(\varphi^\mu, \psi^i, \tau^\sigma) = (\dot{x}^\mu, \dot{y}^i + \rho y^i, \dot{u}^\sigma)$ satisfies the equations (3.6a) and (3.6b). The equations (3.5a) are substituted for

$$\begin{aligned} e^{-\rho t} \frac{d}{dt} \left(\frac{\partial U}{\partial x^\mu} \right) &= \frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial x^\mu} \right) + \rho e^{-\rho t} \frac{\partial U}{\partial x^\mu} \\ &= \frac{d}{dt} \left(\dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu \right) + \rho \left(\dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu \right) \\ &= \frac{d}{dt} (\dot{\pi}_\mu + \rho \pi_\mu) + \frac{\partial f^\nu}{\partial x^\mu} (\dot{\pi}_\nu + \rho \pi_\nu) + \pi_\nu \frac{d}{dt} \left(\frac{\partial f^\nu}{\partial x^\mu} \right), \end{aligned}$$

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and, together with $\dot{\lambda}_i = 0$ in (3.5b) (note that $\partial U / \partial y^i = 0$), the equations (3.5c) are also for

$$\begin{aligned} e^{-\rho t} \frac{d}{dt} \left(\frac{\partial U}{\partial u^\sigma} \right) &= \frac{d}{dt} \left(e^{-\rho t} \frac{\partial U}{\partial u^\sigma} \right) + \rho e^{-\rho t} \frac{\partial U}{\partial u^\sigma} \\ &= \frac{d}{dt} \left(\frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \lambda_i \right) + \rho \left(\frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \lambda_i \right) \\ &= \frac{\partial f^\mu}{\partial u^\sigma} (\dot{\pi}_\mu + \rho \pi_\mu) + \pi_\mu \frac{d}{dt} \left(\frac{\partial f^\mu}{\partial u^\sigma} \right) + e^{-\rho t} \lambda_i \frac{d}{dt} \left(\frac{\partial g^i}{\partial u^\sigma} \right). \end{aligned}$$

These equations are compared respectively with (3.6c) and (3.6e) in viewing that the terms for ψ^i disappear in (3.6c) and (3.6e), and the equations (3.6d) are reduced to $d\zeta_i/dt = 0$ because of $U = U(x, u)$. Consequently we find a solution $(\xi_1^\alpha) = (\dot{\pi}_\mu + \rho \pi_\mu, 0, \dot{x}^\mu, \dot{y}^i + \rho y^i, \dot{u}^\sigma)$.

2. Let $f^\mu(x, u)$ and $g^i(u)$ be homogeneous functions of degree one and $U(x, y, u)$ be a homogeneous function of degree r , i.e., they satisfy respectively

$$(3.8a) \quad f^\mu = \frac{\partial f^\mu}{\partial x^\nu} x^\nu + \frac{\partial f^\mu}{\partial u^\sigma} u^\sigma,$$

$$(3.8b) \quad g^i = \frac{\partial g^i}{\partial u^\sigma} u^\sigma,$$

$$(3.8c) \quad rU = \frac{\partial U}{\partial x^\mu} x^\mu + \frac{\partial U}{\partial y^i} y^i + \frac{\partial U}{\partial u^\sigma} u^\sigma.$$

The relations (3.8a) and (3.8b) are combined respectively with (3.2) and (3.3). Then in the resulting identities

$$\dot{x}^\mu = \frac{\partial f^\mu}{\partial x^\nu} x^\nu + \frac{\partial f^\mu}{\partial u^\sigma} u^\sigma,$$

$$\dot{y}^i = e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} u^\sigma,$$

we find a solution $(\varphi^\mu, \psi^i, \tau^\sigma) = (x^\mu, y^i, u^\sigma)$ of (3.6a) and (3.6b). Together with this solution, the following differentiations of (3.8a) with respect to x^κ or u^ω , and of (3.8b) to u^ω :

$$(3.9a) \quad \frac{\partial^2 f^\mu}{\partial x^\kappa \partial x^\nu} x^\nu + \frac{\partial^2 f^\mu}{\partial x^\kappa \partial u^\sigma} u^\sigma = 0,$$

$$(3.9b) \quad \frac{\partial^2 f^\mu}{\partial u^\omega \partial x^\nu} x^\nu + \frac{\partial^2 f^\mu}{\partial u^\omega \partial u^\sigma} u^\sigma = 0,$$

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$$(3.9c) \quad \frac{\partial^2 g^i}{\partial u^\omega \partial u^\sigma} = 0;$$

and also of (3.8c) with respect to x^ν , y^j or u^ω :

$$(3.10a) \quad (r-1) \frac{\partial U}{\partial x^\nu} = \frac{\partial^2 U}{\partial x^\nu \partial x^\mu} x^\mu + \frac{\partial^2 U}{\partial x^\nu \partial y^i} y^i + \frac{\partial^2 U}{\partial x^\nu \partial u^\sigma} u^\sigma,$$

$$(3.10b) \quad (r-1) \frac{\partial U}{\partial y^j} = \frac{\partial^2 U}{\partial y^j \partial x^\mu} x^\mu + \frac{\partial^2 U}{\partial y^j \partial y^i} y^i + \frac{\partial^2 U}{\partial y^j \partial u^\sigma} u^\sigma,$$

$$(3.10c) \quad (r-1) \frac{\partial U}{\partial u^\omega} = \frac{\partial^2 U}{\partial u^\omega \partial x^\mu} x^\mu + \frac{\partial^2 U}{\partial u^\omega \partial y^i} y^i + \frac{\partial^2 U}{\partial u^\omega \partial u^\sigma} u^\sigma;$$

are substituted for (3.6c), (3.6d) and (3.6e), in detail: (3.9a) and (3.10a) for (3.6c), (3.10b) for (3.6d), and (3.9b), (3.9c), (3.10c) for (3.6e). Then the resulting equations of (3.6b), (3.6c) and (3.6e) are written respectively as

$$(r-1)e^{-\rho t} \frac{\partial U}{\partial x^\mu} = \frac{d\eta_\mu}{dt} + \frac{\partial f^\nu}{\partial x^\mu} \eta_\nu,$$

$$(r-1)e^{-\rho t} \frac{\partial U}{\partial y^i} = \frac{d\zeta_i}{dt},$$

$$(r-1)e^{-\rho t} \frac{\partial U}{\partial u^\sigma} = \frac{\partial f^\mu}{\partial u^\sigma} \eta_\mu + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \zeta_i,$$

whose solution $(\eta_\mu, \zeta_i) = ((r-1)\pi_\mu, (r-1)\lambda_i)$ follows from (3.5a), (3.5b) and (3.5c). Accordingly $(\xi_2^\alpha) = ((r-1)\pi_\mu, (r-1)\lambda_i, x^\mu, y^i, u^\sigma)$ is a solution.

3. Let the function U be of the form $U = -\alpha_i g^i(u)$ (α_i : const.). In this case, the equations (3.5a), (3.5b) and (3.5c) are reduced respectively to

$$(3.5a)' \quad \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu = 0,$$

$$(3.5b)' \quad \dot{\lambda}_i = 0,$$

$$(3.5c)' \quad \frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu + e^{-\rho t} (\lambda_i + \alpha_i) \frac{\partial g^i}{\partial u^\sigma} = 0;$$

and, by putting $\varphi^\mu = 0$, $\psi^i = 0$ and $\tau^\sigma = 0$, the equations (3.6c), (3.6d) and (3.6e) are also respectively to

$$\frac{d\eta_\mu}{dt} + \frac{\partial f^\nu}{\partial x^\mu} \eta_\nu = 0,$$

$$\frac{d\zeta_i}{dt} = 0,$$

$$\frac{\partial f^\mu}{\partial u^\sigma} \eta_\mu + e^{-\rho t} \frac{\partial g^i}{\partial u^\sigma} \zeta_i = 0;$$

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while the others (3.6a) and (3.6b) are distinguished. So that we find immediately a solution $(\xi_3^\alpha) = (\pi_\mu, \lambda_i + \alpha_i, 0, \dots, 0)$.

We have thus obtained the following three types of solutions in the theorem 1 under the respective condition: $U = U(x, u)$; or the homogeneities (3.8a), (3.8b) and (3.8c); or $U = -\alpha_i g^i(u)$ (α_i : const.):

$$(3.11a) \quad (\xi_1^\alpha) = (\dot{\pi}_\mu + \rho\pi_\mu, 0, \dot{x}^\mu, \dot{y}^i + \rho y^i, \dot{u}^\sigma),$$

$$(3.11b) \quad (\xi_2^\alpha) = ((r-1)\pi_\mu, (r-1)\lambda_i, x^\mu, y^i, u^\sigma),$$

$$(3.11c) \quad (\xi_3^\alpha) = (\pi_\mu, \lambda_i + \alpha_i, 0, \dots, 0).$$

in which, a couple of solutions (ξ_1^α) and (ξ_2^α) , or (ξ_1^α) and (ξ_3^α) are substituted respectively for (3.7) to construct the following conserved quantities.

Theorem 3.2. *According to the cases for the functions $f^\mu(x, u)$, $g^i(u)$ and $U(x, y, u)$: (i) $f^\mu(x, u)$ and $g^i(u)$ are homogeneous functions of degree one, and U is a homogeneous function of degree r of the form $U = U(x, u)$, or (ii) the function U is of the form $U = -\alpha_i g^i(u)$ (α_i : const.), there exists the following respective conserved quantity Ω_1 or Ω_2 for the extremal problem of (3.1) under the constraints (3.2) and (3.3):*

$$(3.12a) \quad \Omega_1 = (r-1)(\pi_\mu \dot{x}^\mu + \lambda_i(\dot{y}^i + \rho y^i)) - (\dot{\pi}_\mu + \rho\pi_\mu) x^\mu,$$

$$(3.12b) \quad \Omega_2 = \pi_\mu \dot{x}^\mu + (\lambda_i + \alpha_i)(\dot{y}^i + \rho y^i).$$

Through the conserved quantity Ω_2 of (3.12b), since by (3.3) and (3.5b)':

$$\begin{aligned} \lambda_i(\dot{y}^i + \rho y^i) &= e^{-\rho t} \lambda_i g^i(u(t)) + \rho \int_0^t e^{-\rho s} \lambda_i g^i(u(s)) ds + \text{const.} \\ &= \int_0^t e^{-\rho s} \lambda_i \dot{g}^i(u(s)) ds + \text{const.}, \end{aligned}$$

and since by (3.3) and $U = -\alpha_i g^i$:

$$\begin{aligned} \alpha_i(\dot{y}^i + \rho y^i) &= \alpha_i \left(e^{-\rho t} g^i(u(t)) + \rho \int_0^t e^{-\rho s} g^i(u(s)) ds \right) + \text{const.} \\ &= \alpha_i \int_0^t e^{-\rho s} \dot{g}^i(u(s)) ds + \text{const.}, \end{aligned}$$

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we can conceive the following conserved quantity in the extremal problem for the integration (3.1) with $U = -\alpha_i g^i$ under more general constraints $\dot{x}^\mu = f^\mu(x, u, t)$ and (3.3):

$$(3.13) \quad \Omega = \pi_\mu \dot{x}^\mu + \int_0^t \left(e^{-\rho s} (\lambda_i + \alpha_i) \dot{g}^i(u(s)) - \pi_\mu(s) \frac{\partial f^\mu(x(s), u(s), s)}{\partial s} \right) ds.$$

In fact, the appearances of (3.5a)' and (3.5c)' stay on for $f^\mu = f^\mu(x, u, t)$. They are added after multiplying \dot{x}^μ and \dot{u}^σ and then summing up for the indices μ and σ , respectively. The resulting identity

$$\dot{\pi}_\mu \dot{x}^\mu + e^{-\rho t} (\lambda_i + \alpha_i) \dot{g}^i = -\pi_\mu \left(\frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial f^\mu}{\partial u^\sigma} \dot{u}^\sigma \right),$$

and $\ddot{x}^\mu = \dot{f}^\mu$ are substituted for the total time derivative of (3.13) to conclude

$$\begin{aligned} \dot{\Omega} &= \pi_\mu \ddot{x}^\mu + \dot{\pi}_\mu \dot{x}^\mu + e^{-\rho t} (\lambda_i + \alpha_i) \dot{g}^i - \pi_\mu \frac{\partial f^\mu}{\partial t} \\ &= \pi_\mu \left(\dot{f}^\mu - \frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu - \frac{\partial f^\mu}{\partial u^\sigma} \dot{u}^\sigma - \frac{\partial f^\mu}{\partial t} \right) = 0. \end{aligned}$$

3.3 Reductions of the extremal problem

In the extremal problem, absence of the variables y^i extinguishes the constraints (3.3) and resets an integration

$$(3.1)' \quad \int_0^T e^{-\rho t} U(x, u) dt,$$

under the remaining ones (3.2). Then the relating Lagrangian is given as

$$L = e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - f^\mu),$$

whose Euler-Lagrange equations consist of (3.2) and

$$(3.5a) \quad e^{-\rho t} \frac{\partial U}{\partial x^\mu} = \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu,$$

$$(3.5c)'' \quad e^{-\rho t} \frac{\partial U}{\partial u^\sigma} = \frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu.$$

This situation can be regarded as a particular case with vanishing characters λ_i , y^i , g^i , ζ_i and ψ^i . Accordingly, if $f^\mu(x, u)$ and $U(x, u)$ are homogeneous functions of degree one and

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degree r with respect to x^μ and u^σ , respectively, there exists the following conserved quantity which is observed in part of (3.12a):

$$(3.14) \quad \Xi_1 = (r - 1) \pi_\mu \dot{x}^\mu - (\dot{\pi}_\mu + \rho \pi_\mu) x^\mu.$$

The equations (3.5a) and (3.5c)'' are added after multiplying x^μ and u^σ and then summing up for the indices μ and σ , respectively. Then, through the homogeneities of U and f^μ , it follows that

$$(3.15) \quad r e^{-\rho t} U = \dot{\pi}_\mu x^\mu + \pi_\mu f^\mu = \dot{\pi}_\mu x^\mu + \pi_\mu \dot{x}^\mu,$$

which is used to eliminate $\dot{\pi}_\mu x^\mu$ in Ξ_1 . Consequently Ξ_1 is written as

$$(3.14)' \quad \Xi_1 = -r e^{-\rho t} U + \pi_\mu (r \dot{x}^\mu - \rho x^\mu).$$

Moreover, if U is also a homogeneous function of degree one, i.e., $r = 1$, it is reduced to (cf. (2.9) in Kataoka and Hashimoto [26], while whose λ_i correspond here to $-\pi_\mu$)

$$\Xi_1 = -e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - \rho x^\mu).$$

Particularly let $k = \ell$, i.e., $\sigma = 1, \dots, k$ as well as $\mu = 1, \dots, k$ and consider the integration (3.1)' under the constraints $\dot{x}^\mu = u^\mu$ (put $f^\mu = u^\mu$ in (3.2)), i.e.,

$$\int_0^T e^{-\rho t} U(x, \dot{x}) dt.$$

In this case, the equations (3.5a) and (3.5c)'' are reduced respectively to

$$e^{-\rho t} \frac{\partial U}{\partial x^\mu} = \dot{\pi}_\mu, \quad e^{-\rho t} \frac{\partial U}{\partial u^\mu} = \pi_\mu,$$

in which π_μ are eliminated to see the relations

$$(3.16) \quad \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}^\mu} \right) = \frac{\partial U}{\partial x^\mu} + \rho \frac{\partial U}{\partial \dot{x}^\mu}.$$

Therefore the conserved quantity (3.14) is written as

$$(3.17) \quad \begin{aligned} \Xi_1 &= e^{-\rho t} \left((r - 1) \dot{x}^\mu \frac{\partial U}{\partial \dot{x}^\mu} - x^\mu \left(\frac{\partial U}{\partial x^\mu} + \rho \frac{\partial U}{\partial \dot{x}^\mu} \right) \right) \\ &= e^{-\rho t} \left((r - 1) \dot{x}^\mu \frac{\partial U}{\partial \dot{x}^\mu} - x^\mu \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}^\mu} \right) \right), \end{aligned}$$

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which, by the homogeneity conditions, turns into (cf. [34], Theorem 3')

$$\Xi_1 = e^{-\rho t} \left((r\dot{x}^\mu - \rho x^\mu) \frac{\partial U}{\partial \dot{x}^\mu} - rU \right).$$

Moreover Ξ_1 can be written as $\Xi_1 = r\tilde{H}$, where \tilde{H} is the modified Hamiltonian

$$\tilde{H} = H - \frac{\rho}{r} x^\mu \frac{\partial L}{\partial \dot{x}^\mu} = \left(\dot{x}^\mu - \frac{\rho}{r} x^\mu \right) \frac{\partial L}{\partial \dot{x}^\mu} - L,$$

in which H is the usual Hamiltonian $H = \dot{x}^\mu (\partial L / \partial \dot{x}^\mu) - L$ where $L = e^{-\rho t} U(x, \dot{x})$.

In the extremal problem for the integration (3.1)' under more general constraints $\dot{x}^\mu = f^\mu(x, u, t)$, the conserved quantity (3.14) or (3.14)' can be generalized as

$$(3.18) \quad \Xi = \Xi_1 - r \int_0^t \pi_\mu(s) \frac{\partial f^\mu(x(s), u(s), s)}{\partial s} ds.$$

In fact (3.5a) and (3.5c)'' (which stay on for $f^\mu = f^\mu(x, u, t)$) are added after multiplying \dot{x}^μ and \dot{u}^σ and then summing up for the indices μ and σ , respectively. The resulting identity

$$\begin{aligned} e^{-\rho t} \dot{U} &= \dot{\pi}_\mu \dot{x}^\mu + \pi_\mu \left(\frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial f^\mu}{\partial u^\sigma} \dot{u}^\sigma \right), \\ &= \dot{\pi}_\mu \dot{x}^\mu + \pi_\mu \left(\dot{f}^\mu - \frac{\partial f^\mu}{\partial t} \right), \end{aligned}$$

and $\ddot{x}^\mu = \dot{f}^\mu$ are substituted for the total time derivative of (3.14)' to see

$$\begin{aligned} \dot{\Xi}_1 &= \rho (r e^{-\rho t} U - \dot{\pi}_\mu x^\mu - \pi_\mu \dot{x}^\mu) + r (\pi_\mu \ddot{x}^\mu + \dot{\pi}_\mu \dot{x}^\mu - e^{-\rho t} \dot{U}) \\ &= \rho (r e^{-\rho t} U - \dot{\pi}_\mu x^\mu - \pi_\mu \dot{x}^\mu) + r \pi_\mu \frac{\partial f^\mu}{\partial t}. \end{aligned}$$

So that the total time derivative of (3.18): $\dot{\Xi}_1 - r \pi_\mu \partial f^\mu / \partial t$ vanishes by (3.15).

The characters in **3.2** are organized into $(x^A) = (x^\mu, y^i)$, $(\pi_A) = (\pi_\mu, \lambda_i)$ and $(f^A) = (f^\mu, e^{-\rho t} g^i)$ with the extended index $A = 1, \dots, k+n$, while $\mu = 1, \dots, k$ and $i = 1, \dots, n$. Here the functions f^μ , g^i and U are generalized as $f^\mu = f^\mu(x^A, u^\sigma)$, $g^i = g^i(x^A, u^\sigma)$ and $U = U(x^A, u^\sigma)$; and both f^μ and g^i , or U are assumed to be of homogeneous degree one or r with respect to x^A and u^σ , respectively. Then the conserved quantity from (3.18):

$$\Xi = (r-1) \pi_A \dot{x}^A - (\dot{\pi}_A + \rho \pi_A) x^A - r \int_0^t \pi_A(s) \frac{\partial f^A(x(s), u(s), s)}{\partial s} ds$$

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is arranged in the original variables for

$$(3.19) \quad \Xi = (r - 1)(\pi_\mu \dot{x}^\mu + \lambda_i \dot{y}^i) - (\dot{\pi}_\mu + \rho \pi_\mu) x^\mu - (\dot{\lambda}_i + \rho \lambda_i) y^i + r \rho \int_0^t \lambda_i(s) \dot{y}^i(s) ds,$$

which is a generalization of (3.12a). In fact, particularly let $U = U(x^\mu, u^\sigma)$. Then, since $\dot{\lambda}_i = 0$ in (3.5b)' so that

$$\int_0^t \lambda_i \dot{y}^i(s) ds = \lambda_i y^i + \text{const.},$$

the conserved quantity (3.19) is just reduced to the appearance of (3.12a) up to constants.

3.4 Conservation laws and optimal paths

3.4.1 More general neoclassical growth models

The conservation laws for the model suggested by Samuelson ([41], p.113) in the neo-classical optimal growths can be generalized with an application of the theorem 3.2 to a maximizing problem of an integration

$$(3.20) \quad \int_0^T e^{-\rho t} g(x, \dot{x}) dt,$$

where $g(x, \dot{x})$ is assumed to be a homogeneous production function of degree r with respect to the capital-labour ratio x and the rate of capital accumulation \dot{x} , and ρ ($\rho \geq 0$) is the fixed discount rate; while in the suggested model, $g(x, \dot{x})$ is a second order homogeneous polynomial with respect to x and \dot{x} . Moreover, in accordance with the maximization, the concavity of g is assumed, i.e., $g_{xx} < 0$ and

$$(3.21) \quad g_{xx} g_{\dot{x}\dot{x}} - g_{x\dot{x}}^2 > 0,$$

where $g_x = \partial g / \partial x$, $g_{\dot{x}} = \partial g / \partial \dot{x}$ and $g_{\dot{x}x} = \partial^2 g / \partial x \partial \dot{x}$ (such a convention for the derivatives is used in what follows). In this case, the Euler-Lagrange equation for the Lagrangian $L = e^{-\rho t} g(x, \dot{x})$ is written as

$$(3.22) \quad \dot{g}_{\dot{x}} = g_x + \rho g_{\dot{x}},$$

and the conserved quantity (3.17) reduces to

$$(3.23) \quad \begin{aligned} \Xi_1 &= e^{-\rho t} ((r - 1) \dot{x} g_{\dot{x}} - x (g_x + \rho g_{\dot{x}})) \\ &= e^{-\rho t} ((r - 1) \dot{x} g_{\dot{x}} - x \dot{g}_{\dot{x}}). \end{aligned}$$

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Here recall the homogeneity condition of g :

$$g_x x + g_{\dot{x}} \dot{x} = r g,$$

and its derivative with respect to x and \dot{x} :

$$(3.24a) \quad g_{xx} x + g_{\dot{x}x} \dot{x} = (r-1)g_x,$$

$$(3.24b) \quad g_{\dot{x}x} x + g_{\dot{x}\dot{x}} \dot{x} = (r-1)g_{\dot{x}}.$$

By the identities (3.24b) and $\dot{g}_{\dot{x}} = g_{\dot{x}x} \dot{x} + g_{\dot{x}\dot{x}} \ddot{x}$, the second appearance of Ξ_1 in (3.23) turns finally into

$$(3.25) \quad \Xi_1 = e^{-\rho t} g_{\dot{x}\dot{x}} (\dot{x}^2 - \ddot{x} x).$$

Thus the theorem 3.2 yields the following result.

Theorem 3.3. *For the maximizing problem of (3.20), let $g(x, \dot{x})$ be a homogeneous concave production function of degree r with respect to x and \dot{x} . Then there exists the conserved quantity Ξ_1 of the forms (3.23) and (3.25).*

The conservation law $\Xi_1 = \text{const.}$ can be used effectively to determine the optimal paths completely in the models relative to the following production functions, while the relating integration is assumed to lie in the case of finite horizon $0 < T < \infty$.

Cobb-Douglas production function. Let the homogeneous production function $g(x, \dot{x})$ of degree r be of the form

$$(3.26) \quad g(x, \dot{x}) = x^a \dot{x}^{r-a} \quad (a, r: \text{const.}; 0 < a < 1, a < r < 1),$$

where the conditions on the constants a and r follows from the concavity of g . Then the Euler-Lagrange equation (3.22) is written as

$$(3.27) \quad (r-a)(r-a-1)\ddot{x}x + a(r-a-1)\dot{x}^2 - \rho(r-a)\dot{x}x = 0.$$

And the first appearance of Ξ_1 in (3.23) leads to

$$(3.28) \quad \Xi_1 = e^{-\rho t} \dot{x}^{r-a-1} x^a (r(r-a-1)\dot{x} - \rho(r-a)x),$$

which is, by $y = x^{\frac{r}{r-a}}$, transformed to

$$\Xi_1 = e^{-\rho t} \dot{y}^{r-a-1} ((r-a-1)\dot{y} - \rho y).$$

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Accordingly, from $\dot{\Xi}_1 = 0$, i.e., $d(e^{\rho t}\Xi_1)/dt = \rho e^{\rho t}\Xi_1$:

$$\begin{aligned} & (r-a-1)\ddot{y}y^{r-a-2}((r-a-1)\dot{y}-\rho y) + \dot{y}^{r-a-1}((r-a-1)\ddot{y}-\rho\dot{y}) \\ & = \rho\dot{y}^{r-a-1}((r-a-1)\dot{y}-\rho y), \end{aligned}$$

it follows that

$$((r-a-1)\ddot{y}-\rho\dot{y})((r-a)\dot{y}-\rho y) = 0.$$

Here note that the solution $y = \beta e^{\frac{\rho}{r-a}t}$ (β : const.) of the equation $(r-a)\dot{y}-\rho y = 0$, i.e., $x = y^{\frac{r-a}{r}} = \alpha e^{\frac{\rho}{r}t}$ (α : const.) does not satisfy the Euler-Lagrange equation (3.22). Therefore $(r-a-1)\ddot{y}-\rho\dot{y} = 0$, which is, if $\rho \neq 0$, integrated as $\dot{y} = \gamma e^{\frac{\rho}{r-a-1}t}$ (γ : const.) and moreover as $y = \alpha e^{\frac{\rho}{r-a-1}t} + \beta$ (α, β : const.); while $\ddot{y} = 0$ if $\rho = 0$, so that $y = \alpha t + \beta$ (α, β : const.). Thus, in the maximizing problem, the optimal path $x = x(t)$ ($x = y^{\frac{r-a}{r}}$) is determined completely as

$$(3.29) \quad x(t) = \begin{cases} (\alpha e^{\frac{\rho}{r-a-1}t} + \beta)^{\frac{r-a}{r}} & \text{if } \rho > 0 \\ (\alpha t + \beta)^{\frac{r-a}{r}} & \text{if } \rho = 0 \end{cases} \quad (\alpha, \beta: \text{const.}).$$

In the case of infinite horizon $T = \infty$, the constants α and β in (3.29) for $\rho > 0$ can be given so as to satisfy the transversality condition $\lim_{t \rightarrow \infty} \pi x = 0$ (e.g., see Takayama [47]). In fact, since $\pi = e^{-\rho t}g_{\dot{x}}$ (see (3.4b) with $U = g = x^a u^{r-a}$ and $\dot{x} = f = u$), so that πx is equal to $\alpha e^{\frac{\rho}{r-a-1}t} + \beta$ ($r-a-1 < 0$) up to a constant multiple, it follows that $\lim_{t \rightarrow \infty} \pi x = \alpha + \beta = 0$. Therefore $x(t) = \gamma(1 - e^{\frac{\rho}{r-a-1}t})^{\frac{r-a}{r}}$ (γ : const.) is a feasible path in the case of infinite horizon.

A generalization of the Samuelson's model. Let the homogeneous production function $g(x, \dot{x})$ of degree r be given by

$$(3.30) \quad g_{\dot{x}\dot{x}} = m x^s \quad (m, s: \text{const.}; s \geq 0),$$

(cf. the production function $g(x, \dot{x})$ in the model suggested by Samuelson [41], which is a second order polynomial with respect to x and \dot{x} satisfying $g_{\dot{x}\dot{x}} = -1$). Then the conserved quantity Ξ_1 of (3.25) leads to

$$(3.31) \quad \Xi = e^{-\rho t} x^s (\dot{x}^2 - \ddot{x}x) = a \quad (a: \text{const.}).$$

This equation is, by $y = x^{\frac{s+2}{2}}$, transformed to $e^{-\rho t}(\dot{y}^2 - \ddot{y}y) = \frac{1}{2}(s+2)a$; and then, by $z = e^{-\frac{\rho}{2}t}y$, moreover to

$$(3.32) \quad \dot{z}^2 - \ddot{z}z = \frac{1}{2}(s+2)a.$$

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So put $\dot{z} = p$ and substitute it together with $\ddot{z} = \dot{z}(dp/dz) = p(dp/dz)$ for (3.32). Then the resulting equation

$$\frac{p}{p^2 - \frac{1}{2}(s+2)a} dp = \frac{1}{z} dz$$

is integrated as $\log |p^2 - \frac{1}{2}(s+2)a|/z^2 = \text{const.}$, i.e.,

$$(3.33) \quad \dot{z}^2 - bz^2 - \frac{1}{2}(s+2)a = 0 \quad (a, b: \text{const.}),$$

in which the sign of the constant b can be determined as follows.

In fact, $z = e^{-\frac{\rho}{2}t} x^{\frac{s+2}{2}}$, its derivative $\dot{z} = \frac{1}{2}e^{-\frac{\rho}{2}t} x^{\frac{s}{2}}((s+2)\dot{x} - \rho x)$ and a of (3.31) are substituted for (3.33) to derive

$$(3.34) \quad 4bx^2 = ((s+2)\dot{x} - \rho x)^2 - 2(s+2)(\dot{x}^2 - \ddot{x}x).$$

In turn, for a multiple of the Euler-Lagrange equation (3.22) by $r-1$:

$$(r-1)(g_{\dot{x}\dot{x}}\dot{x} + g_{\dot{x}\ddot{x}}\ddot{x}) = (r-1)(g_x + \rho g_{\dot{x}}),$$

the relations (3.24a) and (3.24b) are substituted to see

$$(r-1)g_{\dot{x}\ddot{x}}\ddot{x} + ((r-2)g_{\dot{x}\dot{x}} - \rho g_{\dot{x}\dot{x}})\dot{x} - (g_{xx} + \rho g_{\dot{x}x})x = 0,$$

which is used to delate the variable \ddot{x} in (3.34). Here note that the homogeneous degree r of g is $r = s+2$ by (3.30). Cosequently it follows that

$$(3.35) \quad 4(s+1)bx^2 = s(s+1)(s+2) \left(\dot{x} - \frac{g_{\dot{x}\dot{x}} + \rho g_{\dot{x}\dot{x}}}{(s+1)g_{\dot{x}\dot{x}}} x \right)^2 + \frac{((s+2)g_{\dot{x}\dot{x}} + \rho g_{\dot{x}\dot{x}})^2}{(s+1)g_{\dot{x}\dot{x}}^2} x^2 + \frac{2(s+2)(g_{xx}g_{\dot{x}\dot{x}} - g_{\dot{x}x}^2)}{g_{\dot{x}\dot{x}}^2} x^2.$$

Therefore, in view of $s \geq 0$ and $x \neq 0$, (3.21) for the concavity of g is carried into (3.35) to conclude that $b > 0$.

Now the equation (3.33) is written as $dz/\sqrt{z^2 \pm \lambda^2} = \beta dt$ where $\lambda = \sqrt{\pm \frac{1}{2}(s+2)a/b}$ (the signs \pm correspond respectively to $a \gtrless 0$) and $\beta = \pm\sqrt{b}$, and then integrated as

$$\begin{cases} \sinh^{-1}(z/\lambda) = \beta t + \gamma & \text{if } a > 0 \\ \cosh^{-1}(z/\lambda) = \beta t + \gamma & \text{if } a < 0 \end{cases} \quad (\lambda, \beta, \gamma: \text{const.}; \lambda > 0),$$

for which $z = e^{-\frac{\rho}{2}t} x^{\frac{s+2}{2}}$ is substituted. Thus the optimal path is determined completely as

$$(3.36) \quad x(t) = \begin{cases} \alpha e^{\frac{\rho}{s+2}t} \sinh^{\frac{2}{s+2}}(\beta t + \gamma) & \text{if } a > 0 \\ \alpha e^{\frac{\rho}{s+2}t} \cosh^{\frac{2}{s+2}}(\beta t + \gamma) & \text{if } a < 0 \end{cases} \quad (\alpha, \beta, \gamma: \text{const.}; \alpha > 0).$$

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3.4.2 The intergenerational problem

The second application of our new derivation of conservation laws is given in the intergenerational problem. Within the variables: extracted amount of n exhaustible resources $u^i = R^i(t)$ with stocks $y^i = S^i(t)$ ($i = 1, \dots, n$), reproducible capital $x = K(t)$ and the consumption $C(t)$, the model of Sato and Kim [45] with single exhaustible resource grew up in (Mimura, Fujiwara and Nôno [28]) to be a minimizing problem of the integration

$$(3.37) \quad \int_0^T e^{-\rho t} \alpha_i R^i dt,$$

under constraints

$$(3.38) \quad \dot{K} = F(R, K, C),$$

$$(3.39) \quad \dot{S}^i = -e^{-\rho t} R^i,$$

where α_i are some constants and ρ is a constant discount rate. Here the implicitly assumed constant consumption path C in [28] is generalized as an exponentially growing one (see Fujiwara, Mimura and Nôno [10] for single exhaustible resource):

$$(3.40) \quad C(t) = ce^{rt} \quad (c, r: \text{const.}).$$

In the problem, since the characters f , g^i and U are $f = F$, $g^i = -R^i$ and $U = -\alpha_i g^i = \alpha_i R^i$, the equations (3.5a), (3.5b) and (3.5c) are written respectively as

$$(3.41a) \quad \dot{\pi} = -\pi F_K,$$

$$(3.41b) \quad \dot{\lambda}_i = 0,$$

$$(3.41c) \quad \lambda_i + \alpha_i = e^{\rho t} \pi F_{R^i},$$

where $F_K = \partial F / \partial K$ and $F_{R^i} = \partial F / \partial R^i$ (such a convention for the derivatives is used in what follows). Accordingly $\lambda_i + \alpha_i$ are constants, so assuming $F_{R^i} \neq 0$ ($i = 1, \dots, n$) that

$$(3.42) \quad \Omega_j^i \equiv \frac{F_{R^i}}{F_{R^j}} = \text{const.}$$

Here note that the following identity from (3.41a), (3.41b) and the total time derivative of (3.41c):

$$\pi \dot{F}_{R^i} + (\rho\pi + \dot{\pi}) F_{R^i} = \pi(\dot{F}_{R^i} + \rho F_{R^i} - F_K F_{R^i}) = 0$$

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yield the generalized Hotelling rule:

$$(3.43) \quad F_K - \rho = \frac{\dot{F}_{R_1}}{F_{R_1}} = \dots = \frac{\dot{F}_{R_n}}{F_{R_n}},$$

in which lie implicitly the conserved quantities (3.42), i.e., (3.43) implies $(d/dt)(F_{R^i}/F_{R^j}) = 0$. The conserved quantity (3.13) is written as

$$(3.44) \quad \Omega = \pi \dot{K} - (\lambda_i + \alpha_i) \int_0^t e^{-\rho s} \dot{R}^i(s) ds - r \int_0^t \pi(s) C(s) F_C(s) ds,$$

where $F(s) = F(R(s), K(s), C(s))$, and so, in view of the identities from (3.39):

$$\begin{aligned} \int_0^t e^{-\rho s} \dot{R}^i(s) ds &= e^{-\rho t} R^i(t) + \rho \int_0^t e^{-\rho s} R^i(s) ds + \text{const.} \\ &= e^{-\rho t} R^i(t) - \rho S^i(t) + \text{const.}, \end{aligned}$$

it is also up to constants as

$$(3.44)' \quad \Omega = \pi \dot{K} - (\lambda_i + \alpha_i) (e^{-\rho t} R^i - \rho S^i) - r \int_0^t \pi(s) C(s) F_C(s) ds.$$

Since the equation $\dot{\pi}/\pi = -F_K$ from (3.41a) is integrated as

$$\pi = A e^{-\int_0^t F_K(s) ds} \quad (A: \text{const.}),$$

accordingly the equations (3.41c) lead to

$$\lambda_i + \alpha_i = A e^{\rho t - \int_0^t F_K(s) ds} F_{R^i},$$

the conserved quantity $\Omega_0 = \Omega/A$ from (3.44) or (3.44)' is written respectively as

$$\begin{aligned} \Omega_0 &= e^{-\int_0^t F_K(s) ds} \left(\dot{K} - e^{\rho t} F_{R^i} \int_0^t e^{\rho s} \dot{R}^i ds \right) - r \int_0^t e^{-\int_0^t F_K(s) ds} C(s) F_C(s) ds, \quad \text{or} \\ \Omega_0 &= e^{-\int_0^t F_K(s) ds} \left(\dot{K} - e^{\rho t} F_{R^i} (e^{-\rho t} R^i - \rho S^i) \right) - r \int_0^t e^{-\int_0^t F_K(s) ds} C(s) F_C(s) ds. \end{aligned}$$

Moreover in $\Omega_j = \Omega/(\lambda_j + \alpha_j)$, (3.41c) and (3.42) are used to deduce (cf. (2.8) or (2.8)' in [28] for constant consumption path, i.e., $r = 0$ in $C = ce^{rt}$)

$$(3.45) \quad \Omega_j = \frac{1}{F_{R^j}} \left(e^{-\rho t} \dot{K} - F_{R^i} \int_0^t e^{-\rho s} \dot{R}^i(s) ds \right) - r \int_0^t e^{-\rho s} \frac{C(s) F_C(s)}{F_{R^j}(s)} ds, \quad \text{or}$$

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$$(3.45)' \quad \Omega_j = \frac{1}{F_{R^j}} \left(e^{-\rho t} \dot{K} - F_{R^i}(e^{-\rho t} R^i - \rho S^i) \right) - r \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_{R^j}(s)} ds;$$

which is, for single exhaustible resource $R^1 = R$, reduced to ((3.10) in [10])

$$\begin{aligned} \Omega_1 &= e^{-\rho t} \frac{\dot{K}}{F_R} - \int_0^t e^{-\rho s} \left(\dot{R}^i(s) + \frac{rC(s)F_C(s)}{F_R(s)} \right) ds, \text{ or} \\ \Omega_1 &= e^{-\rho t} \frac{\dot{K}}{F_R} - e^{-\rho t} R + \rho S - r \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_R(s)} ds. \end{aligned}$$

Finally, in view of the relations from (3.42): $\Omega_j^i F_{R^j} = n F_{R^i}$ and

$$F_{R^i} \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_{R^i}(s)} ds = F_{R^j} \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_{R^j}(s)} ds \text{ (not summing up for } i, j),$$

any one conserved quantity, e.g., Ω_1 turns into the generalized Hartwick's rule (cf. (2.9) or (2.9)' in [28])

$$(3.46) \quad \dot{K} = e^{\rho t} F_{R^i} \left(\int_0^t e^{-\rho s} \dot{R}^i(s) ds + \frac{r}{n} \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_{R^i}(s)} ds + A^i \right), \text{ or}$$

$$(3.46)' \quad \dot{K} = e^{\rho t} F_{R^i} \left(e^{-\rho t} R^i - \rho S^i + \frac{r}{n} \int_0^t e^{-\rho s} \frac{C(s)F_C(s)}{F_{R^i}(s)} ds + A^i \right); \quad (A^i: \text{const.}),$$

where $A^i = \Omega_1 \Omega_i^1 / n$. Particularly in the case where

$$(3.47) \quad F(R, K, C) = f(R, K) - a_i R^i - C \quad (a_i: \text{constant for extraction costs}),$$

if the consumption path C is implicitly assumed to be constant in the extremal problem with zero discount rate, i.e., $\rho = 0$, the above rule is reduced to the Hartwick's IRR (investment resource rents) rule ((3.16) in [20], while in which $A^i = 0$):

$$\dot{K} = (f_{R^i} - a_i)(R^i + A^i) \quad (A^i: \text{const.}).$$

Conversely let assume the generalized Hotelling's rule (3.43) in which lie the conserved quantities (3.42), and the generalized Hartwick's rule (3.46) which is equivalent to the conserved quantities (3.45) under the rule (3.42). Then, through the total time derivative of (3.45) (i.e., $\dot{\Omega}_j = 0$), The rule (3.46) leads to

$$(3.48) \quad \frac{d}{dt} \left(\frac{\dot{K}}{F_{R^j}} \right) - \frac{\rho \dot{K} + F_{R^i} \dot{R}^i + r C F_C}{F_{R^j}} = 0,$$

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in which, by using $\dot{F}_{Rj} = (F_K - \rho)F_{Rj}\dot{K}$ from (3.43), the first term is written as

$$\frac{d}{dt} \left(\frac{\dot{K}}{F_{Rj}} \right) = \frac{F_{Rj}\ddot{K} - \dot{F}_{Rj}\dot{K}}{F_{Rj}^2} = \frac{\ddot{K} - (F_K - \rho)\dot{K}}{F_{Rj}}.$$

Therefore it follows that

$$(3.49) \quad \ddot{K} = F_{Ri}\dot{R}^i + F_K\dot{K} + rCF_C,$$

which is compared with the total time derivative of the constraint (3.38):

$$\ddot{K} = F_{Ri}\dot{R}^i + F_K\dot{K} + F_C\dot{C},$$

to deduce $F_C(\dot{C} - rC) = 0$. So that $\dot{C} = rC$ if $F_C \neq 0$, consequently the consumption path is of the exponential form (3.40).

Under the rule (3.42), if the consumption path is of the form (3.40), the total time derivative of the constraint (3.38) is written as (3.49), which is substituted for the left hand side of (3.48) (i.e., $e^{\rho t}\dot{\Omega}_j$). Then the resulting equations multiplied by F_{Rj}^2 are

$$\begin{aligned} e^{\rho t}F_{Rj}^2\dot{\Omega}_j &= F_{Rj}\ddot{K} - \dot{F}_{Rj}\dot{K} - \rho F_{Rj}\dot{K} - F_{Rj}F_{Ri}\dot{R}^i - rCF_{Rj}F_C \\ &= ((F_K - \rho)F_{Rj} - \dot{F}_{Rj})\dot{K} + (\dot{C} - rC)F_{Rj}F_C, \end{aligned}$$

where j 's of $F_{Rj}^2\dot{\Omega}_j$ in the first term are free indices (not summing up for j). So that the generalized rules (3.43) and (3.46) (i.e., $\dot{\Omega}_j = 0$) are equivalent under the exponentially growing consumption (3.40) and the rule (3.42). Summarizing, we have the following result (cf. Theorem 1 in [28] for constant consumption):

In the minimizing problem of the integration (3.37) under the constraints (3.38) with $F_{Ri}F_C \neq 0$ ($i = 1, \dots, n$) and (3.39), if the consumption path is assumed to be of the exponential form (3.40), there exist the generalized Hotelling's rule (3.43) in which lie implicitly the relations (3.42), and the generalized Hartwick's rule (3.46) which is equivalent to the conserved quantities (3.45) under the relations (3.42) (also (3.45) and (3.46) are equivalent to (3.45)' and (3.46)', respectively). Conversely the two generalized rules make the consumption into the exponential form (3.40), while they are equivalent under the exponential consumption (3.40) and the relations (3.42).

Remark. More generally, in view of the case (ii) in the theorem 3.2, we can see that R^i may be replaced with $G^i(R)$ in the conserved quantity (3.44), which will follow from a minimizing problem of an integration

$$\int_0^T e^{-\rho t} \alpha_i G^i(R) dt,$$

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under constraints

$$\begin{aligned}\dot{K} &= F(R, K, C), \\ \dot{S}^i &= -e^{-\rho t} G^i(R).\end{aligned}$$

The interested reader will find more general Hartwick's rule in [28] and pursue further consideration along the above line of approach.

To determine optimal paths in our generalized model, we consider the function F of the form (3.47) with zero extraction costs and exponentially growing consumption (3.40):

$$(3.47)' \quad F(R, K, C) = f(R, K) + \rho K - ce^{\rho t},$$

where f in (3.47) is replaced with $f + \rho K$, in which the production function f is assumed to be of homogeneous degree one with respect to R^i and K . In this case the generalized Hotelling's rule (3.43) is reduced to

$$(3.43)' \quad f_K = \frac{\dot{f}_{R^1}}{f_{R^1}} = \dots = \frac{\dot{f}_{R^n}}{f_{R^n}}.$$

The total time derivative of the homogeneity condition $f = f_{R^i} R^i + f_K K$:

$$\begin{aligned}\dot{f} &= \dot{f}_{R^i} R^i + \dot{f}_K K + f_{R^i} \dot{R}^i + f_K \dot{K} \\ &= \dot{f}_{R^i} R^i + \dot{f}_K K + \dot{f}\end{aligned}$$

provides the relation

$$\dot{f}_{R^i} R^i + \dot{f}_K K = 0,$$

for which $\dot{f}_{R^i} = f_K f_{R^i}$ from (3.43)' are substituted to derive

$$(3.50) \quad \frac{\dot{f}_K}{f_K} = - \frac{f_{R^i} R^i}{K}.$$

Moreover, we put the production function f into

$$f(R, K) = K^{1-\gamma} \Phi(R),$$

in which Φ is homogeneous of degree γ ($\gamma \neq 1$) with respect to R^i , so that f is homogeneous of degree one with respect to R^i and K . Then, since

$$f_{R^i} R^i = K^{1-\gamma} \Phi_{R^i} R^i = \gamma K^{1-\gamma} \Phi = \gamma f,$$

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the relation (3.50) leads to

$$(3.51) \quad \frac{\dot{f}_K}{f_K} = -\frac{\gamma f}{K}.$$

And the function f satisfies $f_K K = (1 - \gamma)f$, i.e.,

$$\frac{f_K K}{f} = 1 - \gamma = \text{const.},$$

whose total time derivative yields the relation (cf. (1.5) with $\sigma = 1$ in [28]):

$$(3.52) \quad \frac{\dot{f}}{f} - \frac{\dot{K}}{K} = \frac{\dot{f}_K}{f_K}.$$

Therefore, it follows from (3.51) and (3.52) that

$$\begin{aligned} \frac{d}{dt} \left(\frac{K}{f} \right) &= \frac{\dot{K}f - \dot{f}K}{f^2} = -\frac{K}{f} \left(\frac{\dot{f}}{f} - \frac{\dot{K}}{K} \right) \\ &= -\frac{K}{f} \cdot \frac{\dot{f}_K}{f_K} = \gamma, \end{aligned}$$

whose integration

$$(3.53) \quad \frac{K}{f} = K^\gamma \Phi^{-1} = \gamma t + A_0, \quad \text{i.e.,} \quad K^{-\gamma} \Phi = \frac{1}{\gamma t + A_0} \quad (A_0: \text{const.})$$

is substituted for the constraint (3.38), where

$$F(R, K, C) = K^{1-\gamma} \Phi(R) + \rho K - ce^{rt}.$$

Then the resulting equation

$$(3.54) \quad \dot{K} = \frac{K}{\gamma t + A_0} + \rho K - ce^{rt}$$

is transformed, by putting $K = e^{\rho t} \Psi(t)$, into

$$\dot{\Psi} = \frac{\Psi}{\gamma t + A_0} - ce^{(r-\rho)t},$$

for which the solution of $\dot{\Psi} = \Psi/(\gamma t + A_0)$:

$$\Psi = \Psi_0(\gamma t + A_0)^{1/\gamma} \quad (\Psi_0: \text{const.})$$

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is substituted after replacing the constant Ψ_0 with arbitrary function $\Psi_0(t)$. Consequently

$$\dot{\Psi}_0 = -ce^{(r-\rho)t}(\gamma t + A_0)^{-1/\gamma},$$

whose integration completes the solution of (3.54). Thus, assuming that $\gamma > 0$ and $A_0 \geq 0$ (in addition $\rho - r > 0$ if $A_0 = 0$), or $\gamma < 0$ and $A_0 > -\gamma T$, the path of reproducible capital is determined as (cf. [10], also [28] for constant consumption, i.e., $r = 0$)

$$K(t) = -ce^{\rho t}(\gamma t + A_0)^{1/\gamma} \left(\int_0^t e^{(r-\rho)s}(\gamma s + A_0)^{-1/\gamma} ds + A_1 \right).$$

Here assume that $f(R, K) = K^{1-\gamma}\Phi(R)$ is the Cobb-Douglas production function, i.e., $\Phi(R) = (R^1)^{\gamma_1} \dots (R^n)^{\gamma_n}$. In this case, $\gamma = \gamma_1 + \dots + \gamma_n$, i.e.,

$$f(R, K) = K^{1-\gamma_1-\dots-\gamma_n}(R^1)^{\gamma_1} \dots (R^n)^{\gamma_n}.$$

So, the relation from (3.42):

$$\frac{\gamma_i}{\gamma_1} \cdot \frac{f_{R^1}}{f_{R^i}} = \frac{R^i}{R^1} = \kappa_i \quad (\kappa_i: \text{const.}, \kappa_1 = 1),$$

i.e., $R^i = \kappa_i R^1$ are substituted for f in (3.53) to obtain

$$R^1 = (\kappa_2^{\gamma_2} \dots \kappa_n^{\gamma_n})^{-1/\gamma} (\gamma t + A_0)^{-1/\gamma} K.$$

Finally the path of the exhaustible resources are determined as

$$R^i(t) = -c\kappa_i(\kappa_2^{\gamma_2} \dots \kappa_n^{\gamma_n})^{-1/\gamma} e^{\rho t} \left(\int_0^t e^{(r-\rho)s}(\gamma s + A_0)^{-1/\gamma} ds + A_1 \right),$$

in which note that $\kappa_1 = 1$. If f has only one exhaustible resource, i.e., $f(R, K) = K^{1-\gamma}R^\gamma$, the path of exhaustible resource is (cf. [10])

$$R(t) = -ce^{\rho t} \left(\int_0^t e^{(r-\rho)s}(\gamma s + A_0)^{-1/\gamma} ds + A_1 \right).$$

In the case of infinite horizon $T = \infty$, assuming that $\rho - r > 0$, $\gamma > 1$ and $A_0 \geq 0$ (see [10]), the constant A_1 can be given so as to satisfy the transversality condition (e.g., see Takayama [47])

$$\lim_{t \rightarrow \infty} (\pi K + \lambda_i S^i) = 0.$$

In fact, since $f_K = (1 - \gamma)f/K = (1 - \gamma)(\gamma t + A_0)^{-1}$ by (3.53), the equation (3.41a) is written here as

$$\frac{\dot{\pi}}{\pi} + \frac{1 - \gamma}{\gamma t + A_0} = -\rho,$$

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whose integration gives the appearance of π :

$$\pi = \pi_0 e^{-\rho t} (\gamma t + A_0)^{1-1/\gamma} \quad (\pi_0: \text{const.}).$$

Therefore πK is written as $(\gamma t + A_0)(\Gamma(t) + A_1)$ up to a constant multiple, where

$$\Gamma(t) = \int_0^t e^{(r-\rho)s} (\gamma s + A_0)^{-1/\gamma} ds.$$

So by putting $A_1 = -\lim_{t \rightarrow \infty} \Gamma(t)$, since $\lim_{t \rightarrow \infty} (\Gamma(t) + A_1) = 0$ as well as $\lim_{t \rightarrow \infty} (\gamma t + A_0)^{-1} = 0$, it follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} (\gamma t + A_0)(\Gamma(t) + A_1) &= \lim_{t \rightarrow \infty} \frac{\Gamma(t) + A_1}{(\gamma t + A_0)^{-1}} \\ &= \lim_{t \rightarrow \infty} \frac{\dot{\Gamma}(t)}{-\gamma (\gamma t + A_0)^{-2}} \\ &= \lim_{t \rightarrow \infty} \frac{(\gamma t + A_0)^{2-1/\gamma}}{-\gamma e^{(\rho-r)t}} = 0, \end{aligned}$$

i.e., $\lim_{t \rightarrow \infty} \pi K = 0$. Similarly for $\alpha > 1$, $\lim_{t \rightarrow \infty} t^\alpha (\Gamma(t) + A_1) = 0$, which guarantees that $\int_0^\infty (\Gamma(t) + A_1) dt$ exists, i.e., it is a finite constant. So the stocks $S^i(t)$ are given by $\Gamma_1(t) = \Gamma(t) + A_1$ as

$$S^i(t) = -c\kappa_i (\kappa_2^{\gamma_2} \cdots \kappa_n^{\gamma_n})^{-1/\gamma} \left(\int_0^t \Gamma_1(s) ds - \int_0^\infty \Gamma_1(t) dt \right),$$

which satisfy $\lim_{t \rightarrow \infty} S^i = 0$, so that $\lim_{t \rightarrow \infty} \lambda_i S^i = \lambda_i \lim_{t \rightarrow \infty} S^i = 0$ (note that $\lambda_i = \text{const.}$). It is remarkable that the situation around the terminal points ($t = 0, \infty$) follows essentially from the fact for the integration

$$\Gamma(\varepsilon, t) = \int_\varepsilon^t e^{-s} s^{-1/\gamma} ds \quad (0 < \varepsilon < t)$$

that the limits $\lim_{\varepsilon \rightarrow +0} \Gamma(\varepsilon, t)$ and $\lim_{t \rightarrow \infty} \Gamma(\varepsilon, t)$ exist whenever $-1/\gamma > -1$, i.e., $\gamma > 1$ or $\gamma < 0$ (it is well-known that $\lim_{\varepsilon \rightarrow +0, t \rightarrow \infty} \Gamma(\varepsilon, t)$ is the gamma function of γ).

3.4.3 The q -theory of investment

The third application is given with the setting of variables: the capital stock $x = K(t)$, the investment $u^1 = I(t)$ and the labor $u^2 = N(t)$. In growing of utility maximization (e.g.,

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Hayashi [21], Kataoka [25], Takayama [47]), the following integration came up in (Mimura and Nôno [34]):

$$(3.55) \quad \int_0^T e^{-\rho t} U(K, I, N) dt,$$

under the constraint

$$(3.56) \quad \dot{K} = \Phi(K, I, N),$$

where $U(K, I, N)$ and $\Phi(K, I, N)$ are assumed to be homogeneous functions of degree r and degree one with respect to K , I and N , respectively. This case lies in the first reduction in 3.3. The relating equations to (3.5a) and (3.5c)'' are written as

$$e^{-\rho t} U_K - \pi \Phi_K = \dot{\pi},$$

$$e^{-\rho t} U_I - \pi \Phi_I = 0,$$

$$e^{-\rho t} U_N - \pi \Phi_N = 0,$$

here and in what follows note that π corresponds to $-\lambda$ in [34]. These equations are added after multiplying K , I and N , respectively. Then, through the homogeneities of U and Φ , it follows that

$$(3.57) \quad \dot{\pi} K = r e^{-\rho t} U - \pi \Phi.$$

So, for the conserved quantity from (3.14):

$$(3.58) \quad \Xi_1 = (r - 1) \pi \dot{K} - (\dot{\pi} + \rho \pi) K,$$

\dot{K} of (3.56) and $\dot{\pi} K$ of (3.57) are substituted to put Ξ_1 into (cf. (21) in [34])

$$(3.58)' \quad \Xi_1 = -r e^{-\rho t} U + \pi (r \Phi - \rho K),$$

which is just the reduced quantity from (3.14)'. The equation (3.57) is written by (3.56) as

$$(3.59) \quad e^{-\rho t} U = \frac{1}{r} \cdot \frac{d(\pi K)}{dt},$$

and then, in view of the transversality condition $\lim_{t \rightarrow \infty} \pi K = 0$, integrated from t ($t \geq 0$) to infinity as

$$V(t) \equiv \int_t^\infty e^{-\rho s} U(K(s), I(s), N(s)) ds = - \frac{\pi(t) K(t)}{r}.$$

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Accordingly we have the remarkable relation

$$(3.60) \quad q_M(t) \equiv \frac{\partial V}{\partial K} = \frac{V}{K} \equiv q_A(t),$$

where the left $q_M(t) \equiv \partial V / \partial K$ and the right $q_A \equiv V / K$ correspond respectively to Tobin's marginal q and average q at t ; while, if the price of investment goods p_I appears explicitly in U , they are redefined respectively by $q_M \equiv (\partial V / \partial K) / p_I$ and $q_A \equiv (V / K) / p_I$ (e.g., [44]). Therefore, in our model, Hayashi's theorem ([21], p.218; [48], p.131) is generalized as follows.

Marginal q at time t is equal to average q at time t , provided that the cash flow U and the installation function Φ are homogeneous of degrees r ($r \neq 0$) and one, respectively.

Beside the transversality condition $\lim_{t \rightarrow \infty} \pi K = 0$, we place $\lim_{t \rightarrow \infty} \pi \dot{K} = \lim_{t \rightarrow \infty} \dot{\pi} K = 0$ so as to be $\Xi_1 = 0$ in (3.58), i.e.,

$$(3.61) \quad (r - 1)\pi \dot{K} - (\dot{\pi} + \rho\pi)K = 0,$$

which can be put into the form

$$\frac{d}{dt} \left(\frac{\pi}{K^{r-1}} \right) = -\rho \frac{\pi}{K^{r-1}}.$$

Therefore the solution of (3.61) is determined as

$$(3.62) \quad \frac{\pi}{K^{r-1}} = c_0 e^{-\rho t}, \quad \text{i.e.,} \quad \pi = c_0 e^{-\rho t} K^{r-1} \quad (c_0: \text{const.}).$$

So that, by (3.56) and $\Xi_1 = 0$ of (3.58)', i.e.,

$$(3.63) \quad rU = e^{\rho t} \pi (r\Phi - \rho K),$$

we obtain the following appearance of U on the optimal path:

$$rU = c_0 K^{r-1} (r\dot{K} - \rho K),$$

while (3.56) and (3.57) with π of (3.62) yield the same one. Moreover, in view of (3.62) and (3.63), since

$$(3.64) \quad \Theta \equiv \frac{U}{K^{r-1}(r\Phi - \rho K)}$$

is equal to

$$\Theta = \frac{e^{\rho t} \pi}{r K^{r-1}} = \frac{c_0}{r} = \text{const.},$$

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we observe that Θ is a conserved quantity. The identity $\Theta = \pi_0/(rK_0^{r-1})$ where $\pi_0 = \pi(0)$ and $K_0 = K(0)$ suggests us to define the Tobin's average q in (3.60) at $t = 0$, i.e., $q_A^0 = q_A(0) = -\pi_0/r = -K_0^{r-1}\Theta$, by

$$(3.65) \quad q_A^0 = - \left(\frac{K_0}{K} \right)^{r-1} \frac{U}{r\Phi - \rho K}.$$

Particularly in the Hayashi's model, the functions U and Φ are given respectively as

$$U = pR(K, N) - p_I I - wN,$$

$$\Phi = \Psi(K, I) - \delta K,$$

where both of $R(K, N)$ and $\Psi(K, I)$ are homogeneous of degree one with respect to the indicated variables, in which the price of investment goods p_I , the price of output p , the wage rate w and the rate of depreciation δ ($0 < \delta < 1$) are all assumed to be constants. For such U and Φ , Kataoka derived, by Noether theorem [37], the conserved quantity ((3.21) in [25]):

$$e^{-\rho t}(pR - p_I I - wN) + \lambda(\Psi - (\rho + \delta)K),$$

which is obtained immediately by substituting the particular form of U and Φ for the conserved quantity (3.58)' with $\pi = -\lambda$ and the homogeneous degree $r = 1$. His new definition of the Tobin's average q ((4.3) in [25]) is also reduced from (3.65), that is

$$q_A^0 = - \frac{pR - p_I I - wN}{\Psi - (\rho + \delta)K}.$$

More general model can be given by replacing U with

$$U = pR(K, N) - p_I S(K, I) - wT(K, N),$$

where $R(K, N)$, $S(K, I)$ and $T(K, N)$ are all assumed to be homogeneous of degree r ($r \neq 0$) with respect to the indicated variables, while $\Phi = \Psi(K, I) - \delta K$ stays on. Here we remark that $S(K, I) = I$ and $T(K, N) = N$ in (Hayashi [21]), or $S(K, I) = A(K, I)I$ (A : adjustment cost function of homogeneous degree zero) and $T(K, N) = N$ in (Takayama [48]). In this generalized model, the relation (3.63) with (3.62) on the optimal paths leads to

$$rpR(K, N) - rwT(K, N) + c_0(\rho + r\delta)K^r = rp_I S(K, I) + c_0 r K^{r-1} \Psi(K, I).$$

We place here the initial conditions $S(K, 0) = 0$ and $\Psi(K, 0) = 0$ at $I = 0$. Then from the above relation

$$(3.66) \quad rpR(K, N) - rwT(K, N) + c_0(\rho + r\delta)K^r = 0$$

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is separated by putting $I = 0$, accordingly

$$(3.67) \quad p_I S(K, I) + c_0 K^{r-1} \Psi(K, I) = 0.$$

So, by the homogeneities of R, S, T and Ψ :

$$\begin{aligned} R(K, N) &= K^r R\left(1, \frac{N}{K}\right) \equiv K^r \varphi\left(1, \frac{N}{K}\right), \\ S(K, I) &= K^r S\left(1, \frac{I}{K}\right) \equiv K^r \sigma\left(1, \frac{I}{K}\right), \\ T(K, N) &= K^r T\left(1, \frac{N}{K}\right) \equiv K^r \tau\left(1, \frac{N}{K}\right), \\ \Psi(K, I) &= K \Psi\left(1, \frac{I}{K}\right) \equiv K \psi\left(1, \frac{I}{K}\right), \end{aligned}$$

the relations (3.66) and (3.67) are rewritten respectively as

$$\begin{aligned} g(x) &\equiv r p \varphi(x) - r w \tau(x) + c_0(\rho + r \delta) = 0, \\ h(y) &\equiv p_I \sigma(y) + c_0 \psi(y) = 0, \end{aligned}$$

where $x = N/K$ and $y = I/K$. If there exist unique solutions $x = a$ (a : const.) of $g(x) = 0$ and $y = b$ (b : const.) of $h(y) = 0$, so that $N = aK$ and $I = bK$, respectively; $K = K(t)$ is determined by $\dot{K} = \Psi(K, I) - \delta K$ with $y = b$, i.e., $\Psi(K, I) = K \Psi(1, y) = K \psi(b)$:

$$\dot{K} = (\psi(b) - \delta)K.$$

Therefore the optimal path takes the form (cf. [25] for the Hayashi's model)

$$K(t) = K_0 e^{(\psi(b) - \delta)t}, \quad I(t) = b K_0 e^{(\psi(b) - \delta)t}, \quad N(t) = a K_0 e^{(\psi(b) - \delta)t}.$$

The Hayashi's model goes into details. Since $S(K, I) = I$ and $r = 1$ in the model, (3.67) is reduced to $p_I I + c_0 \Psi(K, I) = 0$ where $c_0 = \pi_0 = \pi(0)$ by (3.62), accordingly

$$(3.68) \quad h(y) \equiv p_I y + \pi_0 \psi(y) = 0.$$

Here call back the conditions on Ψ (see [21], [48]): $\partial \Psi / \partial I > 1$ for $I < 0$, $\partial \Psi / \partial I = 1$ for $I = 0$ and $\partial \Psi / \partial I < 1$ for $I > 0$, or on ψ : $\psi'(y) > 1$ for $y < 0$, $\psi'(0) = 1$ and $0 < \psi'(y) < 1$ for $y > 0$; and also $\Psi(K, 0) = 0$, or $\psi(0) = 0$. Then the equation

$$h'(y) = p_I + \pi_0 \psi'(y) = 0, \quad \text{i.e.,} \quad \psi'(y) = -\frac{p_I}{\pi_0}$$

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has a unique solution y_0 :

$$y_0 \gtrless 0 \quad \text{depending upon whether} \quad q_M^0 \gtrless 1,$$

where q_M^0 is the Tobin's marginal q at $t = 0$, i.e., $q_M^0 \equiv -\pi_0/p_I$ which may be assumed to be $q_M^0 > 0$, so that $p_I > 0$ implies $\pi_0 < 0$. And, for the solution, it follows that

$$h'(y) \gtrless 0 \quad \text{for} \quad y \gtrless y_0.$$

So that only solutions of (3.68) are $y = 0$ and $y = b$ (b : const.), where $\psi(b) = -bp_I/\pi_0 = b/q_M^0$ and b stands to y_0 in the relation

$$b \gtrless y_0 \gtrless 0 \quad \text{depending upon whether} \quad q_M^0 \gtrless 1.$$

Therefore, for $y > y_0$ in the case $q_M^0 > 1$, or $y < y_0$ in the case $q_M^0 < 1$, the optimal investment is determined as

$$I(t) = bK_0 e^{(b/q_M^0 - \delta)t},$$

while $I(t) = 0$ if $q_M^0 = 1$. In conclusion we observe that (cf. [48], p.136)

$$I(t) \gtrless 0 \quad \text{depending upon whether} \quad q_M^0 \gtrless 1.$$

Here we may assume that $K_0 > 0$ together with $q_M^0 > 0$. Then, since $b/q_M^0 - \delta \gtrless 0$ if and only if $q_M^0 \lesseqgtr b/\delta \equiv \Delta$ where $\Delta \gtrless 0$ whenever $q_M^0 \gtrless 1$, it follows that

$$\begin{cases} \dot{I}(t) < 0, \ddot{I}(t) > 0 & \text{if } \Delta < 1 < q_M^0, \\ \dot{I}(t) > 0, \ddot{I}(t) < 0 & \text{if } \Delta < q_M^0 < 1, \end{cases} \quad \text{when } \Delta < 1;$$

$$\begin{cases} \dot{I}(t) > 0, \ddot{I}(t) > 0 & \text{if } 1 < q_M^0 < \Delta, \\ \dot{I}(t) < 0, \ddot{I}(t) > 0 & \text{if } 1 < \Delta < q_M^0, \end{cases} \quad \text{when } \Delta > 1.$$

Here note, by $\psi(b) = b/q_M^0$, that

$$q_M^0 \gtrless \Delta \quad \text{if and only if} \quad \psi(b) \lesseqgtr \delta.$$

For the comparison of the rate of depreciation δ with $\psi(b)$, remark the fact that a straight line $z = y/q_M^0$ intersects a curve $z = \psi(y)$ at a point $(b, \psi(b))$ together with the origin.

Thus, in the Hayashi's model, such a belief of Tobin and Brainard in the q -theory of investment ([49], p.238) is available with the following statement ([34], p.20; in which *down/up to 0* should be read *down/up to a constant*).

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When $\Delta \equiv b/\delta < 1$: q_M^0 above 1 discourages decreasingly the investment as far as it goes down to 0, and q_M^0 below 1 stimulates decreasingly the investment as far as it goes up to 0. When $\Delta > 1$: q_M^0 below Δ stimulates increasingly the investment without limit, and q_M^0 above Δ discourages decreasingly the investment, where the rate of investment goes down/up to a constant according as q_M^0 goes up/down to Δ . Consequently an equilibrium value of q_M^0 is $q_M^0 = 1$ when $\Delta < 1$, and $q_M^0 = \Delta$ when $\Delta > 1$. In this statement, the usual belief of Tobin and Brainard can be regarded as a limiting case of $\delta \downarrow 0$, i.e., $\Delta \uparrow \infty$ ($b > 0$) or $\Delta \downarrow -\infty$ ($b < 0$), while the discouragement of q_M^0 below 1 in the belief must be replaced here with the decreasing stimulation. In the limiting case, the rate of investment goes down/up to 0 according as q_M^0 goes up/down to 1.

4 One-sector and external two-sector growth models

4.1 Introduction

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. Sato [44] first pioneered the way of applying the theorem to optimal economic growths. However, to the general neoclassical growth models with positive constant discount rate, he gave a negative answer for the existence of global conservation laws, while he deduced some local ones [44, Chap. 7, III, IV].

In contrast with the Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [32]) and applied it to various economic growth models (Mimura and Nôno [34]; Mimura, Fujiwara and Nôno [29], [30]; Fujiwara, Mimura and Nôno [11]). Particularly in [11], the procedure was so reformed as to make an effective application to more general neoclassical optimal growth models. In this section, the application in the models will be pursued by a reduction of [11, Theorem 1]. In 4.2, the reduction is made, and then in 4.3, it is first applied to a one sector model of Ramsey type (Ramsey [38]) with a constant discount rate relative to a utility (welfare) of consumption. Whenever the consumption grows under an arbitrary linear technology, we can find new conservation law in the model. This new law is a generalization of the constancy of *per capita consumption*, which is an only conservation law derivable from the application of the Noether theorem if and only if the linear technology is a special linear type [44, Chap. 7, Theorem 1]. Through the new conservation law, the optimal paths in the growth are determined and then detailed for a utility of second order polynomial or the logarithm. Moreover in 4.4, the model is generalized in an external two sector version with linear technologies. The growth process relative to the technologies are characterized by a matrix of second order. By the reduced theorem, we can find three types of conservation laws according as the discriminant of the characteristic equation of the matrix is positive, zero or negative, and then establish a way of determining the optimal paths in the external growths. Through the way, the paths are determined completely under growth process characterized by an arbitrary given matrix of second order, while utility is assumed to be of second order polynomial of consumptions.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

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4.2 New derivation of conservation laws

Further application of [32] for the derivation of new conservation laws in economic growth models will begin with a reduction of ([11, Theorem 1], in which the variables u^σ are replaced here with c^μ). So, first discuss an extremal (maximizing or minimizing) problem for the integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(4.1) \quad \int_0^T e^{-\rho t} U(c^1, \dots, c^s) dt,$$

under constraints

$$(4.2) \quad \dot{x}^\mu = f^\mu(x^1, \dots, x^s, c^1, \dots, c^s),$$

where $x^\mu = x^\mu(t)$, $c^\mu = c^\mu(t)$ ($\mu = 1, \dots, s$) and ρ ($\rho \geq 0$) is a constant. In the multiplier technique to the problem, the Lagrangian is given by (π_μ are the multipliers):

$$(4.3) \quad L = e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - f^\mu),$$

whose Euler-Lagrange equations consist of (4.2) and

$$(4.4a) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0 : \quad \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu = 0,$$

$$(4.4b) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{c}^\nu} \right) - \frac{\partial L}{\partial c^\nu} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial c^\mu} - \frac{\partial f^\nu}{\partial c^\mu} \pi_\nu = 0.$$

A conserved quantity (first integral) for the extremal problem is a quantity Ω of the variables $\dot{\pi}^\mu$, \dot{x}^μ , \dot{c}^μ , π^μ , x^μ , c^μ and t whose total time derivative vanishes ($\dot{\Omega} = 0$: conservation law) on the optimal path, i.e., on solutions to the relating Euler-Lagrange equations (4.2), (4.4a) and (4.4b). For a derivation of conserved quantities, the theorem 2.1 in 2.2 is reformulated as follows (cf. the theorems 1 in [11], [29] and [34] (also [30])); which were applied to, e.g., more general neoclassical growth models, the model of von Neumann type and the models in Tobin's q -theory of investment, and the model in the intergenerational problem, respectively).

Theorem 4.1. *For the Lagrangian L of (4.3), let the triples of functions $(\eta_\mu^1, \varphi_1^\mu, \tau_1^\mu)$ and $(\eta_\mu^2, \varphi_2^\mu, \tau_2^\mu)$ of the variables $\dot{\pi}_\mu$, \dot{x}^μ , \dot{c}^μ , π_μ , x^μ , c^μ and t satisfy the equations*

$$(4.5a) \quad \dot{\varphi}^\mu - \left(\frac{\partial f^\mu}{\partial x^\nu} \varphi^\nu + \frac{\partial f^\mu}{\partial c^\nu} \tau^\nu \right) = 0,$$

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$$(4.5b) \quad \dot{\eta}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \eta_\nu + \pi_\nu \left(\frac{\partial^2 f^\nu}{\partial x^\kappa \partial x^\mu} \varphi^\kappa + \frac{\partial^2 f^\nu}{\partial c^\kappa \partial x^\mu} \tau^\kappa \right) = 0,$$

$$(4.5c) \quad \frac{\partial f^\nu}{\partial c^\mu} \eta_\nu + \pi_\nu \left(\frac{\partial^2 f^\nu}{\partial x^\kappa \partial c^\mu} \varphi^\kappa + \frac{\partial^2 f^\nu}{\partial c^\kappa \partial c^\mu} \tau^\kappa \right) = e^{-\rho t} \frac{\partial^2 U}{\partial c^\nu \partial c^\mu} \tau^\nu,$$

on the optimal path for the extremal problem of (4.1) under the constraints (4.2). Then the following conserved quantity Ω is constructed:

$$(4.6) \quad \Omega = \eta_\mu^2 \varphi_1^\mu - \eta_\mu^1 \varphi_2^\mu.$$

In the reduced theorem 1 of [11, Theorem 1] (see also [29], [30]), it follows the following solution satisfying (4.5a), (4.5b) and (4.5c):

$$(4.7) \quad (\eta_\mu, \varphi^\mu, \tau^\mu) = (\dot{\pi}_\mu + \rho \pi_\mu, \dot{x}^\mu, \dot{c}^\mu).$$

In fact, the total time derivative of (4.2):

$$\dot{x}^\mu = \frac{\partial f^\mu}{\partial x^\nu} \dot{x}^\nu + \frac{\partial f^\mu}{\partial c^\nu} \dot{c}^\nu$$

serves the verification of (4.5a). For the verification of (4.5b), a multiple of (4.4a) by ρ and the total time derivative of (4.4a) are added to see

$$\ddot{\pi}_\mu + \rho \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} (\dot{\pi}_\nu + \rho \pi_\nu) + \pi_\nu \left(\frac{\partial^2 f^\nu}{\partial x^\kappa \partial x^\mu} \dot{x}^\kappa + \frac{\partial^2 f^\nu}{\partial c^\kappa \partial x^\mu} \dot{c}^\kappa \right) = 0;$$

and for (4.5c), in the total time derivative of (4.4b), the terms $e^{-\rho t} \partial U / \partial c^\mu$ are replaced with $(\partial f^\nu / \partial c^\mu) \pi_\nu$ to see

$$\frac{\partial f^\nu}{\partial c^\mu} (\dot{\pi}_\nu + \rho \pi_\nu) + \pi_\nu \left(\frac{\partial^2 f^\nu}{\partial x^\kappa \partial c^\mu} \dot{x}^\kappa + \frac{\partial^2 f^\nu}{\partial c^\kappa \partial c^\mu} \dot{c}^\kappa \right) = e^{-\rho t} \frac{\partial^2 U}{\partial c^\nu \partial c^\mu} \dot{c}^\nu.$$

Here let the functions f^μ be first order polynomials of the variables x^μ and c^μ . Then the equations (4.5a), (4.5b) and (4.5c) are reduced respectively to

$$\dot{\varphi}^\mu = \frac{\partial f^\mu}{\partial x^\nu} \varphi^\nu + \frac{\partial f^\mu}{\partial c^\nu} \tau^\nu,$$

$$\dot{\eta}_\mu = -\frac{\partial f^\nu}{\partial x^\mu} \eta_\nu,$$

$$\frac{\partial f^\nu}{\partial c^\mu} \eta_\nu = e^{-\rho t} \frac{\partial^2 U}{\partial c^\nu \partial c^\mu} \tau^\nu;$$

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and, if $\eta_\mu = \tau^\mu = 0$, moreover to

$$(4.5a)' \quad \dot{\varphi}^\mu = \frac{\partial f^\mu}{\partial x^\nu} \varphi^\nu.$$

Together with $\eta_\mu = \tau^\mu = 0$ and φ^μ satisfying the above equations, the solution (4.7) is substituted for (4.6) to construct the following conserved quantity.

Theorem 4.2. *For the extremal problem of (4.1) under the constraints (4.2), let f^μ be first order polynomials of x^μ and c^μ . Then, by the solution φ^μ of (4.5a)', the following conserved quantity Ω is constructed:*

$$(4.8) \quad \Omega = (\dot{\pi}_\mu + \rho \pi_\mu) \varphi^\mu.$$

For the first order polynomials f^μ , since $\partial f^\mu / \partial x^\nu$ are constants, the solution of (4.5a)' can be determined through the eigenvalues of the constant matrix $(\partial f^\mu / \partial x^\nu)$ (see, e.g., [2]). Particularly for a one sector growth model in 4.3 and in an external two sector growth model in 4.4, the solutions are detailed and then the conserved quantities are also.

4.3 One sector growth model of Ramsey type

Generalizing Ramsey's original growth model [38], Sato [44, Chap.7, III] established the application of the Noether theorem [37] to the model characterized by

$$c = g(x) - nx - \dot{x} \quad (n: \text{const.}; n \geq 0),$$

where c is the (per capita) consumption, x the capital-labour ratio, \dot{x} the rate of capital accumulation, $g(x)$ the production function, and the constant n is the sum of the population growth rate and the depreciation rate. Our problem is to find new conservation laws in the objective of society to maximize the following integration (the social welfare functional) over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(4.1)' \quad \int_0^T e^{-\rho t} U(c) dt,$$

under a constraint (growth process)

$$(4.2)' \quad \dot{x} = g(x) - nx - c,$$

where U is a utility (welfare) function satisfying the concavity $U' > 0$ and $U'' < 0$, and ρ ($\rho \geq 0$) is a constant discount rate.

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In this case the equations (4.4a) and (4.4b) are reduced respectively to

$$(4.4a)' \quad \dot{\pi} + (g' - n)\pi = 0,$$

$$(4.4b)' \quad \pi + e^{-\rho t}U' = 0.$$

The above equation (4.4a)' and a multiple of (4.4b)' by ρ are added to see

$$(4.9) \quad \dot{\pi} + \rho\pi = e^{-\rho t}(g' - \rho - n)U',$$

which is combined with the total time derivative of (4.4b)': $\dot{\pi} + \rho\pi = -e^{-\rho t}\dot{U}'$ to derive $\dot{U}' = (\rho + n - g')U'$, i.e.,

$$(4.10) \quad \frac{\dot{U}'}{U'} + g' = \rho + n,$$

or equivalently $(U''/U')\dot{c} = -g' + \rho + n$. So by $U''/U' < 0$ that

$$\dot{c} \gtrless 0 \quad \text{depending upon whether} \quad g' \gtrless \rho + n,$$

which concludes that

Theorem 4.3. *In the maximizing problem of (4.1)', the consumption c is increasing (or decreasing) if and only if the rate of return on capital g' is more (or less) than the sum of the discount rate, the population growth rate and the depreciation rate $\rho + n$.*

In what follows, let the production technology be linear, i.e., $g(x) = \alpha x + \beta$ (α, β : const.). Then the reduced equation of (4.5a)': $\dot{\varphi} = (\alpha - n)\varphi$ is integrated as $\varphi = e^{(\alpha - n)t}$ up to a multiple of constant, which is used together with (4.9) to construct the conserved quantity in the theorem 4.2 (see the remark 1 below).

Theorem 4.4. *In the maximizing problem of (4.1)', let the consumption $c = g(x) - nx - \dot{x}$ grow under linear production technology $g(x) = \alpha x + \beta$. Then there exists the following conserved quantity Ω :*

$$(4.11) \quad \Omega = e^{(\alpha - \rho - n)t}U'.$$

Remark 4.1. The resulting quantity is divided by $\alpha - \rho - n$ ($\neq 0$) to have the final appearance of (4.11), while it becomes identically zero if $\alpha = \rho + n$. However, if $\alpha = \rho + n$, (4.10) is reduced to $\dot{U}' = U''\dot{c} = 0$ so that $c = \text{const.}$ (the constancy of per capita consumption [44], which was derived by the Noether theorem). Thus the constancy of U' (i.e., of c) in

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the special case with $\alpha = \rho + n$ is included in the constancy of $e^{(\alpha-\rho-n)t}U'$, and also in the constancy of (4.10): $\dot{U}'/U' + g' (= \rho + n)$.

Optimal paths in the considering growth can be determined through the conservation law $e^{(\alpha-\rho-n)t}U' = \text{const.}$ from (4.11). The law is written as $U' = ae^{(n-\alpha+\rho)t}$ (a : const.; $a > 0$ by $U' > 0$), in which note the concavity of U , i.e., $U' > 0$ and $U'' < 0$. Since U' is a monotone decreasing function ($U'' < 0$) of c , the consumption c in the law is implicitly of the form

$$(4.12) \quad c = F(ae^{(n-\alpha+\rho)t}),$$

which is substituted for the growth process $\dot{x} = (\alpha - n)x - c + \beta$ from (4.2)'. Then, if $\alpha = n$, it follows immediately the optimal path $x(t)$ in the growth:

$$(4.13) \quad x(t) = -\int F(ae^{\rho t})dt + \beta t + b \quad (a, b: \text{const.}; a > 0).$$

And if $\alpha \neq n$, the solution $x = Ge^{(\alpha-n)t}$ (G : const.) of $\dot{x} = (\alpha - n)x$ is also substituted, after replacing the constant G with arbitrary function $G(t)$, for the growth process. Consequently

$$\dot{G} = -e^{(n-\alpha)t}F(ae^{(n-\alpha+\rho)t}) + \beta e^{(n-\alpha)t},$$

whose integration completes the following optimal path $x(t) = G(t)e^{(\alpha-n)t}$ in the growth:

$$(4.14) \quad x(t) = -e^{(\alpha-n)t} \int e^{(n-\alpha)t} F(ae^{(n-\alpha+\rho)t}) dt + be^{(\alpha-n)t} + \frac{\beta}{n-\alpha} \quad (a, b: \text{const.}; a > 0).$$

Theorem 4.5. *In the maximizing problem of (4.1)', let the consumption $c = g(x) - nx - \dot{x}$ grow under linear production technology $g(x) = \alpha x + \beta$. Then, depending upon whether the marginal product of capital α is equal to n or not, the respective optimal path is determined as (4.13) or (4.14).*

Relative to the following utilities, the optimal path $x(t)$ of (4.13) or (4.14) can be detailed, and then it is substituted for the growth process $\dot{x} = (\alpha - n)x - c + \beta$ to establish the consumption path $c(t)$. In the case of infinite horizon $T = \infty$, the feasibility of the path $x(t)$ necessitates the transversality condition $\lim_{t \rightarrow \infty} \pi x = 0$ (e.g., see [47]). Since $\Omega > 0$ in (4.11) by $U' > 0$, and since $\pi = -\Omega e^{(n-\alpha)t}$ by (4.4b)' and (4.11), the transversality condition leads to

$$(4.15) \quad \lim_{t \rightarrow \infty} e^{(n-\alpha)t}x = 0.$$

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Utility of second order polynomial. Let the utility function $U(c)$ be second order polynomial of c :

$$U(c) = kc^2 + \ell c + m \quad (k < 0, \ell > 0, m \geq 0),$$

which satisfies $U(0) = m \geq 0$, $U'' = 2k < 0$ and $U' = 2kc + \ell \geq 0$ ($c \leq -\frac{1}{2}\ell/k$). In this case, since $U' = 2kc + \ell = ae^{(n-\alpha+\rho)t}$, (4.12) is reduced to $c = \frac{1}{2}(ae^{(n-\alpha+\rho)t} - \ell)/k$. So, it follows that

$$(4.16) \quad e^{(n-\alpha)t}F = \frac{a}{2k}e^{(2n-2\alpha+\rho)t} - \frac{\ell}{2k}e^{(n-\alpha)t}.$$

Therefore, if $\alpha = n$, since $F = \frac{1}{2}(ae^{\rho t} - \ell)/k$ in (4.13), the optimal path $x(t)$ is determined and then, in accordance with the growth process, $c(t)$ is also as

$$\begin{cases} x_{11}(t) = -\frac{a}{2k\rho}e^{\rho t} + \frac{2k\beta + \ell}{2k}t + b, \\ c_{11}(t) = \frac{a}{2k}e^{\rho t} - \frac{\ell}{2k}. \end{cases}$$

If $2(\alpha - n) = \rho$ (> 0), i.e., $\alpha - \rho - n = n - \alpha = -\frac{1}{2}\rho$, since $e^{(n-\alpha)t}F = \frac{1}{2}(a - \ell e^{-\frac{\rho}{2}t})/k$ in (4.14), the optimal paths $x(t)$ and $c(t)$ lead respectively to

$$\begin{cases} x_{12}(t) = -\frac{a}{2k}te^{\frac{\rho}{2}t} + be^{\frac{\rho}{2}t} - \frac{2k\beta + \ell}{k\rho}, \\ c_{12}(t) = \frac{a}{2k}e^{\frac{\rho}{2}t} - \frac{\ell}{2k}. \end{cases}$$

If $2(\alpha - n) \neq \rho$ and $\sigma \equiv n - \alpha \neq 0$, in view of (4.14) with the integrating function (4.16), the optimal paths $x(t)$ and $c(t)$ are written respectively as

$$\begin{cases} x_{13}(t) = -\frac{a}{2k(2\sigma + \rho)}e^{(\sigma+\rho)t} + be^{-\sigma t} + \frac{2k\beta + \ell}{2k\sigma}, \\ c_{13}(t) = \frac{a}{2k}e^{(\sigma+\rho)t} - \frac{\ell}{2k}. \end{cases}$$

Remark 4.2. According to the condition $U' > 0$, the terminal point T of the integration (4.1)' must be placed so as to satisfy $c(T) < -\frac{1}{2}\ell/k$. Therefore the relating integration relative to the considering utility lies in the case of finite horizon.

Utility of logarithm. Let the utility function $U(c)$ be the logarithm:

$$U(c) = k \log(c + \ell) \quad (k, \ell: \text{const.}; k > 0, \ell \geq 1),$$

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which satisfies $U(0) = k \log \ell \geq 0$ and the concavity $U' > 0$, $U'' < 0$. In this case, since $U' = k/(c + \ell)$, (4.11) is reduced to $c = ke^{(\alpha - \rho - n)t}/a - \ell = F$. So, it follows that

$$(4.17) \quad e^{(n-\alpha)t} F = \frac{k}{a} e^{-\rho t} - \ell e^{(n-\alpha)t}.$$

Therefore, if $\alpha = n$, since $F = ke^{-\rho t}/a - \ell$ in (4.13), the optimal paths $x(t)$ and $c(t)$ lead respectively to

$$\begin{cases} x_{21}(t) = \frac{k}{a\rho} e^{-\rho t} + (\beta + \ell)t + b, \\ c_{21}(t) = \frac{k}{a} e^{-\rho t} - \ell; \end{cases}$$

and if $\sigma \equiv n - \alpha \neq 0$, in view of (4.14) with the integrating function (4.17), the optimal paths $x(t)$ and $c(t)$ are written respectively as

$$\begin{cases} x_{22}(t) = \frac{k}{a\rho} e^{-(\sigma+\rho)t} + be^{-\sigma t} + \frac{\beta + \ell}{\sigma}, \\ c_{22}(t) = \frac{k}{a} e^{-(\sigma+\rho)t} - \ell. \end{cases}$$

In the case of infinite horizon, the condition (4.15) for the feasibility of the optimal path requires in x_{21} : $b = \beta + \ell = 0$; and in x_{22} : $b = 0$, and $\sigma < 0$ (i.e., $\alpha > n$) if $\beta + \ell \neq 0$.

Remark 4.3. The property in the theorem 4.3 can be illustrated by the above determined consumptions c_{ij} . In fact, the consumption paths c_{11} , c_{12} , c_{21} with $\dot{c}_{11} < 0$, $\dot{c}_{12} < 0$, $\dot{c}_{21} < 0$ are all lie in the case where $\alpha < \rho + n$; and the consumption paths c_{13} and c_{22} satisfy respectively $\dot{c}_{13} \geq 0$ and $\dot{c}_{22} \geq 0$ depending upon whether $\sigma + \rho \leq 0$, i.e., $\alpha \geq \rho + n$.

4.4 An external two-sector growth model

In a generalization of the one sector growth model of Ramsey type, we discuss the objective of society to maximize the following integration (the social welfare functional) over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(4.1)'' \quad \int_0^T e^{-\rho t} U(c^1, c^2) dt,$$

under constraints (external growth process with respect to the consumption c^μ and the capital-labour ratio x^μ in μ -th ($\mu = 1, 2$) sector):

$$\dot{x}^\mu = g^\mu(x^1, x^2) - n_\nu^\mu x^\nu - c^\mu \quad (n_\nu^\mu: \text{const.}, n_\nu^\mu > 0; \mu, \nu = 1, 2),$$

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where U is a utility (welfare) function satisfying the concavity: $\partial^2 U / \partial c^1 \partial c^1 < 0$ and

$$(4.18) \quad \det \left(\frac{\partial^2 U}{\partial c^\mu \partial c^\nu} \right) = \frac{\partial^2 U}{\partial c^1 \partial c^1} \cdot \frac{\partial^2 U}{\partial c^2 \partial c^2} - \left(\frac{\partial^2 U}{\partial c^1 \partial c^2} \right)^2 > 0,$$

and g^μ are assumed to be linear production technologies

$$g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu \quad (\alpha_\nu^\mu, \beta^\mu: \text{const.}).$$

So that the growth process are written as

$$(4.2)'' \quad \dot{x}^\mu = \alpha_\nu^\mu x^\nu - c^\mu + \beta^\mu \quad (\alpha_\nu^\mu = \alpha_\nu^\mu - n_\nu^\mu: \text{const.}).$$

For the two sector version, (4.5a)' is reduced to a system of first order linear homogeneous differential equations $\dot{\varphi}^\mu = a_\nu^\mu \varphi^\nu$, which are written in the matrix notation $\dot{\varphi} = A\varphi$ where $A = (a_\nu^\mu)$ and $\varphi = {}^t(\varphi^1 \ \varphi^2)$ (t denote the transposition). The solutions can be determined according as the discriminant $D = (a_1^1 - a_2^2)^2 + 4a_2^1 a_1^2$ of the characteristic equation of A is positive, zero or negative, respectively (see, e.g., [2]). Together with the characteristic values of A , we set the vectors \mathbf{p}_1 and \mathbf{p}_2 , as follows.

(i) $D > 0$: The characteristic vectors \mathbf{p}_1 and \mathbf{p}_2 with distinct real characteristic values λ_1 and λ_2 of A , respectively.

(ii) $D = 0$: Assuming that A is not a constant multiple of the identity E , the characteristic vector \mathbf{p}_1 with coincide real characteristic values λ of A and a vector \mathbf{p}_2 satisfying $A\mathbf{p}_2 = \mathbf{p}_1 + \lambda\mathbf{p}_2$, i.e., $(A - \lambda E)\mathbf{p}_2 = \mathbf{p}_1$. In fact such a vector \mathbf{p}_2 exists; since it can be verified that all determinants of the submatrices of second order in $(A - \lambda E \ \mathbf{p}_1)$ vanish (note that $a_1^1 - \lambda = \lambda - a_2^2$ for the verification), so that $\text{rank}(A - \lambda E) = \text{rank}(A - \lambda E \ \mathbf{p}_1) = 1$.

(iii) $D < 0$: The characteristic vector $\mathbf{p}_1 + i\mathbf{p}_2$ ($\mathbf{p}_1, \mathbf{p}_2$: real) with the complex characteristic value $\lambda + i\theta$ (λ, θ : real, $\theta \neq 0$) of A .

It is easy to show that the vectors \mathbf{p}_1 and \mathbf{p}_2 are linearly independent. Therefore the matrix $P = (\mathbf{p}_1 \ \mathbf{p}_2)$ is nonsingular. In the above setting, the following independent solutions φ_1 and φ_2 of $\dot{\varphi} = A\varphi$, i.e., $\Phi = (\varphi_1 \ \varphi_2)$, are provided:

$$\Phi = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \quad \text{for } D > 0,$$

$$\Phi = e^{\lambda t} P \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{for } D = 0,$$

$$\Phi = e^{\lambda t} P \begin{pmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{pmatrix} \quad \text{for } D < 0.$$

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The solutions are used to construct the conserved quantities Ω_1 and Ω_2 in the theorem 4.2:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = {}^t\Phi \begin{pmatrix} \dot{\pi}_1 + \rho\pi_1 \\ \dot{\pi}_2 + \rho\pi_2 \end{pmatrix};$$

for which, the relations from (4.4a) and (4.4b): $\dot{\pi}_\mu = -a_\mu^\nu \pi_\nu$ and $\pi_\mu = -e^{-\rho t} U_\mu$ ($U_\mu = \partial U / \partial c^\mu$) are arranged:

$$(4.19) \quad \begin{pmatrix} \dot{\pi}_1 + \rho\pi_1 \\ \dot{\pi}_2 + \rho\pi_2 \end{pmatrix} = e^{-\rho t} ({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

and then substituted to see

$$(4.20) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = e^{-\rho t} {}^t\Phi ({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

Moreover, in ${}^t\Phi ({}^tA - \rho E)$ with above Φ , the transpositions of the following relations are used:

$$(4.21a) \quad (A - \rho E)P = P \begin{pmatrix} \lambda_1 - \rho & 0 \\ 0 & \lambda_2 - \rho \end{pmatrix} \quad \text{for } D > 0,$$

$$(4.21b) \quad (A - \rho E)P = P \begin{pmatrix} \lambda - \rho & 1 \\ 0 & \lambda - \rho \end{pmatrix} \quad \text{for } D = 0,$$

$$(4.21c) \quad (A - \rho E)P = P \begin{pmatrix} \lambda - \rho & \theta \\ -\theta & \lambda - \rho \end{pmatrix} \quad \text{for } D < 0.$$

Consequently the conserved quantities Ω_1 and Ω_2 have the following appearances

$$(4.22a) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} & 0 \\ 0 & e^{(\lambda_2 - \rho)t} \end{pmatrix} \begin{pmatrix} \lambda_1 - \rho & 0 \\ 0 & \lambda_2 - \rho \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D > 0,$$

$$(4.22b) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = e^{(\lambda - \rho)t} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda - \rho & 0 \\ 1 & \lambda - \rho \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D = 0,$$

$$(4.22c) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = e^{(\lambda - \rho)t} \begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} \lambda - \rho & -\theta \\ \theta & \lambda - \rho \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D < 0.$$

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Particularly let $\lambda_\mu \neq \rho$ in (4.22a) and $\lambda \neq \rho$ in (4.22b), while $\theta \neq 0$ is guaranteed in (4.22c). Then, since the product of the diagonal matrices in (4.22a) are commutative, and since the products in (4.22b) and (4.22c) are written respectively as

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \lambda - \rho & 0 \\ 1 & \lambda - \rho \end{pmatrix} = \begin{pmatrix} 0 & \lambda - \rho \\ \lambda - \rho & 1 \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} \lambda - \rho & -\theta \\ \theta & \lambda - \rho \end{pmatrix} = \begin{pmatrix} -\theta & \lambda - \rho \\ \lambda - \rho & \theta \end{pmatrix} \begin{pmatrix} \sin \theta t & \cos \theta t \\ \cos \theta t & -\sin \theta t \end{pmatrix},$$

the conserved quantities Ω_1 and Ω_2 in (4.22a), (4.22b) and (4.22c) may be put respectively as

$$(4.23a) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} & 0 \\ 0 & e^{(\lambda_2 - \rho)t} \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D > 0,$$

$$(4.23b) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = e^{(\lambda - \rho)t} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D = 0,$$

$$(4.23c) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = e^{(\lambda - \rho)t} \begin{pmatrix} \sin \theta t & \cos \theta t \\ \cos \theta t & -\sin \theta t \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D < 0.$$

Since the relations from (4.4b): $\pi_\mu = -e^{-\rho t} U_\mu$ and their total time derivatives yield

$$\begin{pmatrix} \dot{\pi}_1 + \rho \pi_1 \\ \dot{\pi}_2 + \rho \pi_2 \end{pmatrix} = -e^{-\rho t} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix},$$

the relation (4.19) leads to

$$(4.24) \quad \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = -({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix};$$

which is multiplied by ${}^tP = {}^t(\mathbf{p}_1 \ \mathbf{p}_2)$, and then the transpositions of (4.21a) and (4.21b) are used respectively to see

$$\begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = - \begin{pmatrix} \lambda_1 - \rho & 0 \\ 0 & \lambda_2 - \rho \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D > 0,$$

$$\begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = - \begin{pmatrix} \lambda - \rho & 0 \\ 1 & \lambda - \rho \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \text{for } D = 0.$$

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Therefore it follows that

$${}^t\mathbf{p}_1 \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = 0 \text{ if } \lambda_1 = \rho, \quad {}^t\mathbf{p}_2 \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = 0 \text{ if } \lambda_2 = \rho, \text{ for } D > 0;$$

$${}^t\mathbf{p}_1 \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = 0, \quad {}^t\mathbf{p}_2 \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = -{}^t\mathbf{p}_1 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \text{ if } \lambda = \rho, \text{ for } D = 0;$$

accordingly

$${}^t\mathbf{p}_1 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \text{const.} \text{ if } \lambda_1 = \rho, \quad {}^t\mathbf{p}_2 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \text{const.} \text{ if } \lambda_2 = \rho, \text{ for } D > 0;$$

$${}^t\mathbf{p}_1 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \text{const.}, \quad ({}^t\mathbf{p}_1 + {}^t\mathbf{p}_2) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \text{const.} \text{ if } \lambda = \rho, \text{ for } D = 0;$$

which may be included respectively in (4.23a) and (4.23b) to delete the conditions $\lambda_\mu \neq \rho$ of (4.23a) and $\lambda \neq \rho$ of (4.23b). Thus the conserved quantities are summarized as follows.

Theorem 4.6. *In the maximizing problem of (4.1)'', let the consumptions $c^\mu = g^\mu(x^1, x^2) - n_\nu^\mu x^\nu - \dot{x}^\nu$ grow externally under the linear production technologies $g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu$. Then, in the setting of (i), (ii) and (iii) according as the discriminant D of the characteristic equation of the matrix $A = (\alpha_\nu^\mu - n_\nu^\mu)$ is positive, zero and negative, there exist the respective conserved quantities (4.23a), (4.23b) and (4.23c), in which $U_1 = \partial U / \partial c^1$ and $U_2 = \partial U / \partial c^2$; while in (4.23b), the matrix A is assumed not to be a constant multiple of E .*

Now, by putting $\mathbf{p}_\mu = (p_1^\mu, p_2^\mu)$ and

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} p_1^1 U_1 + p_2^1 U_2 \\ p_1^2 U_1 + p_2^2 U_2 \end{pmatrix},$$

the conserved quantities Ξ_1 and Ξ_2 have the following appearances, respectively:

$$\begin{cases} \Xi_1 = e^{(\lambda_1 - \rho)t} \xi_1 \\ \Xi_2 = e^{(\lambda_2 - \rho)t} \xi_2 \end{cases} \quad \text{for } D > 0,$$

$$\begin{cases} \Xi_1 = e^{(\lambda - \rho)t} (t \xi_1 + \xi_2) \\ \Xi_2 = e^{(\lambda - \rho)t} \xi_1 \end{cases} \quad \text{for } D = 0,$$

$$\begin{cases} \Xi_1 = e^{(\lambda - \rho)t} (\xi_1 \sin \theta t + \xi_2 \cos \theta t) \\ \Xi_2 = e^{(\lambda - \rho)t} (\xi_1 \cos \theta t - \xi_2 \sin \theta t) \end{cases} \quad \text{for } D < 0;$$

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which yield, by $\Xi_3 = \Xi_1/\Xi_2$, the another ones: $\Xi_3 = e^{(\lambda_1 - \lambda_2)t} \xi_1/\xi_2$ for $D > 0$, $\Xi_3 = \xi_2/\xi_1 + t$ for $D = 0$ and $\Xi_3 = (\xi_1 \sin \theta t + \xi_2 \cos \theta t)/(\xi_1 \cos \theta t - \xi_2 \sin \theta t)$ for $D < 0$; and in addition, $\Xi_4 = (\Xi_1)^2 + (\Xi_2)^2 = e^{2(\lambda - \rho)t}((\xi_1)^2 + (\xi_2)^2)$ for $D < 0$.

Remark 4.4. Assume that the utility function $U(c^1, c^2)$ is homogeneous of degree r and $\beta^\mu = 0$ in the linear technology, i.e., $g^\mu = \alpha_\nu^\mu x^\nu$, so as to recall the conserved quantity of ([11, Eq. (10)], also [29, Eq. (14)] or [30, Eq. (14)]):

$$\Xi = -(r-1)\pi_\mu \dot{x}^\mu + (\dot{\pi}_\mu + \rho\pi_\mu)x^\mu.$$

Since $\pi_\mu = -e^{-\rho t}U_\mu$ from (4.4b) and $\dot{x}^\mu = \alpha_\nu^\mu x^\nu - c^\mu$ from (4.2)'', the quantity is written by (4.19) as

$$\Xi = e^{-\rho t}(r^t \dot{x} - \rho^t x + {}^t c) \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

in which the vector $r^t \dot{x} - \rho^t x + {}^t c$ is identical to $r^t x^t A - \rho^t x - (r-1)^t c$, where $x = {}^t(x^1 \ x^2)$ and $c = {}^t(c^1 \ c^2)$.

Remark 4.5. If the growth process (4.2)'' have no externality, the matrix $A = (a_\nu^\mu)$ becomes the diagonal form, i.e., $a_1^1 = \lambda_1$, $a_2^2 = \lambda_2$ and $a_1^2 = a_2^1 = 0$. Such a matrix A includes the exceptional case of $A = \lambda E$ (λ : const.) in (ii). For the matrix A , if $\lambda_1 \neq \lambda_2$, the characteristic vectors p_μ with the characteristic values λ_μ may be put as $p_1 = {}^t(1 \ 0)$ and $p_2 = {}^t(0 \ 1)$, so that $(p_1 \ p_2) = E$. Therefore (4.23a) is reduced to

$$\begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} & 0 \\ 0 & e^{(\lambda_2 - \rho)t} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

which is valid even if $\lambda_1 = \lambda_2 = \lambda$ (note that the equations in the consideration are reduced to $\dot{\varphi}^\mu = \lambda\varphi^\mu$); and (4.24) is also to

$$\begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \end{pmatrix} = \begin{pmatrix} \rho - \lambda_1 & 0 \\ 0 & \rho - \lambda_2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

which implies that $U_1 = \text{const.}$ if $\lambda_1 = \rho$, and $U_2 = \text{const.}$ if $\lambda_2 = \rho$. Finally we have the following conserved quantities Ξ_1 and Ξ_2 :

$$\begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} U_1 \\ e^{(\lambda_2 - \rho)t} U_2 \end{pmatrix} \quad \text{for } A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

which can be regarded as a generalization of (4.11) into the two sector version without the externality.

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Since the matrix $P = (p_1 \ p_2)$ is nonsingular, (4.23a), (4.23b) and (4.23c) lead respectively to

$$(4.23a)' \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \end{pmatrix} \quad \text{for } D > 0,$$

$$(4.23b)' \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = e^{(\rho-\lambda)t} {}^tP^{-1} \begin{pmatrix} \Xi_2 \\ \Xi_1 - \Xi_2 t \end{pmatrix} \quad \text{for } D = 0,$$

$$(4.23c)' \quad \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = e^{(\rho-\lambda)t} {}^tP^{-1} \begin{pmatrix} \Xi_1 \sin \theta t + \Xi_2 \cos \theta t \\ \Xi_1 \cos \theta t - \Xi_2 \sin \theta t \end{pmatrix} \quad \text{for } D < 0.$$

Therefore, since $\det(\partial U_\mu / \partial c^\nu) = \det(\partial^2 U / \partial c^\mu \partial c^\nu) \neq 0$ by (4.18), the optimal path $c(t) = {}^t(c^1(t) \ c^2(t))$ is implicitly determined as $c^\mu(t) = F^\mu(\psi_1(t), \psi_2(t))$, where ${}^t(\psi_1(t) \ \psi_2(t))$ is the right hand side of (4.23a)', (4.23b)' or (4.23c)' according as $D \gtrless 0$, respectively. And then $c^\mu(t) = F^\mu$ are substituted for the growth process (4.2)'' to have the first order linear differential equations with respect to x^μ , i.e.,

$$(4.2)''' \quad \dot{x} = Ax - F + \beta,$$

where $x = {}^t(x^1 \ x^2)$, $F = {}^t(F^1 \ F^2)$ and $\beta = {}^t(\beta^1 \ \beta^2)$. The general solution x of the subsidiary equation $\dot{x} = Ax$ of (4.2)''' are give as a linear combination of the independent solutions φ_μ with constant coefficients G^μ , i.e., $x = G^1 \varphi_1 + G^2 \varphi_2 = \Phi G$ where $G = {}^t(G^1 \ G^2)$. And then, after replacing the constants G^μ with arbitrary functions $G^\mu(t)$, the solution x is substituted for (4.2)''' to have the equation $\Phi \dot{G} = -F + \beta$, i.e.,

$$\dot{G} = -\Phi^{-1}(F - \beta).$$

Thus the optimal path $x(t)$ is implicitly determined as

$$(4.25) \quad x(t) = -\Phi \int \Phi^{-1}(F - \beta) dt.$$

Utility of second order polynomial. Let the utility function $U(c^1, c^2)$ with the concavity be a second order polynomial of consumptions:

$$(4.26) \quad U(c^1, c^2) = -\frac{1}{2}(c^1)^2 - mc^1 c^2 - \frac{1}{2}(c^2)^2 \quad (m: \text{const.}, -1 < m < 1).$$

Then the explicit appearances of (4.8) can be determined as follows.

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(i) When $D > 0$, by (4.7a)' i.e.,

$$-\begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

the optimal path $\mathbf{c}(t) = \mathbf{F}(t)$ leads to

$$(4.27) \quad \mathbf{c}(t) = \frac{1}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

where Ξ_i ($i = 1, 2$) are some constants. Accordingly, by putting

$$(4.28) \quad -\frac{1}{1-m^2} {}^tP^{-1} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P^{-1}\boldsymbol{\beta} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix},$$

$\Phi^{-1}(\mathbf{F} - \boldsymbol{\beta})$ is written as

$$\Phi^{-1}(\mathbf{F} - \boldsymbol{\beta}) = - \begin{pmatrix} a\Xi_1 e^{(\rho-2\lambda_1)t} + b\Xi_2 e^{(\rho-\lambda_1-\lambda_2)t} \\ c\Xi_1 e^{(\rho-\lambda_1-\lambda_2)t} + d\Xi_2 e^{(\rho-2\lambda_2)t} \end{pmatrix} - \begin{pmatrix} \kappa_1 e^{-\lambda_1 t} \\ \kappa_2 e^{-\lambda_2 t} \end{pmatrix}.$$

The above appearance of $\Phi^{-1}(\mathbf{F} - \boldsymbol{\beta})$ can be integrated immediately, where ρ ($\rho \geq 0$) is assumed not to be $2\lambda_1$, $2\lambda_2$ and $\lambda_1 + \lambda_2$ (the integration can be made also when ρ equals $2\lambda_1$, $2\lambda_2$ or $\lambda_1 + \lambda_2$). Therefore, the optimal path $\mathbf{x}(t)$ of (4.8) is determined completely as

$$(4.29) \quad \mathbf{x}(t) = P \begin{pmatrix} \frac{a\Xi_1}{\rho-2\lambda_1} e^{(\rho-\lambda_1)t} + \frac{b\Xi_2}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_2)t} + \gamma_1 e^{\lambda_1 t} + \delta_1 \\ \frac{c\Xi_1}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_1)t} + \frac{d\Xi_2}{\rho-2\lambda_2} e^{(\rho-\lambda_2)t} + \gamma_2 e^{\lambda_2 t} + \delta_2 \end{pmatrix},$$

where $\delta_i = -\kappa_i/\lambda_i$ ($i = 1, 2$) and γ_i ($i = 1, 2$) are some constants.

Similarly, for the case $D = 0$ or $D < 0$, the optimal path is determined by (4.7b)' or (4.7c)' respectively.

(ii) When $D = 0$:

$$(4.30) \quad \mathbf{c}(t) = \frac{e^{(\rho-\lambda)t}}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} \begin{pmatrix} \Xi_2 \\ \Xi_1 - \Xi_2 t \end{pmatrix},$$

$$(4.31) \quad \mathbf{x}(t) = P \begin{pmatrix} e^{(\rho-\lambda)t}(A_{11} + A_{12}t) + e^{\lambda t}(\gamma_1 + \gamma_2 t) + \delta_1 \\ e^{(\rho-\lambda)t}(A_{21} + A_{22}t) + \gamma_2 e^{\lambda t} + \delta_2 \end{pmatrix};$$

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where $\delta_1 = \kappa_2/\lambda^2 - \kappa_1/\lambda$, $\delta_2 = -\kappa_2/\lambda$, γ_i ($i = 1, 2$) are some constants and A_{ij} ($i, j = 1, 2$) are the constants of the following form with the constants Ξ_1 and Ξ_2 , while $\rho \neq 2\lambda$:

$$\begin{aligned} A_{11} &= \left(\frac{b}{\rho-2\lambda} + \frac{d}{(\rho-2\lambda)^2} \right) \Xi_1 + \left(\frac{a}{\rho-2\lambda} + \frac{b+c}{(\rho-2\lambda)^2} + \frac{2d}{(\rho-2\lambda)^3} \right) \Xi_2, \\ A_{12} &= - \left(\frac{b}{\rho-2\lambda} + \frac{d}{(\rho-2\lambda)^2} \right) \Xi_2, \\ A_{21} &= \frac{d}{\rho-2\lambda} \Xi_1 + \left(\frac{c}{\rho-2\lambda} + \frac{d}{(\rho-2\lambda)^2} \right) \Xi_2, \\ A_{22} &= -\frac{d}{\rho-2\lambda} \Xi_2. \end{aligned}$$

(iii) When $D < 0$:

$$(4.32) \quad \mathbf{c}(t) = \frac{e^{(\rho-\lambda)t}}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1}R \begin{pmatrix} \Xi_2 \\ \Xi_1 \end{pmatrix},$$

$$(4.33) \quad \begin{aligned} \mathbf{x}(t) &= \frac{e^{(\rho-\lambda)t}}{(\rho-2\lambda)^2+4\theta^2} PR \begin{pmatrix} A_1 \sin^2 \theta t + B_1 \cos^2 \theta t + C_1 \sin \theta t \cos \theta t + D_1 \\ A_2 \sin^2 \theta t + B_2 \cos^2 \theta t + C_2 \sin \theta t \cos \theta t + D_2 \end{pmatrix} \\ &\quad - \frac{1}{\lambda^2+\theta^2} P \begin{pmatrix} \kappa_1 \lambda - \kappa_2 \theta \\ \kappa_1 \theta + \kappa_2 \lambda \end{pmatrix} + e^{\lambda t} PR \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}; \end{aligned}$$

where R is the matrix:

$$R = \begin{pmatrix} \cos \theta t & \sin \theta t \\ -\sin \theta t & \cos \theta t \end{pmatrix},$$

and moreover γ_i ($i = 1, 2$) are some constants and A_i , B_i , C_i , D_i ($i = 1, 2$) are the constants of the following form with the constants Ξ_1 and Ξ_2 , while $\rho \neq 2\lambda$:

$$\begin{aligned} A_1 &= (\rho - 2\lambda)(d\Xi_2 - c\Xi_1) + \theta[(a - d)\Xi_1 - (b + c)\Xi_2], \\ A_2 &= (\rho - 2\lambda)(a\Xi_1 - b\Xi_2) + \theta[(b + c)\Xi_1 + (a - d)\Xi_2], \\ B_1 &= (\rho - 2\lambda)(a\Xi_2 + b\Xi_1) - \theta[(a - d)\Xi_1 - (b + c)\Xi_2], \\ B_2 &= (\rho - 2\lambda)(c\Xi_2 + d\Xi_1) - \theta[(b + c)\Xi_1 + (a - d)\Xi_2], \\ C_1 &= (\rho - 2\lambda)[(a - d)\Xi_1 - (b + c)\Xi_2] + 2\theta[(b + c)\Xi_1 + (a - d)\Xi_2], \\ C_2 &= (\rho - 2\lambda)[(b + c)\Xi_1 + (a - d)\Xi_2] - 2\theta[(a - d)\Xi_1 - (b + c)\Xi_2], \\ D_1 &= \frac{2\theta^2}{\rho-2\lambda}[(b - c)\Xi_1 + (a + d)\Xi_2], \\ D_2 &= \frac{2\theta^2}{\rho-2\lambda}[(a + d)\Xi_1 - (b - c)\Xi_2]. \end{aligned}$$

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Theorem 4.7. *In the maximizing problem of (4.1)', let the utility function $U(c^1, c^2)$ with the concavity be given as (4.26) and the consumptions $c^\mu = g^\mu(x^1, x^2) - n_\nu^\mu - \dot{x}^\nu$ grow externally under the linear production technologies $g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu$. Then, in the case of finite horizon $T < \infty$, the optimal paths $c(t)$ and $x(t)$ are determined completely as (4.27) and (4.29) when $D > 0$ and ρ is not equal to $2\lambda_1$, $2\lambda_2$ and $\lambda_1 + \lambda_2$; as (4.30) and (4.31) when $D = 0$ and $\rho \neq 2\lambda$; as (4.32) and (4.33) when $D < 0$ and $\rho \neq 2\lambda$.*

Remark 4.6. The interested reader will find the optimal paths when ρ takes the exceptional values in the theorem 4.7.

(i) To look the case of infinite horizon $T = \infty$, the relation (4.4b): $\pi_\mu = -e^{-\rho t} U_\mu$, i.e.,

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = e^{-\rho t} \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \end{pmatrix},$$

are written, when $D > 0$, by (4.27) as

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = -e^{-\rho t} {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

and then used, together with (4.29), for the transversality condition $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$. In view of the resulting appearance of $\pi_\mu x^\mu$:

$$\begin{aligned} \pi_\mu x^\mu = & -\frac{a}{\rho-2\lambda_1} \Xi_1^2 e^{(\rho-2\lambda_1)t} - \frac{d}{\rho-2\lambda_2} \Xi_2^2 e^{(\rho-2\lambda_2)t} - \frac{b+c}{\rho-\lambda_1-\lambda_2} \Xi_1 \Xi_2 e^{(\rho-\lambda_1-\lambda_2)t} \\ & -(\delta_1 \Xi_1 e^{-\lambda_1 t} + \delta_2 \Xi_2 e^{-\lambda_2 t}) - (\gamma_1 \Xi_1 + \gamma_2 \Xi_2), \end{aligned}$$

we can find the optimal paths satisfying the transversality condition. For example, let $0 < 2\lambda_1 \leq \rho < 2\lambda_2$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\Xi_1 = 0$ in the coefficient of $e^{(\rho-2\lambda_1)t}$; so that $\gamma_2 \Xi_2 = 0$, i.e., $\gamma_2 = 0$ or $\Xi_2 = 0$. Therefore, by putting $\Xi_1 = 0$ and $\gamma_2 = 0$, the optimal paths (4.27) and (4.29) take the forms respectively:

$$(4.27)' \quad c(t) = \frac{1}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} \begin{pmatrix} 0 \\ \Xi_2 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

$$(4.29)' \quad x(t) = P \begin{pmatrix} \frac{b\Xi_2}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_2)t} + \gamma_1 e^{\lambda_1 t} + \delta_1 \\ \frac{d\Xi_2}{\rho-2\lambda_2} e^{(\rho-\lambda_2)t} + \delta_2 \end{pmatrix},$$

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while $c(t) = 0$ if $\Xi_1 = 0$ and $\Xi_2 = 0$.

Similarly, we can have the optimal paths satisfying the transversality condition for the case $D = 0$ or $D < 0$.

(ii) When $D = 0$, by (4.30) and (4.31), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -e^{(\rho-2\lambda)t} [A_{11}\Xi_2 + A_{21}\Xi_1 + (A_{12}\Xi_2 + A_{22}\Xi_1 - A_{21}\Xi_2)t - A_{22}\Xi_2 t^2] \\ & -e^{-\lambda t} (\delta_1\Xi_2 + \delta_2\Xi_1 - \delta_2\Xi_2 t) - (\gamma_1\Xi_2 + \gamma_2\Xi_1). \end{aligned}$$

So, let $0 < \rho < 2\lambda$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\gamma_1\Xi_2 + \gamma_2\Xi_1 = 0$. Therefore, by putting $\Xi_1 = 0$ and $\gamma_1 = 0$, the optimal paths (4.30) and (4.31) take the forms respectively:

$$(4.30)' \quad c(t) = \frac{e^{(\rho-\lambda)t}}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} \begin{pmatrix} \Xi_2 \\ -\Xi_2 t \end{pmatrix},$$

$$(4.31)' \quad x(t) = P \begin{pmatrix} \Xi_2 e^{(\rho-\lambda)t} \left(\frac{a\sigma^2 + (b+c)\sigma + 2d}{\sigma^3} - \frac{b\sigma + d}{\sigma^2} t \right) + \gamma_2 t e^{\lambda t} + \delta_1 \\ \Xi_2 e^{(\rho-\lambda)t} \left(\frac{c\sigma + d}{\sigma^2} - \frac{d}{\sigma} t \right) + \gamma_2 e^{\lambda t} + \delta_2 \end{pmatrix}.$$

(iii) When $D < 0$, by (4.32) and (4.33), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -\frac{e^{(\rho-2\lambda)t}}{(\rho-2\lambda)^2 + 4\theta^2} [(A_1\Xi_2 + A_2\Xi_1) \sin^2 \theta t + (B_1\Xi_2 + B_2\Xi_1) \cos^2 \theta t \\ & + (C_1\Xi_2 + C_2\Xi_1) \sin \theta t \cos \theta t + D_1\Xi_2 + D_2\Xi_1] \\ & -\frac{e^{-\lambda t}}{\lambda^2 + \theta^2} [((\delta_1\theta + \delta_2\lambda)\Xi_2 + (\delta_2\theta - \delta_1\lambda)\Xi_1) \sin \theta t \\ & + ((\delta_2\theta - \delta_1\lambda)\Xi_2 - (\delta_1\theta + \delta_2\lambda)\Xi_1) \cos \theta t] \\ & -(\gamma_1\Xi_2 + \gamma_2\Xi_1), \end{aligned}$$

in which, let $0 < \rho < 2\lambda$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\gamma_1\Xi_2 + \gamma_2\Xi_1 = 0$. Therefore, by putting $\Xi_1 = 0$ and $\gamma_1 = 0$, the optimal paths (4.32) and (4.33) take the forms respectively:

$$(4.32)' \quad c(t) = \frac{e^{(\rho-\lambda)t}}{1-m^2} \begin{pmatrix} -1 & m \\ m & -1 \end{pmatrix} {}^tP^{-1} R \begin{pmatrix} \Xi_2 \\ 0 \end{pmatrix},$$

$$\begin{aligned} (4.33)' \quad x(t) = & \frac{e^{(\rho-\lambda)t}}{(\rho-2\lambda)^2 + 4\theta^2} P R \begin{pmatrix} A_1 \sin^2 \theta t + B_1 \cos^2 \theta t + C_1 \sin \theta t \cos \theta t + D_1 \\ A_2 \sin^2 \theta t + B_2 \cos^2 \theta t + C_2 \sin \theta t \cos \theta t + D_2 \end{pmatrix} \\ & -\frac{1}{\lambda^2 + \theta^2} P \begin{pmatrix} \kappa_1\lambda - \kappa_2\theta \\ \kappa_1\theta + \kappa_2\lambda \end{pmatrix} + P R \begin{pmatrix} 0 \\ \gamma_2 e^{\lambda t} \end{pmatrix}; \end{aligned}$$

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where A_i, B_i, C_i, D_i ($i = 1, 2$) are the constants given before, while in which Ξ_1 is placed as $\Xi_1 = 0$. We can see immediately the following limits of the optimal paths (4.27)' and (4.29)'–(4.33)'.

Theorem 4.8. *In the case of infinite horizon $T = \infty$, there exist the feasible optimal paths (4.27)' and (4.29)' with $0 < 2\lambda_1 \leq \rho < 2\lambda_2$ for $D > 0$; (4.30)' and (4.31)' with $0 < \rho < 2\lambda$ for $D = 0$; (4.32)' and (4.33)' with $0 < \rho < 2\lambda$ for $D < 0$. Paticularly, let $0 < 2\lambda_1 \leq \rho < \lambda_2$ and $\gamma_1 = 0$ in the determined optimal paths $\mathbf{c}(t)$ and $\mathbf{x}(t)$ for $D > 0$; or $0 < \rho < \lambda$ and $\gamma_2 = 0$ in those for $D \leq 0$. Then $\lim_{t \rightarrow \infty} \mathbf{c}(t) = \mathbf{0}$ and $\mathbf{x}_\infty = \lim_{t \rightarrow \infty} \mathbf{x}(t)$ are the following constants:*

$$\mathbf{x}_\infty = P \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \quad \text{for } D \geq 0; \quad \mathbf{x}_\infty = -\frac{1}{\lambda^2 + \theta^2} P \begin{pmatrix} \kappa_1 \lambda - \kappa_2 \theta \\ \kappa_1 \theta + \kappa_2 \lambda \end{pmatrix} \quad \text{for } D < 0.$$

Remark 4.7. More feasible optimal paths for $D \leq 0$ can be derived from (4.30) and (4.31), or (4.32) and (4.33), with the condition $\gamma_1 \Xi_2 + \gamma_2 \Xi_1 = 0$ (e.g., put $\Xi_2 = 0$ and $\gamma_2 = 0$).

Remark 4.8. If $\lambda_2 < \rho < 2\lambda_2$ for $D > 0$, $\lambda < \rho < 2\lambda$ for $D = 0$, or $\lambda < \rho < 2\lambda$ for $D < 0$, there exist feasible optimal paths $\mathbf{c}(t)$ and $\mathbf{x}(t)$ of the forms (4.27)' and (4.29)', (4.30)' and (4.31)', or (4.32)' and (4.33)' respectively; and both of whose limits are not finite constant vectors.

The feasible optimal path $\mathbf{x}(t)$ of (4.33)' goes into details with $\gamma_2 = 0$. In view of the identities $B_1 - A_1 = C_2$ and $A_2 - B_2 = C_1$, since

$$\begin{aligned} & \begin{pmatrix} A_1 \sin^2 \theta t + B_1 \cos^2 \theta t + C_1 \sin \theta t \cos \theta t + D_1 \\ A_2 \sin^2 \theta t + B_2 \cos^2 \theta t + C_2 \sin \theta t \cos \theta t + D_2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} B_1 - A_1 & C_1 \\ B_2 - A_2 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \frac{A_1+B_1}{2} + D_1 \\ \frac{A_2+B_2}{2} + D_2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} C_2 & C_1 \\ -C_1 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \frac{A_1+B_1}{2} + D_1 \\ \frac{A_2+B_2}{2} + D_2 \end{pmatrix}, \end{aligned}$$

the appearance of (4.33)' can be arranged as

$$(4.34) \quad R^{-1} \mathbf{y}(t) = e^{(\rho-\lambda)t} \left[\frac{1}{2} \begin{pmatrix} C_2 & C_1 \\ -C_1 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \frac{A_1+B_1}{2} + D_1 \\ \frac{A_2+B_2}{2} + D_2 \end{pmatrix} \right],$$

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where the vector $\mathbf{y}(t) = {}^t(y^1 \ y^2)$ is given by

$$(4.35) \quad \mathbf{y}(t) = [(\rho - 2\lambda)^2 + 4\theta^2] \left[P^{-1} \mathbf{x}(t) + \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \kappa_1 \lambda - \kappa_2 \theta \\ \kappa_1 \theta + \kappa_2 \lambda \end{pmatrix} \right].$$

The constants C_1 and C_2 are written in the matrix form

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Delta \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix},$$

where the matrix

$$\Delta = \begin{pmatrix} (\rho - 2\lambda)(a - d) + 2\theta(b + c) & -(\rho - 2\lambda)(b + c) + 2\theta(a - d) \\ (\rho - 2\lambda)(b + c) - 2\theta(a - d) & (\rho - 2\lambda)(a - d) + 2\theta(b + c) \end{pmatrix}$$

has the determinant

$$|\Delta| = [(\rho - 2\lambda)(a - d) + 2\theta(b + c)]^2 + [(\rho - 2\lambda)(b + c) - 2\theta(a - d)]^2,$$

which vanishes if and only if $(\rho - 2\lambda)(a - d) + 2\theta(b + c) = 0$ and $(\rho - 2\lambda)(b + c) - 2\theta(a - d) = 0$, i.e.,

$$\begin{pmatrix} \rho - 2\lambda & 2\theta \\ -2\theta & \rho - 2\lambda \end{pmatrix} \begin{pmatrix} a - d \\ b + c \end{pmatrix} = 0.$$

Therefore, since $\theta \neq 0$ by $D < 0$, i.e.,

$$\begin{vmatrix} \rho - 2\lambda & 2\theta \\ -2\theta & \rho - 2\lambda \end{vmatrix} = (\rho - 2\lambda)^2 + 4\theta^2 \neq 0,$$

the determinant $|\Delta|$ vanishes if and only if $a - d = 0$ and $b + c = 0$. So, assuming $a - d \neq 0$ or $b + c \neq 0$, choose the constants Ξ_1 and Ξ_2 such that ${}^t(\Xi_1 \ \Xi_2) \neq \mathbf{0}$. Then ${}^t(C_1 \ C_2) \neq \mathbf{0}$ so that

$$(4.36) \quad \begin{vmatrix} C_2 & C_1 \\ -C_1 & C_2 \end{vmatrix} = C_1^2 + C_2^2 \neq 0.$$

Therefore, the vector ${}^t(\cos 2\theta t \ \sin 2\theta t)$ in (4.34) is written as

$$\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} = \frac{2e^{-(\rho-\lambda)t}}{C_1^2 + C_2^2} \begin{pmatrix} C_2 & -C_1 \\ C_1 & C_2 \end{pmatrix} \begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} - \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

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where n_i ($i = 1, 2$) are the following constants, respectively:

$$n_1 = \frac{-(A_2+B_2)C_1+(A_1+B_1)C_2-2(C_1D_2-C_2D_1)}{C_1^2+C_2^2}, \quad n_2 = \frac{(A_1+B_1)C_1+(A_2+B_2)C_2+2(C_1D_1+C_2D_2)}{C_1^2+C_2^2},$$

which can be put as $n_1 = n_2 = 0$ by a suitable choice of D_1 and D_2 (this fact is guaranteed by (4.36)). Thus, the identity $\cos^2 2\theta t + \sin^2 2\theta t = 1$ yields the equation of the spiral

$$(4.37) \quad (y^1)^2 + (y^2)^2 = \frac{C_1^2+C_2^2}{4} e^{2(\rho-\lambda)t}.$$

In conclusion, we have the following result.

Theorem 4.9. *Let a , b , c and d in the matrix in (4.28) satisfy $a - d \neq 0$ or $b + c \neq 0$. Then, for $D < 0$ in the case of infinite horizon $T = \infty$, the feasible optimal path $\mathbf{x}(t)$ of the form (4.33)' with $\gamma_2 = 0$ is transformed by (4.35), under a suitable choice of the constants D_i ($i = 1, 2$), to $\mathbf{y}(t) = {}^t(y^1 \ y^2)$ which satisfies the equation of the spiral (4.37).*

5 External three-sector growth model

5.1 Introduction

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. Sato [44] first pioneered the way of applying the theorem to optimal economic growths.

In contrast with the Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [32]) and applied it to various economic growth models (Mimura and Nôno [34]; Mimura, Fujiwara and Nôno [29], [30]; Fujiwara, Mimura and Nôno [11]; which were applied to, e.g., the model of von Neumann type and the models in Tobin's q -theory of investment, and the model in the intergenerational problem, respectively) to discover new economic conservation laws including non-Noether ones. Particularly in [11], the procedure was so reformed as to make an effective application to more general neoclassical optimal growth models. And in (Fujiwara, Mimura and Nôno [12]), by a reduction of the theorem 1 in [3], the application was pursued to a one sector model of Ramsey type (Ramsey [38]) with a constant discount rate relative to a utility (welfare) of consumption, and then the model was generalized in an external two sector version with linear technologies. The growth process relative to the technologies were characterized by a matrix of second order. By the reduced theorem, we have found three types of conservation laws according as the discriminant of the characteristic equation of the matrix is positive, zero or negative. And in (Fujiwara, Mimura and Nôno [14]), optimal paths were determined completely through the three types of conservation laws, while the utility is assumed to be of second order polynomial of consumptions.

In this section, more application of the reduced theorem can be made to an external three-sector growth model with linear technology. We will find six types of triple conservation laws in 5.2.1. In 5.2.2, through the conservation laws, optimal paths are determined completely for finite horizon and then detailed for infinite horizon under a given utility of second order polynomial of consumptions.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

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5.2 An external three-sector growth model

5.2.1 Derivation of conserved quantities

The external two-sector growth model of Ramsey type [12] can be extended to a three-sector version. We discuss the objective of society to maximize the following integration (the social welfare functional) over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(5.1) \quad \int_0^T e^{-\rho t} U(c^1, c^2, c^3) dt,$$

under constraints (external growth process with respect to the consumption c^μ and the capital-labour ratio x^μ in μ -th ($\mu = 1, 2, 3$) sector):

$$\dot{x}^\mu = g^\mu(x^1, x^2, x^3) - n_\nu^\mu x^\nu - c^\mu \quad (n_\nu^\mu: \text{const.}, n_\nu^\mu > 0; \mu, \nu = 1, 2, 3),$$

where U is a utility (welfare) function provided with the concavity (see, e.g., [46]), i.e., the successive principal minors D_k ($k = 1, 2, 3$) of Hessian matrix of U satisfy $D_1 < 0$, $D_2 > 0$ and

$$(5.2) \quad D_3 = \det \left(\frac{\partial^2 U}{\partial c^\mu \partial c^\nu} \right) < 0;$$

and g^μ are assumed to be linear production technologies

$$g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu \quad (\alpha_\nu^\mu, \beta^\mu: \text{const.}).$$

So that the growth process are written as

$$(5.3) \quad \dot{x}^\mu = a_\nu^\mu x^\nu - c^\mu + \beta^\mu \quad (a_\nu^\mu = \alpha_\nu^\mu - n_\nu^\mu: \text{const.}).$$

For the three-sector version, (4.5a) in 4 is reduced to a system of first order linear homogeneous differential equations $\dot{\varphi}^\mu = a_\nu^\mu \varphi^\nu$, which are written in the matrix notation $\dot{\varphi} = A\varphi$ where $A = (a_\nu^\mu)$ and $\varphi = {}^t(\varphi^1 \ \varphi^2 \ \varphi^3)$ (t denote the transposition).

When A has real characteristic values λ_μ ($\mu = 1, 2, 3$), we set the vectors \mathbf{p}_μ ($\mu = 1, 2, 3$) in the following three cases (i), (ii) and (iii), respectively.

(i) The case of $\lambda_\mu \neq \lambda_\nu$ ($\mu \neq \nu$): The characteristic vectors \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 of A with respective characteristic values λ_1 , λ_2 and λ_3 .

(ii) The case of $\lambda_1 \neq \lambda_2 = \lambda_3$ is divided into two subcases.

(ii-1) $\text{rank}(A - \lambda_2 E) = 1$: The characteristic vector \mathbf{p}_1 of A with characteristic value λ_1 and linearly independent characteristic vectors \mathbf{p}_2 and \mathbf{p}_3 of A with characteristic value λ_2 .

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(ii-2) $\text{rank}(A - \lambda_2 E) = 2$: The characteristic vectors \mathbf{p}_1 and \mathbf{p}_2 of A with respective characteristic values λ_1 and λ_2 , and a vector \mathbf{p}_3 satisfying $A\mathbf{p}_3 = \mathbf{p}_2 + \lambda_2\mathbf{p}_3$, i.e., $(A - \lambda_2 E)\mathbf{p}_3 = \mathbf{p}_2$. Such a vector \mathbf{p}_3 exists by $\text{rank}(A - \lambda_2 E) = 2$.

(iii) The case of $\lambda \equiv \lambda_1 = \lambda_2 = \lambda_3$ is divided into two subcases with an assumption that A is not a constant multiple of the identity E . Here remark that whenever A is a constant multiple of E , there is no externality, i.e., each of three sectors behaves independently of the others.

(iii-1) $\text{rank}(A - \lambda E) = 1$: Linearly independent characteristic vectors \mathbf{p}_1 and \mathbf{p}_2 with characteristic value λ , and a vector \mathbf{p}_3 satisfying $A\mathbf{p}_3 = \mathbf{p}_2 + \lambda\mathbf{p}_3$, i.e., $(A - \lambda E)\mathbf{p}_3 = \mathbf{p}_2$.

(iii-2) $\text{rank}(A - \lambda E) = 2$: The characteristic vector \mathbf{p}_1 with characteristic value λ , and vectors \mathbf{p}_2 and \mathbf{p}_3 satisfying $A\mathbf{p}_2 = \mathbf{p}_1 + \lambda\mathbf{p}_2$ and $A\mathbf{p}_3 = \mathbf{p}_2 + \lambda\mathbf{p}_3$, i.e., $(A - \lambda E)\mathbf{p}_2 = \mathbf{p}_1$ and $(A - \lambda E)\mathbf{p}_3 = \mathbf{p}_2$, respectively.

In fact, the vectors \mathbf{p}_3 in (iii-1), \mathbf{p}_2 and \mathbf{p}_3 in (iii-2) exist by the respective condition of the rank of the matrix $A - \lambda E$.

When A has real characteristic value λ_1 and complex characteristic values $\lambda \pm i\theta$ (λ, θ : real, $\theta \neq 0$), we set the vectors \mathbf{p}_μ ($\mu = 1, 2, 3$) such that

(iv) The characteristic vectors \mathbf{p}_1 and $\mathbf{p}_2 \pm i\mathbf{p}_3$ ($\mathbf{p}_2, \mathbf{p}_3$: real) of A with respective characteristic values λ_1 and $\lambda \pm i\theta$.

In each cases, it is easy to show that the vectors $\mathbf{p}_1, \mathbf{p}_2$ and \mathbf{p}_3 are linearly independent. Therefore the matrix $P = (\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3)$ is nonsingular. In the above setting, the following independent solutions φ_1, φ_2 and φ_3 of $\dot{\varphi} = A\varphi$, i.e., $\Phi = (\varphi_1 \ \varphi_2 \ \varphi_3)$, are provided:

$$\begin{aligned} \Phi &= P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} && \text{for (i),} \\ \Phi &= P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{pmatrix} && \text{for (ii-1),} \\ \Phi &= P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & e^{\lambda_2 t} t \\ 0 & 0 & e^{\lambda_2 t} \end{pmatrix} && \text{for (ii-2),} \\ \Phi &= e^{\lambda t} P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} && \text{for (iii-1),} \end{aligned}$$

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$$\Phi = e^{\lambda t} P \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for (iii - 2),}$$

$$\Phi = P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda t} \cos \theta t & e^{\lambda t} \sin \theta t \\ 0 & -e^{\lambda t} \sin \theta t & e^{\lambda t} \cos \theta t \end{pmatrix} \quad \text{for (iv).}$$

The solutions Φ are used to construct the conserved quantities Ω_1 , Ω_2 and Ω_3 in (4.8) in 4:

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = {}^t\Phi \begin{pmatrix} \dot{\pi}_1 + \rho\pi_1 \\ \dot{\pi}_2 + \rho\pi_2 \\ \dot{\pi}_3 + \rho\pi_3 \end{pmatrix};$$

for which, the relations from (4.4a) and (4.4b): $\dot{\pi}_\mu = -a_\mu^\nu \pi_\nu$ and $\pi_\mu = -e^{-\rho t} U_\mu$ ($U_\mu = \partial U / \partial c^\mu$) are substituted after arranging:

$$(5.4) \quad \begin{pmatrix} \dot{\pi}_1 + \rho\pi_1 \\ \dot{\pi}_2 + \rho\pi_2 \\ \dot{\pi}_3 + \rho\pi_3 \end{pmatrix} = e^{-\rho t} ({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}.$$

Consequently it follows that

$$(5.5) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = e^{-\rho t} {}^t\Phi ({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}.$$

In ${}^t\Phi ({}^tA - \rho E)$ of (5.5) with the above solutions Φ , the transpositions of the following relations are used:

$$(5.6a) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \quad \text{for (i),}$$

$$(5.6b) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix} \quad \text{for (ii - 1),}$$

$$(5.6c) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 1 \\ 0 & 0 & \Lambda_2 \end{pmatrix} \quad \text{for (ii - 2),}$$

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$$(5.6d) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 1 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{for (iii-1),}$$

$$(5.6e) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda & 1 & 0 \\ 0 & \Lambda & 1 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{for (iii-2),}$$

$$(5.6f) \quad (A - \rho E)P = P \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda & \theta \\ 0 & -\theta & \Lambda \end{pmatrix} \quad \text{for (iv),}$$

where $\Lambda \equiv \lambda - \rho$ and $\Lambda_\mu \equiv \lambda_\mu - \rho$ ($\mu = 1, 2, 3$). Then the conserved quantities Ω_1 , Ω_2 and Ω_3 in (5.5) have the following appearances

$$(5.7a) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} e^{\Lambda_1 t} & 0 & 0 \\ 0 & e^{\Lambda_2 t} & 0 \\ 0 & 0 & e^{\Lambda_3 t} \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (i),}$$

$$(5.7b) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} e^{\Lambda_1 t} & 0 & 0 \\ 0 & e^{\Lambda_2 t} & 0 \\ 0 & 0 & e^{\Lambda_2 t} \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-1),}$$

$$(5.7c) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = e^{\Lambda_2 t} \begin{pmatrix} e^{(\lambda_1 - \lambda_2)t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-2),}$$

$$(5.7d) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 1 & \Lambda \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-1),}$$

$$(5.7e) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & 0 \\ 1 & \Lambda & 0 \\ 0 & 1 & \Lambda \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-2),}$$

$$(5.7f) \quad \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} e^{(\lambda_1 - \lambda)t} & 0 & 0 \\ 0 & \cos \theta t & -\sin \theta t \\ 0 & \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda & -\theta \\ 0 & \theta & \Lambda \end{pmatrix} \begin{pmatrix} {}^t \mathbf{p}_1 \\ {}^t \mathbf{p}_2 \\ {}^t \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iv).}$$

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Particularly let $\Lambda_\mu \neq 0$ ($\mu = 1, 2, 3$) in (5.7a), $\Lambda_\mu \neq 0$ ($\mu = 1, 2$) in (5.7b)-(5.7c), $\Lambda \neq 0$ in (5.7d)-(5.7e), and $\Lambda_1 \neq 0$ in (5.7f). Then, since the first two matrices of the right hand side of (5.7a), (5.7b), (5.7d) and (5.7e) are commutative respectively, and the products in (5.7c) and (5.7f) are written respectively as

$$\begin{pmatrix} e^{(\lambda_1-\lambda_2)t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 1 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & 0 & \Lambda_2 \\ 0 & \Lambda_2 & 1 \end{pmatrix} \begin{pmatrix} e^{(\lambda_1-\lambda_2)t} & 0 & 0 \\ 0 & t & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} e^{(\lambda_1-\lambda)t} & 0 & 0 \\ 0 & \cos \theta t & -\sin \theta t \\ 0 & \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda & -\theta \\ 0 & \theta & \Lambda \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & -\theta & \Lambda \\ 0 & \Lambda & \theta \end{pmatrix} \begin{pmatrix} e^{(\lambda_1-\lambda)t} & 0 & 0 \\ 0 & \sin \theta t & \cos \theta t \\ 0 & \cos \theta t & -\sin \theta t \end{pmatrix};$$

the conserved quantities Ω_1 , Ω_2 and Ω_3 in (5.7a)-(5.7f) may be put respectively by multiplying the inverse of the first constant matrix in the orderly arrangement of the product of the right hand side of (5.7a)-(5.7f):

$$(5.8a) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = \begin{pmatrix} e^{\Lambda_1 t} & 0 & 0 \\ 0 & e^{\Lambda_2 t} & 0 \\ 0 & 0 & e^{\Lambda_3 t} \end{pmatrix} \begin{pmatrix} {}^t p_1 \\ {}^t p_2 \\ {}^t p_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (i),}$$

$$(5.8b) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = \begin{pmatrix} e^{\Lambda_1 t} & 0 & 0 \\ 0 & e^{\Lambda_2 t} & 0 \\ 0 & 0 & e^{\Lambda_2 t} \end{pmatrix} \begin{pmatrix} {}^t p_1 \\ {}^t p_2 \\ {}^t p_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-1),}$$

$$(5.8c) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{\Lambda_2 t} \begin{pmatrix} e^{(\lambda_1-\lambda_2)t} & 0 & 0 \\ 0 & t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} {}^t p_1 \\ {}^t p_2 \\ {}^t p_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-2),}$$

$$(5.8d) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} {}^t p_1 \\ {}^t p_2 \\ {}^t p_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-1),}$$

$$(5.8e) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{pmatrix} \begin{pmatrix} {}^t p_1 \\ {}^t p_2 \\ {}^t p_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-2),}$$

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$$(5.8f) \quad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{\Lambda t} \begin{pmatrix} e^{(\lambda_1 - \lambda)t} & 0 & 0 \\ 0 & \sin \theta t & \cos \theta t \\ 0 & \cos \theta t & -\sin \theta t \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iv).}$$

The relations from (4.4b): $\pi_\mu = -e^{-\rho t} U_\mu$ and their total time derivatives yield

$$\begin{pmatrix} \dot{\pi}_1 + \rho\pi_1 \\ \dot{\pi}_2 + \rho\pi_2 \\ \dot{\pi}_3 + \rho\pi_3 \end{pmatrix} = -e^{-\rho t} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix}.$$

So that the relation (5.4) leads to

$$\begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = -({}^tA - \rho E) \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

which is multiplied by ${}^tP = {}^t(\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3)$, and then the transpositions of (5.6a)-(5.6f) are used respectively to see

$$(5.9a) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (i),}$$

$$(5.9b) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-1),}$$

$$(5.9c) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-2),}$$

$$(5.9d) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 1 & \Lambda \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-1),}$$

$$(5.9e) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda & 0 & 0 \\ 1 & \Lambda & 0 \\ 0 & 1 & \Lambda \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-2),}$$

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$$(5.9f) \quad \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{U}_3 \end{pmatrix} = - \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda & -\theta \\ 0 & \theta & \Lambda \end{pmatrix} \begin{pmatrix} {}^t\mathbf{p}_1 \\ {}^t\mathbf{p}_2 \\ {}^t\mathbf{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iv).}$$

Integration of the above equations will give six types of triple conserved quantities which are equivalent respectively to (5.8a)-(5.8f); while particular integration for some $\Lambda_\mu = 0$ ($\mu = 1, 2, 3$) in (5.9a), some $\Lambda_\mu = 0$ ($\mu = 1, 2$) in (5.9b)-(5.9c), $\Lambda = 0$ in (5.9d)-(5.9e) and $\Lambda_1 = 0$ in (5.9f) are used to delete the respective condition $\Lambda_\mu \neq 0$ ($\mu = 1, 2, 3$) in (5.8a), $\Lambda_\mu \neq 0$ ($\mu = 1, 2$) in (5.8b)-(5.8c), $\Lambda \neq 0$ in (5.8d)-(5.8e) and $\Lambda_1 \neq 0$ in (5.8f). Thus the conserved quantities in question are summarized as follows.

Theorem 5.1. *In the maximizing problem of (5.1), let the consumptions $c^\mu = g^\mu(x^1, x^2, x^3) - n_\nu^\mu x^\nu - \dot{x}^\nu$ grow externally under the linear production technologies $g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu$. Then, in the setting of (i), (ii-1), (ii-2), (iii-1), (iii-2) and (iv) according to the classification of the characteristic values of the matrix $A = (\alpha_\nu^\mu - n_\nu^\mu)$, there exist the respective conserved quantities (5.8a)-(5.8f) in which $U_i = \partial U / \partial c^i$ ($i = 1, 2, 3$); while in (5.8d)-(5.8e), the matrix A is assumed not to be a constant multiple of E .*

5.2.2 Determination of optimal paths

Since the matrix $P = (\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3)$ is nonsingular, (5.8a)-(5.8f) lead respectively to

$$(5.10a) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{-\Lambda_1 t} \\ \Xi_2 e^{-\Lambda_2 t} \\ \Xi_3 e^{-\Lambda_3 t} \end{pmatrix} \quad \text{for (i),}$$

$$(5.10b) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{-\Lambda_1 t} \\ \Xi_2 e^{-\Lambda_2 t} \\ \Xi_3 e^{-\Lambda_2 t} \end{pmatrix} \quad \text{for (ii-1),}$$

$$(5.10c) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{-\Lambda_2 t} {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\Lambda_2 - \Lambda_1)t} \\ \Xi_2 \\ \Xi_2 - \Xi_3 t \end{pmatrix} \quad \text{for (ii-2),}$$

$$(5.10d) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{-\Lambda t} {}^tP^{-1} \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ -\Xi_2 t + \Xi_3 \end{pmatrix} \quad \text{for (iii-1),}$$

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$$(5.10e) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{-\Lambda t} {}^t P^{-1} \begin{pmatrix} \Xi_1 \\ -\Xi_1 t + \Xi_2 \\ \frac{1}{2}\Xi_1 t^2 - \Xi_2 t + \Xi_3 \end{pmatrix} \quad \text{for (iii-2),}$$

$$(5.10f) \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{-\Lambda t} {}^t P^{-1} \begin{pmatrix} \Xi_1 e^{(\lambda-\lambda_1)t} \\ \Xi_2 \sin \theta t + \Xi_3 \cos \theta t \\ \Xi_2 \cos \theta t - \Xi_3 \sin \theta t \end{pmatrix} \quad \text{for (iv).}$$

Therefore, since $\det(\partial U_\mu / \partial c^\nu) = D_3 \neq 0$ by (5.2), the optimal path $\mathbf{c}(t) = ({}^t c^1(t) \ {}^t c^2(t) \ {}^t c^3(t))$ is implicitly determined as $c^\mu(t) = F^\mu(\psi_1(t), \psi_2(t), \psi_3(t))$, where $({}^t \psi_1(t) \ {}^t \psi_2(t) \ {}^t \psi_3(t))$ is the right hand side of (5.10a)-(5.10f), respectively. And then $c^\mu(t) = F^\mu$ are substituted for the growth process (5.3) to have the first order linear differential equations with respect to x^μ , i.e.,

$$(5.3)' \quad \dot{\mathbf{x}} = A\mathbf{x} - \mathbf{F} + \boldsymbol{\beta},$$

where $\mathbf{x} = ({}^t x^1 \ {}^t x^2 \ {}^t x^3)$, $\mathbf{F} = ({}^t F^1 \ {}^t F^2 \ {}^t F^3)$ and $\boldsymbol{\beta} = ({}^t \beta^1 \ {}^t \beta^2 \ {}^t \beta^3)$. The general solution \mathbf{x} of the subsidiary equation $\dot{\mathbf{x}} = A\mathbf{x}$ of (2)'' are give as a linear combination of the independent solutions φ_μ with constant coefficients G^μ , i.e., $\mathbf{x} = G^1 \varphi_1 + G^2 \varphi_2 + G^3 \varphi_3 = \Phi \mathbf{G}$ where $\mathbf{G} = ({}^t G^1 \ {}^t G^2 \ {}^t G^3)$. And then, after replacing the constants G^μ with arbitrary functions $G^\mu(t)$, the solution \mathbf{x} is substituted for (2)'' to have the equation $\Phi \dot{\mathbf{G}} = -\mathbf{F} + \boldsymbol{\beta}$, i.e.,

$$\dot{\mathbf{G}} = -\Phi^{-1}(\mathbf{F} - \boldsymbol{\beta}).$$

Thus the optimal path $\mathbf{x}(t)$ is implicitly determined as

$$(5.11) \quad \mathbf{x}(t) = -\Phi \int \Phi^{-1}(\mathbf{F} - \boldsymbol{\beta}) dt.$$

To go into detail, let the utility function $U(c^1, c^2, c^3)$ be a second order polynomial of consumptions of the form

$$(5.12) \quad U(c^1, c^2, c^3) = -\frac{1}{2}((c^1)^2 + (c^2)^2 + (c^3)^2) - \alpha c^1 c^2 - \beta c^2 c^3 - \gamma c^3 c^1 \quad (\alpha, \beta, \gamma: \text{const.});$$

where in view of

$$\left(\frac{\partial^2 U}{\partial c^\mu \partial c^\nu} \right) = \begin{pmatrix} -1 & -\alpha & -\gamma \\ -\alpha & -1 & -\beta \\ -\gamma & -\beta & -1 \end{pmatrix} \equiv M,$$

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the constants α , β and γ are assumed for the concavity to satisfy

$$D_2 = \begin{vmatrix} -1 & -\alpha \\ -\alpha & -1 \end{vmatrix} = 1 - \alpha^2 > 0,$$

$$D_3 = \begin{vmatrix} -1 & -\alpha & -\gamma \\ -\alpha & -1 & -\beta \\ -\gamma & -\beta & -1 \end{vmatrix} = -1 - 2\alpha\beta\gamma + \alpha^2 + \beta^2 + \gamma^2 < 0,$$

while $D_1 = -1 < 0$. Then the explicit appearances of (5.11) can be determined as follows.

The case (i): By (5.10a), i.e.,

$$M \mathbf{c}(t) = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix},$$

the optimal path $\mathbf{c}(t) = \mathbf{F}(t)$ leads to

$$(5.13) \quad \mathbf{c}(t) = M^{-1} {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix},$$

where Ξ_μ ($\mu = 1, 2, 3$) are some constants, while

$$M^{-1} = \frac{1}{D_3} \begin{pmatrix} 1 - \beta^2 & \beta\gamma - \alpha & \alpha\beta - \gamma \\ \beta\gamma - \alpha & 1 - \gamma^2 & \gamma\alpha - \beta \\ \alpha\beta - \gamma & \gamma\alpha - \beta & 1 - \alpha^2 \end{pmatrix}.$$

Accordingly, after substituting the solution Φ in the case (i) for $\Phi^{-1}(\mathbf{F} - \beta)$, by putting

$$(5.14) \quad -P^{-1}M^{-1} {}^tP^{-1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad P^{-1}\beta = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix},$$

it follows that

$$\Phi^{-1}(\mathbf{F} - \beta) = - \begin{pmatrix} a_1 \Xi_1 e^{(\rho-2\lambda_1)t} + b_1 \Xi_2 e^{(\rho-\lambda_1-\lambda_2)t} + c_1 \Xi_3 e^{(\rho-\lambda_1-\lambda_3)t} + \kappa_1 e^{-\lambda_1 t} \\ a_2 \Xi_1 e^{(\rho-\lambda_1-\lambda_2)t} + b_2 \Xi_2 e^{(\rho-2\lambda_2)t} + c_2 \Xi_3 e^{(\rho-\lambda_2-\lambda_3)t} + \kappa_2 e^{-\lambda_2 t} \\ a_3 \Xi_1 e^{(\rho-\lambda_1-\lambda_3)t} + b_3 \Xi_2 e^{(\rho-\lambda_2-\lambda_3)t} + c_3 \Xi_3 e^{(\rho-2\lambda_3)t} + \kappa_3 e^{-\lambda_3 t} \end{pmatrix}.$$

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The above appearance of $\Phi^{-1}(\mathbf{F} - \beta)$ can be integrated immediately, where ρ ($\rho \geq 0$) is assumed that $\rho \neq \lambda_\mu + \lambda_\nu$ ($\mu, \nu = 1, 2, 3$), i.e., $\rho \neq 2\lambda_1$, $\rho \neq 2\lambda_2$, $\rho \neq 2\lambda_3$, $\rho \neq \lambda_1 + \lambda_2$, $\rho \neq \lambda_2 + \lambda_3$ and $\rho \neq \lambda_3 + \lambda_1$ (the integration can be made also when $\rho = \lambda_\mu + \lambda_\nu$ ($\mu, \nu = 1, 2, 3$)). Therefore, the optimal path $\mathbf{x}(t)$ of (5.11) is determined completely as

$$(5.15) \quad \mathbf{x}(t) = P \begin{pmatrix} \frac{a_1 \Xi_1}{\rho - 2\lambda_1} e^{(\rho - \lambda_1)t} + \frac{b_1 \Xi_2}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_2)t} + \frac{c_1 \Xi_3}{\rho - \lambda_1 - \lambda_3} e^{(\rho - \lambda_3)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ \frac{a_2 \Xi_1}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_1)t} + \frac{b_2 \Xi_2}{\rho - 2\lambda_2} e^{(\rho - \lambda_2)t} + \frac{c_2 \Xi_3}{\rho - \lambda_2 - \lambda_3} e^{(\rho - \lambda_3)t} + \delta_2 e^{\lambda_2 t} - \frac{\kappa_2}{\lambda_2} \\ \frac{a_3 \Xi_1}{\rho - \lambda_1 - \lambda_3} e^{(\rho - \lambda_1)t} + \frac{b_3 \Xi_2}{\rho - \lambda_2 - \lambda_3} e^{(\rho - \lambda_2)t} + \frac{c_3 \Xi_3}{\rho - 2\lambda_3} e^{(\rho - \lambda_3)t} + \delta_3 e^{\lambda_3 t} - \frac{\kappa_3}{\lambda_3} \end{pmatrix},$$

where δ_μ ($\mu = 1, 2, 3$) are some constants.

Similarly, for the cases (ii-1), (ii-2), (iii-1), (iii-2) or (iv), the optimal paths can be determined by (5.10b), (5.10c), (5.10d), (5.10e) or (5.10f) respectively.

The case (ii-1):

$$(5.16) \quad \mathbf{c}(t) = M^{-1} {}^t P^{-1} \begin{pmatrix} \Xi_1 e^{(\rho - \lambda_1)t} \\ \Xi_2 e^{(\rho - \lambda_2)t} \\ \Xi_3 e^{(\rho - \lambda_2)t} \end{pmatrix},$$

$$(5.17) \quad \mathbf{x}(t) = P \begin{pmatrix} \frac{a_1 \Xi_1}{\rho - 2\lambda_1} e^{(\rho - \lambda_1)t} + \frac{b_1 \Xi_2 + c_1 \Xi_3}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_2)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ \frac{a_2 \Xi_1}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_1)t} + \frac{b_2 \Xi_2 + c_2 \Xi_3}{\rho - 2\lambda_2} e^{(\rho - \lambda_2)t} + \delta_2 e^{\lambda_2 t} - \frac{\kappa_2}{\lambda_2} \\ \frac{a_3 \Xi_1}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_1)t} + \frac{b_3 \Xi_2 + c_3 \Xi_3}{\rho - 2\lambda_2} e^{(\rho - \lambda_2)t} + \delta_3 e^{\lambda_2 t} - \frac{\kappa_3}{\lambda_2} \end{pmatrix};$$

where δ_μ ($\mu = 1, 2, 3$) are some constants, while $\rho \neq \lambda_i + \lambda_j$ ($i, j = 1, 2, 3$).

The case (ii-2):

$$(5.18) \quad \mathbf{c}(t) = M^{-1} {}^t P^{-1} \begin{pmatrix} \Xi_1 e^{(\rho - \lambda_1)t} \\ \Xi_2 e^{(\rho - \lambda_2)t} \\ (\Xi_2 - \Xi_3 t) e^{(\rho - \lambda_2)t} \end{pmatrix},$$

$$(5.19) \quad \mathbf{x}(t) = P \begin{pmatrix} A_{11} e^{(\rho - \lambda_1)t} + (A_{12} - A_{13} t) e^{(\rho - \lambda_2)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ A_{21} e^{(\rho - \lambda_1)t} + (A_{22} - A_{23} t) e^{(\rho - \lambda_2)t} + (\delta_2 + \delta_3 t) e^{\lambda_2 t} - \frac{\kappa_2}{\lambda_2} + \frac{\kappa_3}{\lambda_2} \\ A_{31} e^{(\rho - \lambda_1)t} + (A_{32} - A_{33} t) e^{(\rho - \lambda_2)t} + \delta_3 e^{\lambda_2 t} - \frac{\kappa_3}{\lambda_2} \end{pmatrix};$$

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where δ_μ ($\mu = 1, 2, 3$) are some constants and $A_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) are the constants of the following form with the constants Ξ_1 , Ξ_2 and Ξ_3 , while $\rho \neq \lambda_i + \lambda_j$ ($i, j = 1, 2$):

$$\begin{aligned}
A_{11} &= \frac{a_1}{\rho - 2\lambda_1} \Xi_1, \\
A_{12} &= \frac{(b_1 + c_1)}{\rho - \lambda_1 - \lambda_2} \Xi_2 + \frac{c_1}{(\rho - \lambda_1 - \lambda_2)^2} \Xi_3, \\
A_{13} &= \frac{c_1}{\rho - \lambda_1 - \lambda_2} \Xi_3, \\
A_{21} &= \left(\frac{a_2}{\rho - \lambda_1 - \lambda_2} + \frac{a_3}{(\rho - \lambda_1 - \lambda_2)^2} \right) \Xi_1, \\
A_{22} &= \left(\frac{b_2 + c_2}{\rho - 2\lambda_2} + \frac{b_3 + c_3}{(\rho - 2\lambda_2)^2} \right) \Xi_2 + \left(\frac{c_2}{(\rho - 2\lambda_2)^2} + \frac{2c_3}{(\rho - 2\lambda_2)^3} \right) \Xi_3, \\
A_{23} &= \left(\frac{c_2}{\rho - 2\lambda_2} + \frac{c_3}{(\rho - 2\lambda_2)^2} \right) \Xi_3, \\
A_{31} &= \frac{a_3}{\rho - \lambda_1 - \lambda_2} \Xi_1, \\
A_{32} &= \frac{b_3 + c_3}{\rho - 2\lambda_2} \Xi_2 + \frac{c_3}{(\rho - 2\lambda_2)^2} \Xi_3, \\
A_{33} &= \frac{c_3}{\rho - 2\lambda_2} \Xi_3.
\end{aligned}$$

The case (iii-1):

$$\begin{aligned}
(5.20) \quad \mathbf{c}(t) &= e^{(\rho - \lambda)t} M^{-1} {}^t P^{-1} \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 - \Xi_2 t \end{pmatrix}, \\
(5.21) \quad \mathbf{x}(t) &= P \begin{pmatrix} (B_{11} - B_{12}t)e^{(\rho - \lambda)t} + \delta_1 e^{\lambda t} - \frac{\kappa_1}{\lambda} \\ (B_{21} - B_{22}t)e^{(\rho - \lambda)t} + (\delta_2 + \delta_3 t)e^{\lambda t} + \frac{\kappa_3}{\lambda^2} - \frac{\kappa_2}{\lambda} \\ (B_{31} - B_{32}t)e^{(\rho - \lambda)t} + \delta_3 e^{\lambda t} - \frac{\kappa_3}{\lambda} \end{pmatrix};
\end{aligned}$$

where δ_μ ($\mu = 1, 2, 3$) are some constants and B_{ij} ($i = 1, 2, 3; j = 1, 2$) are the constants of the following form with the constants Ξ_1 , Ξ_2 and Ξ_3 , while $\rho \neq 2\lambda$:

$$\begin{aligned}
B_{11} &= \frac{a_1}{\rho - 2\lambda} \Xi_1 + \left(\frac{b_1}{\rho - 2\lambda} + \frac{c_1}{(\rho - 2\lambda)^2} \right) \Xi_2 + \frac{c_1}{\rho - 2\lambda} \Xi_3, \\
B_{12} &= \frac{c_1}{\rho - 2\lambda} \Xi_2, \\
B_{21} &= \left(\frac{a_2}{\rho - 2\lambda} + \frac{a_3}{(\rho - 2\lambda)^2} \right) \Xi_1 + \left(\frac{b_2}{\rho - 2\lambda} + \frac{b_3 + c_2}{(\rho - 2\lambda)^2} + \frac{2c_3}{(\rho - 2\lambda)^3} \right) \Xi_2 + \left(\frac{c_2}{\rho - 2\lambda} + \frac{c_3}{(\rho - 2\lambda)^2} \right) \Xi_3,
\end{aligned}$$

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$$\begin{aligned} B_{22} &= \frac{c_3}{(\rho-2\lambda)^2} \Xi_2, \\ B_{31} &= \frac{a_3}{\rho-2\lambda} \Xi_1 + \left(\frac{b_3}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_2 + \frac{c_3}{\rho-2\lambda} \Xi_3, \\ B_{32} &= \frac{c_3}{\rho-2\lambda} \Xi_2. \end{aligned}$$

The case (iii-2):

$$(5.22) \quad c(t) = e^{(\rho-\lambda)t} M^{-1} t P^{-1} \begin{pmatrix} \Xi_1 \\ \Xi_2 - \Xi_1 t \\ \Xi_3 - \Xi_2 t + \frac{1}{2} \Xi_1 t^2 \end{pmatrix},$$

$$(5.23) \quad x(t) = P \begin{pmatrix} (C_{11} - C_{12}t + C_{13}t^2)e^{(\rho-\lambda)t} + (\delta_1 + \delta_2 t + \frac{1}{2}\delta_3 t^2)e^{\lambda t} - \frac{\kappa_1}{\lambda} + \frac{\kappa_2}{\lambda^2} - \frac{\kappa_3}{\lambda^3} \\ (C_{21} - C_{22}t + C_{23}t^2)e^{(\rho-\lambda)t} + (\delta_2 + \delta_3 t)e^{\lambda t} - \frac{\kappa_2}{\lambda} + \frac{\kappa_3}{\lambda^2} \\ (C_{31} - C_{32}t + C_{33}t^2)e^{(\rho-\lambda)t} + \delta_3 e^{\lambda t} - \frac{\kappa_3}{\lambda} \end{pmatrix};$$

where δ_μ ($\mu = 1, 2, 3$) are some constants and $C_{\mu\nu}$ ($\mu, \nu = 1, 2, 3$) are the constants of the following form with the constants Ξ_1 , Ξ_2 and Ξ_3 , while $\rho \neq 2\lambda$:

$$\begin{aligned} C_{11} &= \left(\frac{a_1}{\rho-2\lambda} + \frac{a_2+b_1}{(\rho-2\lambda)^2} + \frac{a_3+2b_2+c_1}{(\rho-2\lambda)^3} + \frac{3(b_3+c_2)}{(\rho-2\lambda)^4} + \frac{6c_3}{(\rho-2\lambda)^5} \right) \Xi_1 \\ &\quad + \left(\frac{b_1}{\rho-2\lambda} + \frac{b_2+c_1}{(\rho-2\lambda)^2} + \frac{b_3+2c_2}{(\rho-2\lambda)^3} + \frac{3c_3}{(\rho-2\lambda)^4} \right) \Xi_2 + \left(\frac{c_1}{\rho-2\lambda} + \frac{c_2}{(\rho-2\lambda)^2} + \frac{c_3}{(\rho-2\lambda)^3} \right) \Xi_3, \\ C_{12} &= \left(\frac{2(a_2+b_1)}{\rho-2\lambda} + \frac{b_2+c_1}{(\rho-2\lambda)^2} + \frac{b_3+2c_2}{(\rho-2\lambda)^3} + \frac{3c_3}{(\rho-2\lambda)^4} \right) \Xi_1 \\ &\quad + \left(\frac{2(b_2+c_1)}{\rho-2\lambda} + \frac{c_2}{(\rho-2\lambda)^2} + \frac{c_3}{(\rho-2\lambda)^3} \right) \Xi_2 + \frac{2c_2}{\rho-2\lambda} \Xi_3, \\ C_{13} &= \frac{1}{2} \left(\frac{c_1}{\rho-2\lambda} + \frac{c_2}{(\rho-2\lambda)^2} + \frac{c_3}{(\rho-2\lambda)^3} \right) \Xi_1, \\ C_{21} &= \left(\frac{a_2}{\rho-2\lambda} + \frac{a_3+b_2}{(\rho-2\lambda)^2} + \frac{2b_3+c_2}{(\rho-2\lambda)^3} + \frac{3c_3}{(\rho-2\lambda)^4} \right) \Xi_1 \\ &\quad + \left(\frac{b_2}{\rho-2\lambda} + \frac{b_3+c_2}{(\rho-2\lambda)^2} + \frac{2c_3}{(\rho-2\lambda)^3} \right) \Xi_2 + \left(\frac{c_2}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_3, \\ C_{22} &= \left(\frac{a_3+b_2}{\rho-2\lambda} + \frac{b_3+c_2}{(\rho-2\lambda)^2} + \frac{2c_3}{(\rho-2\lambda)^3} \right) \Xi_1 + \left(\frac{b_3+c_2}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_2 + \frac{c_3}{\rho-2\lambda} \Xi_3, \\ C_{23} &= \frac{1}{2} \left(\frac{c_2}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_1, \\ C_{31} &= \left(\frac{a_3}{\rho-2\lambda} + \frac{b_3}{(\rho-2\lambda)^2} + \frac{c_3}{(\rho-2\lambda)^3} \right) \Xi_1 + \left(\frac{b_3}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_2 + \frac{c_3}{\rho-2\lambda} \Xi_3, \\ C_{32} &= \left(\frac{b_3}{\rho-2\lambda} + \frac{c_3}{(\rho-2\lambda)^2} \right) \Xi_1 + \frac{c_3}{\rho-2\lambda} \Xi_2, \\ C_{33} &= \frac{c_3}{2(\rho-2\lambda)} \Xi_1. \end{aligned}$$

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The case (iv):

$$(5.24) \quad \mathbf{c}(t) = M^{-1} t P^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ (\Xi_2 \sin \theta t + \Xi_3 \cos \theta t) e^{(\rho-\lambda)t} \\ (\Xi_2 \cos \theta t - \Xi_3 \sin \theta t) e^{(\rho-\lambda)t} \end{pmatrix},$$

$$(5.25) \quad \mathbf{x}(t) = PR \begin{pmatrix} \frac{a_1}{\rho-2\lambda_1} \Xi_1 e^{(\rho-2\lambda_1)t} + \delta_1 \\ (B_1 \sin^2 \theta t + B_2 \sin \theta t \cos \theta t + B_3 \cos^2 \theta t + B_4) e^{(\rho-2\lambda)t} + \delta_2 \\ (C_1 \sin^2 \theta t + C_2 \sin \theta t \cos \theta t + C_3 \cos^2 \theta t + C_4) e^{(\rho-2\lambda)t} + \delta_3 \end{pmatrix} \\ + P \begin{pmatrix} (A_1 \sin \theta t + A_2 \cos \theta t) e^{(\rho-\lambda)t} - \frac{\kappa_1}{\lambda_1} \\ B_5 e^{(\rho-\lambda_1)t} + B_6 \\ C_5 e^{(\rho-\lambda_1)t} + C_6 \end{pmatrix};$$

where R is the matrix:

$$R = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda t} \cos \theta t & e^{\lambda t} \sin \theta t \\ 0 & -e^{\lambda t} \sin \theta t & e^{\lambda t} \cos \theta t \end{pmatrix},$$

and moreover δ_μ ($\mu = 1, 2, 3$) are some constants and A_i , B_j , C_j ($i = 1, 2; j = 1, 2, \dots, 6$) are the constants of the following form with the constants Ξ_1 , Ξ_2 and Ξ_3 , while $\rho \neq 2\lambda$ and $\rho \neq 2\lambda_1$:

$$\begin{aligned} A_1 &= \frac{(\rho-\lambda-\lambda_1)(b_1\Xi_2-c_1\Xi_3)+\theta(b_1\Xi_3+c_1\Xi_2)}{(\rho-\lambda-\lambda_1)^2+\theta^2}, \\ A_2 &= \frac{(\rho-\lambda-\lambda_1)(c_1\Xi_2+b_1\Xi_3)-\theta(b_1\Xi_2-c_1\Xi_3)}{(\rho-\lambda-\lambda_1)^2+\theta^2}, \\ B_1 &= \frac{(\rho-2\lambda)(c_3\Xi_3-b_3\Xi_2)+\theta[(b_2-c_3)\Xi_2-(b_3+c_2)\Xi_3]}{(\rho-2\lambda)^2+4\theta^2}, \\ B_2 &= \frac{(\rho-2\lambda)[(b_2-c_3)\Xi_2-(b_3+c_2)\Xi_3]+2\theta[(b_2-c_3)\Xi_3+(b_3+c_2)\Xi_2]}{(\rho-2\lambda)^2+4\theta^2}, \\ B_3 &= \frac{(\rho-2\lambda)(c_2\Xi_2+b_2\Xi_3)-\theta[(b_2-c_3)\Xi_2-(b_3+c_2)\Xi_3]}{(\rho-2\lambda)^2+4\theta^2}, \\ B_4 &= \frac{4\theta^2/(\rho-2\lambda)}{(\rho-2\lambda)^2+4\theta^2}, \\ B_5 &= \frac{(\rho-\lambda-\lambda_1)a_2\Xi_1+\theta a_3\Xi_1}{(\rho-\lambda-\lambda_1)^2+\theta^2}, \\ B_6 &= \frac{\kappa_3\theta-\kappa_2\lambda}{\lambda^2+\theta^2}, \end{aligned}$$

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$$\begin{aligned}
C_1 &= \frac{(\rho-2\lambda)(b_2\Xi_2-c_2\Xi_3)+\theta[(b_3+c_2)\Xi_2+(b_2-c_3)\Xi_3]}{(\rho-2\lambda)^2+4\theta^2}, \\
C_2 &= \frac{(\rho-2\lambda)[(b_3+c_2)\Xi_2+(b_2-c_3)\Xi_3]+2\theta[(c_3-b_2)\Xi_2+(b_3+c_2)\Xi_3]}{(\rho-2\lambda)^2+4\theta^2}, \\
C_3 &= \frac{(\rho-2\lambda)(c_3\Xi_2+b_3\Xi_3)-\theta[(b_3+c_2)\Xi_2+(b_2-c_3)\Xi_3]}{(\rho-2\lambda)^2+4\theta^2}, \\
C_4 &= \frac{4\theta^2/(\rho-2\lambda)}{(\rho-2\lambda)^2+4\theta^2}, \\
C_5 &= \frac{(\rho-\lambda-\lambda_1)a_3\Xi_1-\theta a_2\Xi_1}{(\rho-\lambda-\lambda_1)^2+\theta^2}, \\
C_6 &= -\frac{\kappa_3\lambda+\kappa_2\theta}{\lambda^2+\theta^2}.
\end{aligned}$$

Theorem 5.2. *In the maximizing problem of (1), let the utility function $U(c^1, c^2, c^3)$ with the concavity be given as (5.12) and the consumptions $c^\mu = g^\mu(x^1, x^2, x^3) - n_\nu^\mu x^\nu - \dot{x}^\mu$ grow externally under the linear production technologies $g^\mu = \alpha_\nu^\mu x^\nu + \beta^\mu$. Then, in the case of finite horizon $T < \infty$, the optimal paths $c(t)$ and $x(t)$ are determined completely as (5.13) and (5.15) in the case (i) with $\rho \neq \lambda_\mu + \lambda_\nu$ ($\mu, \nu = 1, 2, 3$); as (5.16) and (5.17) in the case (ii-1) with $\rho \neq \lambda_i + \lambda_j$ ($i, j = 1, 2$); as (5.18) and (5.19) in the case (ii-2) with $\rho \neq \lambda_i + \lambda_j$ ($i, j = 1, 2$); as (5.20) and (5.21) in the case (iii-1) with $\rho \neq 2\lambda$; as (5.22) and (5.23) in the case (iii-2) with $\rho \neq 2\lambda$; as (5.24) and (5.25) in the case (iv) with $\rho \neq 2\lambda$ and $\rho \neq 2\lambda_1$.*

Remark. The interested reader will find the optimal paths when ρ takes the exceptional values in the theorem 1.

The case (i): To look the case of infinite horizon $T = \infty$, the relation (4.4b): $\pi_\mu = -e^{-\rho t}U_\mu$ are written by (5.10a) as

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = -{}^tP^{-1} \begin{pmatrix} \Xi_1 e^{-\lambda_1 t} \\ \Xi_2 e^{-\lambda_2 t} \\ \Xi_3 e^{-\lambda_3 t} \end{pmatrix},$$

and then used, together with (5.15), for the transversality condition $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$. In view of the resulting appearance of $\pi_\mu x^\mu$:

$$\begin{aligned}
\pi_\mu x^\mu &= -\frac{a_1\Xi_1^2}{\rho-2\lambda_1}e^{(\rho-2\lambda_1)t} - \frac{b_2\Xi_2^2}{\rho-2\lambda_2}e^{(\rho-2\lambda_2)t} - \frac{c_3\Xi_3^2}{\rho-2\lambda_3}e^{(\rho-2\lambda_3)t} \\
&\quad - \frac{(a_2+b_1)\Xi_1\Xi_2}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_1-\lambda_2)t} - \frac{(b_3+c_2)\Xi_2\Xi_3}{\rho-\lambda_2-\lambda_3}e^{(\rho-\lambda_2-\lambda_3)t} - \frac{(c_1+a_3)\Xi_1\Xi_3}{\rho-\lambda_1-\lambda_3}e^{(\rho-\lambda_1-\lambda_3)t} \\
&\quad + \frac{\kappa_1\Xi_1}{\lambda_1}e^{-\lambda_1 t} + \frac{\kappa_2\Xi_2}{\lambda_2}e^{-\lambda_2 t} + \frac{\kappa_3\Xi_3}{\lambda_3}e^{-\lambda_3 t} - (\delta_1\Xi_1 + \delta_2\Xi_2 + \delta_3\Xi_3),
\end{aligned}$$

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we can find the optimal paths satisfying the transversality condition. For example, let $0 < 2\lambda_1 \leq \rho < 2\lambda_2$ and $0 < 2\lambda_1 \leq \rho < 2\lambda_3$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\Xi_1 = 0$ in the coefficient of $e^{(\rho-2\lambda_1)t}$; so that $\delta_2\Xi_2 + \delta_3\Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (5.13) and (5.15) take the forms respectively:

$$(5.13)' \quad c(t) = M^{-1} {}^tP^{-1} \begin{pmatrix} 0 \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix},$$

$$(5.15)' \quad x(t) = P \begin{pmatrix} \frac{b_1\Xi_2}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_2)t} + \frac{c_1\Xi_3}{\rho-\lambda_1-\lambda_3} e^{(\rho-\lambda_3)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ \frac{b_2\Xi_2}{\rho-2\lambda_2} e^{(\rho-\lambda_2)t} + \frac{c_2\Xi_3}{\rho-\lambda_2-\lambda_3} e^{(\rho-\lambda_3)t} - \frac{\kappa_2}{\lambda_2} \\ \frac{b_3\Xi_2}{\rho-\lambda_2-\lambda_3} e^{(\rho-\lambda_2)t} + \frac{c_3\Xi_3}{\rho-2\lambda_3} e^{(\rho-\lambda_3)t} - \frac{\kappa_3}{\lambda_3} \end{pmatrix}.$$

Similarly, we can have the optimal paths satisfying the transversality condition for the case (ii-1), (ii-2), (iii-1), (iii-2) and (iv).

The case (ii-1): By (5.10b) and (5.17), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -\frac{a_1\Xi_1^2}{\rho-2\lambda_1} e^{(\rho-2\lambda_1)t} - \frac{b_2\Xi_2^2 + c_3\Xi_3^2 + (b_3+c_2)\Xi_2\Xi_3}{\rho-2\lambda_2} e^{(\rho-2\lambda_2)t} - \frac{(a_2+b_1)\Xi_1\Xi_2 + (c_1+a_3)\Xi_1\Xi_3}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_1-\lambda_2)t} \\ & + \frac{\kappa_1\Xi_1}{\lambda_1} e^{-\lambda_1 t} + \frac{\kappa_2\Xi_2 + \kappa_3\Xi_3}{\lambda_2} e^{-\lambda_2 t} - (\delta_1\Xi_1 + \delta_2\Xi_2 + \delta_3\Xi_3), \end{aligned}$$

in which, let $0 < 2\lambda_1 \leq \rho < 2\lambda_2$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\Xi_1 = 0$ in the coefficient of $e^{(\rho-2\lambda_1)t}$; so that $\delta_2\Xi_2 + \delta_3\Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (5.16) and (5.17) take the forms respectively:

$$(5.16)' \quad c(t) = M^{-1} {}^tP^{-1} \begin{pmatrix} 0 \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

$$(5.17)' \quad x(t) = P \begin{pmatrix} \frac{b_1\Xi_2 + c_1\Xi_3}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_2)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ \frac{b_2\Xi_2 + c_2\Xi_3}{\rho-2\lambda_2} e^{(\rho-\lambda_2)t} - \frac{\kappa_2}{\lambda_2} \\ \frac{b_3\Xi_2 + c_3\Xi_3}{\rho-2\lambda_2} e^{(\rho-\lambda_2)t} - \frac{\kappa_3}{\lambda_2} \end{pmatrix}.$$

The case (ii-2): By (5.10c) and (5.19), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -[A_{11}\Xi_1 + (A_{21} + A_{31})\Xi_2 - A_{31}\Xi_3]e^{(\rho-2\lambda_1)t} \\ & -[A_{12}\Xi_1 + (A_{22} + A_{32})\Xi_2 + (A_{13}\Xi_1 + (A_{23} + A_{33})\Xi_2 - A_{32}\Xi_3)t - A_{33}\Xi_3 t^2]e^{(\rho-2\lambda_2)t} \\ & + \frac{\kappa_1\Xi_1}{\lambda_1} e^{-\lambda_1 t} + [(\frac{\kappa_2 + \kappa_3}{\lambda_2} - \frac{\kappa_3}{\lambda_2^2})\Xi_2 - \frac{\kappa_3\Xi_3}{\lambda_2} t]e^{-\lambda_2 t} - (\delta_3\Xi_2 - \delta_3\Xi_3)t - \delta_1\Xi_1 - (\delta_2 + \delta_3)\Xi_2. \end{aligned}$$

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So, let $0 < \rho < 2\lambda_1$ and $0 < \rho < 2\lambda_2$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\delta_3 \Xi_2 - \delta_3 \Xi_3 = 0$ and $\delta_1 \Xi_1 + (\delta_2 + \delta_3) \Xi_2 = 0$. Therefore, by putting $\delta_1 = 0$, $\Xi_2 = 0$ and $\Xi_3 = 0$, the optimal paths (5.18) and (5.19) take the forms respectively:

$$(5.18)' \quad c(t) = M^{-1} {}^t P^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ 0 \\ 0 \end{pmatrix},$$

$$(5.19)' \quad x(t) = P \begin{pmatrix} A_{11} e^{(\rho-\lambda_1)t} - \frac{\kappa_1}{\lambda_1} \\ A_{21} e^{(\rho-\lambda_1)t} + (\delta_2 + \delta_3 t) e^{\lambda_2 t} - \frac{\kappa_2}{\lambda_2} + \frac{\kappa_3}{\lambda_2^2} \\ A_{31} e^{(\rho-\lambda_1)t} + \delta_3 e^{\lambda_2 t} - \frac{\kappa_3}{\lambda_2} \end{pmatrix};$$

where A_{11} , A_{21} and A_{31} are the constants given before, while in which Ξ_2 and Ξ_3 are placed as $\Xi_2 = \Xi_3 = 0$.

The case (iii-1): By (5.10d) and (5.21), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -[B_{11} \Xi_1 + B_{21} \Xi_2 + B_{31} \Xi_3 + (B_{12} \Xi_1 + (B_{22} - B_{31}) \Xi_2 + B_{32} \Xi_3) t - B_{32} \Xi_2 t^2] e^{(\rho-2\lambda)t} \\ & + \frac{1}{\lambda} [\kappa_1 \Xi_1 + (\kappa_2 - \kappa_3) \Xi_2 + \kappa_3 \Xi_3 - \kappa_3 \Xi_2 t] e^{-\lambda t} - (\delta_1 \Xi_1 + \delta_2 \Xi_2 + \delta_3 \Xi_3), \end{aligned}$$

in which, let $0 < \rho < 2\lambda$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\delta_1 \Xi_1 + \delta_2 \Xi_2 + \delta_3 \Xi_3 = 0$. Therefore, by putting $\delta_1 = 0$, $\Xi_2 = 0$ and $\Xi_3 = 0$, the optimal paths (5.20) and (5.21) take the forms respectively:

$$(5.20)' \quad c(t) = M^{-1} {}^t P^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda)t} \\ 0 \\ 0 \end{pmatrix},$$

$$(5.21)' \quad x(t) = P \begin{pmatrix} B_{11} e^{(\rho-\lambda)t} - \frac{\kappa_1}{\lambda} \\ B_{21} e^{(\rho-\lambda)t} + (\delta_2 + \delta_3 t) e^{\lambda_2 t} + \frac{\kappa_3}{\lambda^2} - \frac{\kappa_2}{\lambda} \\ B_{31} e^{(\rho-\lambda)t} + \delta_3 e^{\lambda t} - \frac{\kappa_3}{\lambda} \end{pmatrix};$$

where B_{11} , B_{21} and B_{31} are the constants given before, while in which Ξ_2 and Ξ_3 are placed as $\Xi_2 = \Xi_3 = 0$.

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The case (iii-2): By (5.10e) and (5.23), $\pi_\mu x^\mu$ is written as

$$\begin{aligned}\pi_\mu x^\mu = & -[\Xi_1(C_{11} + C_{12}t + C_{13}t^2) + (\Xi_2 - \Xi_1 t)(C_{21} + C_{22}t + C_{23}t^2) \\ & + (\tfrac{1}{2}\Xi_1 t^2 - \Xi_2 t + \Xi_3)(C_{31} + C_{32}t + C_{33}t^2)]e^{(\rho-2\lambda)t} \\ & - [(-\tfrac{\kappa_1}{\lambda} + \tfrac{\kappa_2}{\lambda^2} - \tfrac{\kappa_3}{\lambda^3})\Xi_1 + (-\tfrac{\kappa_2}{\lambda} + \tfrac{\kappa_3}{\lambda^2})(\Xi_2 - \Xi_1 t) - \tfrac{\kappa_3}{\lambda}(\tfrac{1}{2}\Xi_1 t^2 - \Xi_2 t + \Xi_3)]e^{-\lambda t} \\ & - (\delta_1 \Xi_1 + \delta_2 \Xi_2 + \delta_3 \Xi_3),\end{aligned}$$

in which, let $0 < \rho < 2\lambda$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\delta_1 \Xi_1 + \delta_2 \Xi_2 + \delta_3 \Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (5.22) and (5.23) take the forms respectively:

$$(5.22)' \quad c(t) = e^{(\rho-\lambda)t} M^{-1} t P^{-1} \begin{pmatrix} 0 \\ \Xi_2 \\ \Xi_3 - \Xi_2 t \end{pmatrix},$$

$$(5.23)' \quad x(t) = P \begin{pmatrix} (C_{11} + C_{12}t)e^{(\rho-\lambda)t} + \delta_1 e^{\lambda t} - \tfrac{\kappa_1}{\lambda} + \tfrac{\kappa_2}{\lambda^2} - \tfrac{\kappa_3}{\lambda^3} \\ (C_{21} + C_{22}t)e^{(\rho-\lambda)t} - \tfrac{\kappa_2}{\lambda} + \tfrac{\kappa_3}{\lambda^2} \\ (C_{31} + C_{32}t)e^{(\rho-\lambda)t} - \tfrac{\kappa_3}{\lambda} \end{pmatrix};$$

where C_{ij} ($i = 1, 2, 3; j = 1, 2$) are the constants given before, while in which Ξ_1 is placed as $\Xi_1 = 0$.

The case (iv): By (5.10f) and (5.25), $\pi_\mu x^\mu$ is written as

$$\begin{aligned}\pi_\mu x^\mu = & -\tfrac{a_1}{\rho-2\lambda_1} \Xi_1^2 e^{(\rho-2\lambda_1)t} - [(B_1 \Xi_3 + C_1 \Xi_2) \sin^2 \theta t \\ & + (B_2 \Xi_3 + C_2 \Xi_2) \sin \theta t \cos \theta t + (B_3 \Xi_3 + C_3 \Xi_2) \cos^2 \theta t + B_4 \Xi_3 + C_4 \Xi_2] e^{(\rho-2\lambda)t} \\ & - [(A_1 \Xi_1 + B_5 \Xi_2 - C_5 \Xi_3) \sin \theta t + (A_2 \Xi_1 + C_5 \Xi_2 + B_5 \Xi_3) \cos \theta t] e^{(\rho-\lambda-\lambda_1)t} \\ & + \tfrac{\kappa_1 \Xi_1}{\lambda_1} e^{-\lambda_1 t} - [(B_6 \Xi_2 - C_6 \Xi_3) \sin \theta t + (C_6 \Xi_2 + B_6 \Xi_3) \cos \theta t] e^{-\lambda t} \\ & - (\delta_1 \Xi_1 + \delta_3 \Xi_2 + \delta_2 \Xi_3),\end{aligned}$$

in which, let $0 < \rho < 2\lambda_1$ and $0 < \rho < 2\lambda$. Then $\lim_{t \rightarrow \infty} \pi_\mu x^\mu = 0$ requires $\delta_1 \Xi_1 + \delta_3 \Xi_2 + \delta_2 \Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (5.24) and (5.25) take the forms respectively:

$$(5.24)' \quad c(t) = M^{-1} t P^{-1} \begin{pmatrix} 0 \\ (\Xi_2 \sin \theta t + \Xi_3 \cos \theta t) e^{(\rho-\lambda)t} \\ (\Xi_2 \cos \theta t - \Xi_3 \sin \theta t) e^{(\rho-\lambda)t} \end{pmatrix},$$

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$$(5.25)' \quad \begin{aligned} \mathbf{x}(t) = PR & \begin{pmatrix} 0 \\ (B_1 \sin^2 \theta t + B_2 \sin \theta t \cos \theta t + B_3 \cos^2 \theta t + B_4)e^{(\rho-2\lambda)t} \\ (C_1 \sin^2 \theta t + C_2 \sin \theta t \cos \theta t + C_3 \cos^2 \theta t + C_4)e^{(\rho-2\lambda)t} \end{pmatrix} \\ & + P \begin{pmatrix} (A_1 \sin \theta t + A_2 \cos \theta t)e^{(\rho-\lambda)t} + \delta_1 e^{\lambda_1 t} - \frac{\kappa_1}{\lambda_1} \\ B_6 \\ C_6 \end{pmatrix}; \end{aligned}$$

where A_i , B_j and C_j ($i = 1, 2; j = 1, 2, 3, 4, 6$) are the constants given before, while in which Ξ_1 is placed as $\Xi_1 = 0$.

Theorem 5.3. *In the case of infinite horizon $T = \infty$, there exist the feasible optimal paths (5.13)' and (5.15)' with $0 < 2\lambda_1 \leq \rho < 2 \min(\lambda_2, \lambda_3)$ for the case (i); (5.16)' and (5.17)' with $0 < 2\lambda_1 \leq \rho < 2\lambda_2$ for the case (ii-1); (5.18)' and (5.19)' with $0 < \rho < 2 \min(\lambda_1, \lambda_2)$ for the case (ii-2); (5.20)' and (5.21)' with $0 < \rho < 2\lambda$ for the case (iii-1); (5.22)' and (5.23)' with $0 < \rho < 2\lambda$ for the case (iii-2); (5.24)' and (5.25)' with $0 < \rho < 2 \min(\lambda, \lambda_1)$ for the case (iv). Particularly, let $0 < 2\lambda_1 \leq \rho < \min(\lambda_2, \lambda_3)$ and $\delta_1 = 0$ in the determined optimal paths $\mathbf{c}(t)$ and $\mathbf{x}(t)$ for the case (i); $0 < 2\lambda_1 \leq \rho < \lambda_2$ and $\delta_1 = 0$ in those for the case (ii-1); $0 < \rho < \min(\lambda_1, 2\lambda_2)$ and $\delta_2 = \delta_3 = 0$ in those for the case (ii-2); $0 < \rho < \lambda$ and $\delta_2 = \delta_3 = 0$ in those for the case (iii-1); $0 < \rho < \lambda$ and $\delta_1 = 0$ in those for the cases (iii-2) and (iv). Then, as $t \rightarrow \infty$, the optimal paths $\mathbf{c}(t)$ and $\mathbf{x}(t)$ converge to zero and some constant vectors, respectively.*

The feasible optimal path $\mathbf{x}(t)$ of (5.25)' goes into details with $\delta_1 = 0$. In view of the identities $B_3 - B_1 = C_2$ and $C_1 - C_3 = B_2$, since

$$\begin{aligned} & \begin{pmatrix} B_1 \sin^2 \theta t + B_2 \sin \theta t \cos \theta t + B_3 \cos^2 \theta t + B_4 \\ C_1 \sin^2 \theta t + C_2 \sin \theta t \cos \theta t + C_3 \cos^2 \theta t + C_4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} B_3 - B_1 & B_2 \\ C_3 - C_1 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \frac{B_1+B_3}{2} + B_4 \\ \frac{C_1+C_3}{2} + C_4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \frac{B_1+B_3}{2} + B_4 \\ \frac{C_1+C_3}{2} + C_4 \end{pmatrix}, \end{aligned}$$

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the appearance of (5.25)' can be arranged as

$$(5.26) \quad \begin{cases} y^1 = (A_1 \sin \theta t + A_2 \cos \theta t)e^{(\rho-\lambda)t} \\ \begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} y^2 \\ y^3 \end{pmatrix} = \frac{1}{2}e^{(\rho-\lambda)t} \left(\begin{pmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{pmatrix} \begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} n_2 \\ n_3 \end{pmatrix} \right), \end{cases}$$

where $n_2 = B_1 + B_3 + 2B_4$, $n_3 = C_1 + C_3 + 2C_4$ and the vector $\mathbf{y}(t) = {}^t(y^1 \ y^2 \ y^3)$ is given by

$$(5.27) \quad \mathbf{y}(t) = P^{-1}\mathbf{x}(t) - \begin{pmatrix} -\frac{\kappa_1}{\lambda_1} \\ B_6 \\ C_6 \end{pmatrix}.$$

The constants B_2 and C_2 are written in the matrix form

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \Delta \begin{pmatrix} \Xi_2 \\ \Xi_3 \end{pmatrix},$$

where the matrix

$$\Delta = \frac{1}{(\rho-2\lambda)^2 + 4\theta^2} \begin{pmatrix} (\rho-2\lambda)(b_2-c_3) + 2\theta(b_3+c_2) & -(\rho-2\lambda)(b_3+c_2) + 2\theta(b_2-c_3) \\ (\rho-2\lambda)(b_3+c_2) - 2\theta(b_2-c_3) & (\rho-2\lambda)(b_2-c_3) + 2\theta(b_3+c_2) \end{pmatrix}$$

has the determinant

$$|\Delta| = \frac{[(\rho-2\lambda)(b_2-c_3) + 2\theta(b_3+c_2)]^2 + [(\rho-2\lambda)(b_3+c_2) - 2\theta(b_2-c_3)]^2}{[(\rho-2\lambda)^2 + 4\theta^2]^2},$$

which vanishes if and only if

$$\begin{cases} (\rho-2\lambda)(b_2-c_3) + 2\theta(b_3+c_2) = 0, \\ (\rho-2\lambda)(b_3+c_2) - 2\theta(b_2-c_3) = 0, \end{cases} \quad \text{i.e.,} \quad \begin{pmatrix} \rho-2\lambda & 2\theta \\ -2\theta & \rho-2\lambda \end{pmatrix} \begin{pmatrix} b_2-c_3 \\ b_3+c_2 \end{pmatrix} = \mathbf{0}.$$

Therefore, since $\theta \neq 0$, i.e.,

$$\begin{vmatrix} \rho-2\lambda & 2\theta \\ -2\theta & \rho-2\lambda \end{vmatrix} = (\rho-2\lambda)^2 + 4\theta^2 \neq 0,$$

the determinant $|\Delta|$ vanishes if and only if $b_2 - c_3 = 0$ and $b_3 + c_2 = 0$. So, assuming $b_2 - c_3 \neq 0$ or $b_3 + c_2 \neq 0$, choose the constants Ξ_2 and Ξ_3 such that ${}^t(\Xi_2 \ \Xi_3) \neq \mathbf{0}$. Then ${}^t(B_2 \ C_2) \neq \mathbf{0}$ so that

$$(5.28) \quad \begin{vmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{vmatrix} = B_2^2 + C_2^2 \neq 0.$$

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Therefore, the vector ${}^t(\cos 2\theta t \ \sin 2\theta t)$ in (5.26) is written as

$$\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} = \frac{2}{B_2^2 + C_2^2} e^{-(\rho - \lambda)t} \begin{pmatrix} C_2 & -B_2 \\ B_2 & C_2 \end{pmatrix} \begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix} \begin{pmatrix} y^2 \\ y^3 \end{pmatrix} - \begin{pmatrix} \ell_2 \\ \ell_3 \end{pmatrix},$$

where ℓ_i ($i = 2, 3$) are the following constants, respectively:

$$\ell_2 = \frac{(B_1 + B_3 + 2B_4)C_2 - (C_1 + C_3 + 2C_4)B_2}{B_2^2 + C_2^2}, \quad \ell_3 = \frac{(B_1 + B_3 + 2B_4)B_2 + (C_1 + C_3 + 2C_4)C_2}{B_2^2 + C_2^2},$$

which can be put as $\ell_2 = \ell_3 = 0$ by a suitable choice of B_4 and C_4 (this fact is guaranteed by (5.28)). Thus, the identity $\cos^2 2\theta t + \sin^2 2\theta t = 1$ yields the equation of the spiral

$$(5.29) \quad (y^2)^2 + (y^3)^2 = \frac{B_2^2 + C_2^2}{4} e^{2(\rho - \lambda)t}.$$

In conclusion, we have the following result.

Theorem 5.4. *Let a_μ , b_μ and c_μ ($\mu = 1, 2, 3$) in the matrix in (5.14) satisfy $b_2 - c_3 \neq 0$ or $b_3 + c_2 \neq 0$. Then, for (iv) in the case of infinite horizon $T = \infty$, the feasible optimal path $\mathbf{x}(t)$ of the form (5.25)' with $\delta_1 = 0$ is transformed by (5.27), under a suitable choice of the constants B_4 and C_4 , to $\mathbf{y}(t) = {}^t(y^1 \ y^2 \ y^3)$ in which y^2 and y^3 satisfy the equation of the spiral (5.29).*

6 External two-sector growth model with utility polynomial

6.1 Introduction

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with the Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [32]) and applied it to various economic growth models (Mimura and Nôno [34]; Mimura, Fujiwara and Nôno [29], [30]; Fujiwara, Mimura and Nôno [11], [12], [13]) to discover new economic conservation laws including non-Noether ones. Particularly in [11], the procedure was so reformed as to make an effective application to more general neoclassical optimal growth models. And in [13], by a reduction of the theorem 1.1 in [11], the application was pursued to a one sector model of Ramsey type (Ramsey [38]) with a constant discount rate relative to a utility (welfare) of consumptions, and then the model was generalized in an external two-sector version with linear constraints.

In this section, we inquire further into an application of the theorem 1.1 in [11] to the two-sector growth model with a utility function of state and control variables, while in ([12], [14]) the application of the theorem was made with a utility of homogeneous second order polynomial of control variables only. In 6.2, the theorem is reviewed briefly. In 6.3, there is given a utility of non-homogeneous second order polynomial of a state variable and two control variables. Then, to discover conserved quantities, the theorem is applied in the maximizing problem for an integration (over a period of time) of the utility function with a constant discount rate ρ ($\rho \geq 0$) under a linear constraint with respect to the variables. Moreover, leaving the linear constraint as it is, more general utility function is obtained in an inverse problem which gives rise to the obtained conserved quantities. Finally, in 6.4, through the obtained conserved quantities in 6.3, the optimal paths are established completely in finite and infinite horizons. The determined optimal paths will illustrate the Pareto-efficient steady-state (Fershtman and Nitzan [2]).

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

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6.2 New derivation of conservation laws

We discussed the following extremal (maximizing or minimizing) problem for the integration over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(6.1) \quad \int_0^T e^{-\rho t} U(x, u) dt,$$

under constraints

$$(6.2) \quad \dot{x}^\mu = f^\mu(x, u),$$

where $x = (x^\mu(t))$, $u = (u^\sigma(t))$ ($\mu = 1, \dots, k$; $\sigma = 1, \dots, \ell$) and ρ ($\rho \geq 0$) is a constant. In the multiplier technique to the problem, the Lagrangian is given by (π_μ are the multipliers):

$$(6.3) \quad L = e^{-\rho t} U + \pi_\mu (\dot{x}^\mu - f^\mu),$$

whose Euler-Lagrange equations consist of (6.2) and

$$(6.4a) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial x^\mu} = \dot{\pi}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \pi_\nu,$$

$$(6.4b) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}^\sigma} \right) - \frac{\partial L}{\partial u^\sigma} = 0 : \quad e^{-\rho t} \frac{\partial U}{\partial u^\sigma} = \frac{\partial f^\mu}{\partial u^\sigma} \pi_\mu.$$

A conserved quantity (first integral) for the extremal problem is a quantity Ω of the variables $\dot{\pi}_\mu$, \dot{x}^μ , \dot{u}^σ , π_μ , x^μ , u^σ and t whose total time derivative vanishes ($\dot{\Omega} = 0$: conservation law) on the optimal paths, i.e., on solutions to the relating Euler-Lagrange equations (6.2), (6.4a) and (6.4b).

Here let f^μ in (6.2) be linear in x^μ and u^σ . Then, for derivation of conserved quantities, the theorem 2.1 in 2.2 ([11], Theorem 1.1; [30], Theorem 1) is reformulated as follows:

For the Lagrangian L of (6.3), let the functions $(\xi_1^\alpha) = (\eta_\mu^1, \varphi_1^\mu, \tau_1^\sigma)$ and $(\xi_2^\alpha) = (\eta_\mu^2, \varphi_2^\mu, \tau_2^\sigma)$ of the variables $\dot{\pi}_\mu$, \dot{x}^μ , \dot{u}^σ , π_μ , x^μ , u^σ and t satisfy the equations

$$(6.5a) \quad \dot{\varphi}^\mu = \frac{\partial f^\mu}{\partial x^\nu} \varphi^\nu + \frac{\partial f^\mu}{\partial u^\sigma} \tau^\sigma,$$

$$(6.5b) \quad \dot{\eta}_\mu + \frac{\partial f^\nu}{\partial x^\mu} \eta_\nu = e^{-\rho t} \left(\frac{\partial^2 U}{\partial x^\nu \partial x^\mu} \varphi^\nu + \frac{\partial^2 U}{\partial u^\sigma \partial x^\mu} \tau^\sigma \right),$$

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$$(6.5c) \quad \frac{\partial f^\mu}{\partial u^\sigma} \eta_\mu = e^{-\rho t} \left(\frac{\partial^2 U}{\partial x^\mu \partial u^\sigma} \varphi^\mu + \frac{\partial^2 U}{\partial u^\omega \partial u^\sigma} \tau^\omega \right),$$

on the optimal paths for the extremal problem of (6.1) under the constraints (6.2). Then the following conserved quantity Ω is constructed:

$$(6.6) \quad \Omega = \eta_\mu^2 \varphi_1^\mu - \eta_\mu^1 \varphi_2^\mu.$$

Moreover, by substituting the solution $(\xi_1^\alpha) = (\eta_\mu, \varphi^\mu, \tau^\sigma) = (\dot{\pi}_\mu + \rho\pi_\mu, \dot{x}^\mu, u^\sigma)$ of (6.5a), (6.5b) and (6.5c), while (ξ_2^α) is left as $(\xi_2^\alpha) = (\xi^\alpha)$, the conserved quantity (6.6) is reduced to

$$(6.7) \quad \Omega = \dot{x}^\mu \eta_\mu - (\dot{\pi}_\mu + \rho\pi_\mu) \varphi^\mu.$$

6.3 Utility of second order polynomial and conservation laws

Generalizing the utility of homogeneous second order polynomial which was first introduced by Samuelson [41] (see also [12], [14]):

$$U(u^1, u^2) = -\frac{1}{2}(u^1)^2 - mu^1u^2 - \frac{1}{2}(u^2)^2 \quad (m: \text{const.}, -1 < m < 1),$$

the discussion begins with the utility polynomial

$$(6.8) \quad U(x, u^1, u^2) = a_1x + \frac{1}{2}a_2x^2 + b_1u^1 + b_2u^2 + \frac{1}{2}b_{11}(u^1)^2 + b_{12}u^1u^2 + \frac{1}{2}b_{22}(u^2)^2, \\ (a_i, b_i, b_{ij}: \text{const.}, i, j = 1, 2; a_1 > 0),$$

where U is also assumed to be provided with the concavity (see, e.g., [47]), i.e., the successive principal minors D_k ($k = 1, 2, 3$) of Hessian matrix of U satisfy $D_1 = a_2 < 0$, $D_2 = a_2b_{11} > 0$ and $D_3 = a_2(b_{11}b_{22} - b_{12}^2) < 0$. So it follows that

$$(6.9) \quad a_2 < 0, \quad b_{11} < 0, \quad b_{22} < 0, \quad b_{11}b_{22} - b_{12}^2 > 0.$$

Now we consider an extremal problem of (6.1) for the utility polynomial U of (6.8) under a constraint $\dot{x} = f(x, u^1, u^2)$, where

$$(6.10) \quad f(x, u^1, u^2) = \alpha x + \beta_1u^1 + \beta_2u^2 \quad (\alpha, \beta_i: \text{const.}, i = 1, 2; \alpha < 0).$$

Then, by (6.8) and (6.10), a part of Euler-Lagrange equations (6.4a) and (6.4b) are reduced to

$$(6.11a) \quad \dot{\pi} + \alpha\pi = e^{-\rho t}(a_2x + a_1),$$

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$$(6.11b) \quad \beta_1 \pi = e^{-\rho t} (b_{11} u^1 + b_{12} u^2 + b_1),$$

$$(6.11c) \quad \beta_2 \pi = e^{-\rho t} (b_{12} u^1 + b_{22} u^2 + b_2);$$

and the equations (6.5a), (6.5b) and (6.5c) are also to

$$(6.12a) \quad \dot{\varphi} = \alpha \varphi + \beta_1 \tau^1 + \beta_2 \tau^2,$$

$$(6.12b) \quad \dot{\eta} + \alpha \eta = a_2 e^{-\rho t} \varphi,$$

$$(6.12c) \quad \beta_1 \eta = e^{-\rho t} (b_{11} \tau^1 + b_{12} \tau^2),$$

$$(6.12d) \quad \beta_2 \eta = e^{-\rho t} (b_{12} \tau^1 + b_{22} \tau^2).$$

Since $b_{11}b_{22} - b_{12}^2 \neq 0$ in (6.9), it follows from (6.12c) and (6.12d) that

$$(6.13) \quad \tau^1 = n_1 e^{\rho t} \eta, \quad \tau^2 = n_2 e^{\rho t} \eta;$$

which are substituted for (6.12a) to see

$$(6.12a)' \quad \dot{\varphi} = \alpha \varphi + (n_1 \beta_1 + n_2 \beta_2) e^{\rho t} \eta,$$

where n_1 and n_2 are the constants:

$$n_1 = \frac{b_{22}\beta_1 - b_{12}\beta_2}{b_{11}b_{22} - b_{12}^2}, \quad n_2 = \frac{b_{11}\beta_2 - b_{12}\beta_1}{b_{11}b_{22} - b_{12}^2}.$$

By putting

$$(6.14) \quad \eta = g(t) e^{-\alpha t}, \quad \varphi = h(t) e^{\alpha t},$$

(6.12a)' and (6.12b) are arranged respectively as

$$(6.15) \quad \dot{h} = (n_1 \beta_1 + n_2 \beta_2) e^{(\rho-2\alpha)t} g, \quad \dot{g} = a_2 e^{-(\rho-2\alpha)t} h;$$

which are combined to obtain

$$(6.16) \quad \ddot{g} + (\rho - 2\alpha) \dot{g} - a_2 (\beta_1 n_1 + \beta_2 n_2) g = 0.$$

By means of (6.9) which guarantees

$$(6.17) \quad n_1 \beta_1 + n_2 \beta_2 = \frac{(b_{22}\beta_1 - b_{12}\beta_2)^2}{b_{22}(b_{11}b_{22} - b_{12}^2)} + \frac{\beta_2^2}{b_{22}} < 0,$$

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the discriminant D of the subsidiary equation of (6.16) becomes positive:

$$D = (2\alpha - \rho)^2 + 4a_2(n_1\beta_1 + n_2\beta_2) > 0.$$

So, the solutions $g(t)$ of (6.16) and then $h(t)$ of (6.15) are determined respectively as

$$\begin{aligned} g(t) &= C_1 e^{(2\alpha - \rho + \sqrt{D})t/2} + C_2 e^{(2\alpha - \rho - \sqrt{D})t/2} \\ h(t) &= \frac{2\alpha - \rho + \sqrt{D}}{2a_2} C_1 e^{(-2\alpha + \rho + \sqrt{D})t/2} + \frac{2\alpha - \rho - \sqrt{D}}{2a_2} C_2 e^{(-2\alpha + \rho - \sqrt{D})t/2} \end{aligned} \quad (C_1, C_2: \text{const.}).$$

Therefore, by (6.13) and (6.14), we have the following solution $(\eta, \varphi, \tau^1, \tau^2)$ of (6.12a)–(6.12d):

$$\begin{aligned} \eta &= C_1 e^{(-\rho + \sqrt{D})t/2} + C_2 e^{(-\rho - \sqrt{D})t/2}, \\ \varphi &= \frac{2\alpha - \rho + \sqrt{D}}{2a_2} C_1 e^{(\rho + \sqrt{D})t/2} + \frac{2\alpha - \rho - \sqrt{D}}{2a_2} C_2 e^{(\rho - \sqrt{D})t/2}, \\ \tau^1 &= n_1 (C_1 e^{(\rho + \sqrt{D})t/2} + C_2 e^{(\rho - \sqrt{D})t/2}), \\ \tau^2 &= n_2 (C_1 e^{(\rho + \sqrt{D})t/2} + C_2 e^{(\rho - \sqrt{D})t/2}); \end{aligned}$$

in which η and φ are used to construct the conserved quantity Ω of the form (6.7):

$$\begin{aligned} \Omega &= C_1 \left(\dot{x} e^{(-\rho + \sqrt{D})t/2} - \frac{2\alpha - \rho + \sqrt{D}}{2a_2} (\dot{\pi} + \rho\pi) e^{(\rho + \sqrt{D})t/2} \right) \\ &\quad + C_2 \left(\dot{x} e^{(-\rho - \sqrt{D})t/2} - \frac{2\alpha - \rho - \sqrt{D}}{2a_2} (\dot{\pi} + \rho\pi) e^{(\rho - \sqrt{D})t/2} \right). \end{aligned}$$

Since C_1 and C_2 in the above Ω are arbitrary constants, it is deduced:

Theorem 6.1. *In the extremal problem of (6.1) for the utility polynomial (6.8) satisfying (6.9) under the constraint (6.2) with (6.10), there exists the following two conserved quantities*

$$(6.18) \quad \Omega_1 = \dot{x} e^{(-\rho + \sqrt{D})t/2} - \frac{2\alpha - \rho + \sqrt{D}}{2a_2} (\dot{\pi} + \rho\pi) e^{(\rho + \sqrt{D})t/2},$$

$$(6.19) \quad \Omega_2 = \dot{x} e^{(-\rho - \sqrt{D})t/2} - \frac{2\alpha - \rho - \sqrt{D}}{2a_2} (\dot{\pi} + \rho\pi) e^{(\rho - \sqrt{D})t/2}.$$

In the converse, U of (6.8) is replaced with an arbitrary function $U(x, u^1, u^2)$, while f of (6.10) remains unchanged. Then the above determined solution $(\eta, \varphi, \tau^1, \tau^2)$ is substituted for the reduced equations of (6.5b) and (6.5c) to derive

$$(6.20a) \quad \frac{\partial}{\partial x} \left(\frac{2\alpha - \rho + \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} \right) = \frac{2\alpha - \rho + \sqrt{D}}{2},$$

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$$(6.20b) \quad \frac{\partial}{\partial x} \left(\frac{2\alpha - \rho - \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} \right) = \frac{2\alpha - \rho - \sqrt{D}}{2},$$

$$(6.21a) \quad \frac{\partial}{\partial u^\sigma} \left(\frac{2\alpha - \rho + \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} \right) = \beta_\sigma,$$

$$(6.21b) \quad \frac{\partial}{\partial u^\sigma} \left(\frac{2\alpha - \rho - \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} \right) = \beta_\sigma.$$

From (6.20a) and (6.21a), or (6.20b) and (6.21b), it follows respectively that

$$(6.22) \quad \frac{2\alpha - \rho + \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} = \frac{2\alpha - \rho + \sqrt{D}}{2} x + \beta_\sigma u^\sigma + C_1 \quad (C_1: \text{const.}),$$

$$(6.23) \quad \frac{2\alpha - \rho - \sqrt{D}}{2a_2} \frac{\partial U}{\partial x} + n_1 \frac{\partial U}{\partial u^1} + n_2 \frac{\partial U}{\partial u^2} = \frac{2\alpha - \rho - \sqrt{D}}{2} x + \beta_\sigma u^\sigma + C_2 \quad (C_2: \text{const.}),$$

whose difference leads to

$$\frac{\partial U}{\partial x} = a_1 + a_2 x \quad \left(a_1 = \frac{a_2(C_1 - C_2)}{\sqrt{D}} : \text{const.} \right).$$

Therefore U is of the form

$$U = a_1 x + \frac{1}{2} a_2 x^2 + \Psi(u^1, u^2),$$

which is substituted for (6.22) (or equivalently for (6.23)) to have

$$(6.24) \quad n_1 \frac{\partial \Psi}{\partial u^1} + n_2 \frac{\partial \Psi}{\partial u^2} = \beta_1 u^1 + \beta_2 u^2 + \gamma \quad \left(\gamma = C_1 - \frac{(C_1 - C_2)(2\alpha - \rho + \sqrt{D})}{2\sqrt{D}} : \text{const.} \right).$$

Since the subsidiary equation of (6.24):

$$\frac{du^1}{n_1} = \frac{du^2}{n_2} = \frac{d\Psi}{\beta_1 u^1 + \beta_2 u^2 + \gamma}$$

has the following independent solutions:

$$c_1 \equiv n_2 u^1 - n_1 u^2 = \text{const.},$$

$$c_2 \equiv n_2 \gamma u^2 + n_2 \beta_1 u^1 u^2 + \frac{1}{2} (n_2 \beta_2 - n_1 \beta_1) (u^2)^2 - n_2^2 \Psi = \text{const.},$$

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the function Ψ is determined, through a functional relation $\Phi(c_1, c_2) = 0$, as

$$\Psi = \frac{\gamma}{n_2} u^2 + \frac{\beta_1}{n_2} u^1 u^2 + \frac{\beta_2 n_2 - \beta_1 n_1}{2n_2^2} (u^2)^2 + \psi(n_2 u^1 - n_1 u^2),$$

in which $\psi = \psi(n_2 u^1 - n_1 u^2)$ is an arbitrary function of $n_2 u^1 - n_1 u^2$. Thus, by replacing $\psi(n_2 u^1 - n_1 u^2)$ with $\psi(n_2 u^1 - n_1 u^2) + m_2(n_2 u^1 - n_1 u^2)^2 + m_1(n_2 u^1 - n_1 u^2)$ (m_1, m_2 : const.), the utility function U is determined as

$$(6.25) \quad U = a_1 x + \frac{1}{2} a_2 x^2 + m_2 n_2 u^1 + \frac{\gamma - m_2 n_1 n_2}{n_2} u^2 + m_1 n_2^2 (u^1)^2 + \frac{\beta_1 - 2m_1 n_1 n_2^2}{n_2} u^1 u^2 + \frac{2m_1 n_1^2 n_2^2 + n_2 \beta_2 - n_1 \beta_1}{2n_2^2} (u^2)^2 + \psi(n_2 u^1 - n_1 u^2),$$

which can be written as follows by arranging the arbitrary constants m_1, m_2, n_1, n_2 and $\gamma = C_1 - (C_1 - C_2)(2\alpha - \rho + \sqrt{D})/(2\sqrt{D})$:

$$(6.26) \quad U = a_1 x + \frac{1}{2} a_2 x^2 + b_1 u^1 + b_2 u^2 + \frac{1}{2} b_{11} (u^1)^2 + b_{12} u^1 u^2 + \frac{1}{2} b_{22} (u^2)^2 + \psi(n_2 u^1 - n_1 u^2).$$

In view of Hessian matrix of (6.26), the concavity of (6.26) is guaranteed if

$$(6.27) \quad a_2 < 0, \quad b_{11} + n_2^2 \psi'' < 0, \quad (b_{11} b_{22} - b_{12}^2) + (b_{11} n_1^2 + b_{22} n_2^2 + 2b_{12} n_1 n_2) \psi'' > 0.$$

Theorem 6.2. *The theorem 6.1 is valid even if the utility polynomial (6.8) satisfying (6.9) is replaced with the function (6.26) satisfying (6.27) (the difference between (6.8) and (6.26) is the function ψ only).*

Remark 6.1. Since U of (6.8) is concave, the conditions in (6.9) can be used to see

$$b_{11} n_1^2 + b_{22} n_2^2 + 2b_{12} n_1 n_2 = b_{11} \left(\left(n_1 + \frac{b_{12}}{b_{11}} n_2 \right)^2 + (b_{11} b_{22} - b_{12}^2) \frac{n_2^2}{b_{11}^2} \right) < 0.$$

Therefore, all of the conditions in (6.27) are valid, if the function ψ in (6.26) satisfies $\psi'' < 0$.

6.4 Utility of second order polynomial and optimal paths

In the conserved quantities (6.18) and (6.19), the term $\dot{\pi} + \rho\pi$ can be eliminated to obtain the differential equation with respect to x :

$$\dot{x} = -\frac{(2\alpha - \rho - \sqrt{D})\Omega_1}{2\sqrt{D}} e^{(\rho - \sqrt{D})t/2} + \frac{(2\alpha - \rho + \sqrt{D})\Omega_2}{2\sqrt{D}} e^{(\rho + \sqrt{D})t/2},$$

whose integration is

$$(6.28) \quad x(t) = -\frac{(2\alpha - \rho - \sqrt{D})\Omega_1}{\rho\sqrt{D} - D} e^{(\rho - \sqrt{D})t/2} + \frac{(2\alpha - \rho + \sqrt{D})\Omega_2}{\rho\sqrt{D} + D} e^{(\rho + \sqrt{D})t/2} + A_1 \quad (A_1: \text{const.}).$$

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Similarly, \dot{x} is eliminated to have

$$(\dot{\pi} + \rho\pi)e^{\rho t} = -\frac{a_2\Omega_1}{\sqrt{D}}e^{(\rho-\sqrt{D})t/2} + \frac{a_2\Omega_2}{\sqrt{D}}e^{(\rho+\sqrt{D})t/2},$$

whose integration is

$$(6.29) \quad \pi(t) = -\frac{2a_2\Omega_1}{\rho\sqrt{D}-D}e^{(-\rho-\sqrt{D})t/2} + \frac{2a_2\Omega_2}{\rho\sqrt{D}+D}e^{(-\rho+\sqrt{D})t/2} + A_2e^{-\rho t} \quad (A_2: \text{const.});$$

which is substituted for (6.11b) and (6.11c) to obtain

$$(6.30) \quad u^1(t) = n_1 \left(-\frac{2a_2\Omega_1}{\rho\sqrt{D}-D}e^{(\rho-\sqrt{D})t/2} + \frac{2a_2\Omega_2}{\rho\sqrt{D}+D}e^{(\rho+\sqrt{D})t/2} + A_2 \right) + \frac{b_{12}b_2 - b_{22}b_1}{b_{11}b_{22} - b_{12}^2},$$

$$(6.31) \quad u^2(t) = n_2 \left(-\frac{2a_2\Omega_1}{\rho\sqrt{D}-D}e^{(\rho-\sqrt{D})t/2} + \frac{2a_2\Omega_2}{\rho\sqrt{D}+D}e^{(\rho+\sqrt{D})t/2} + A_2 \right) + \frac{b_{12}b_1 - b_{11}b_2}{b_{11}b_{22} - b_{12}^2}.$$

The equations (6.28)–(6.31) are substituted for (6.2) with (6.10), and for (6.11a), to see

$$\alpha A_1 + (n_1\beta_1 + n_2\beta_2)A_2 = n_1b_1 + n_2b_2,$$

$$a_2A_1 + (\rho - \alpha)A_2 = -a_1.$$

Here remark the constants $\rho \geq 0$, $\alpha < 0$, $a_2 < 0$ and $n_1\beta_1 + n_2\beta_2 < 0$ (see (6.17)) imply $\alpha(\rho - \alpha) \leq 0$ and $a_2(n_1\beta_1 + n_2\beta_2) > 0$, so that

$$(6.32) \quad \alpha(\rho - \alpha) - a_2(n_1\beta_1 + n_2\beta_2) < 0.$$

Therefore the constants A_1 and A_2 of the above equations are determined respectively as

$$(6.33) \quad A_1 = \frac{(\rho - \alpha)(n_1b_1 + n_2b_2) + a_1(n_1\beta_1 + n_2\beta_2)}{\alpha(\rho - \alpha) - a_2(n_1\beta_1 + n_2\beta_2)},$$

$$A_2 = -\frac{a_2(n_1b_1 + n_2b_2) + a_1\alpha}{\alpha(\rho - \alpha) - a_2(n_1\beta_1 + n_2\beta_2)},$$

which complete the optimal paths $x(t)$, $u^1(t)$ and $u^2(t)$.

Theorem 6.3. *In the extremal problem of (6.1) for the utility polynomial (6.8) satisfying (6.9), or more generally for the utility function (6.26) satisfying (6.27), under the constraint (6.2) with (6.10); the optimal paths $x(t)$, $u^1(t)$ and $u^2(t)$ are determined, in the case of finite horizon $T < \infty$, completely as (6.28), (6.30) and (6.31), in which A_1 and A_2 are the constants of (6.33).*

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In the case of infinite horizon $T = \infty$, the optimal paths have to satisfy the transversality condition $\lim_{t \rightarrow \infty} \pi x = 0$. Since the apperance of πx with (6.28) and (6.29) is

$$\begin{aligned} \pi x = & A_1 A_2 e^{-\rho t} + \frac{2a_2(2\alpha - \rho - \sqrt{D})\Omega_1^2}{(\rho\sqrt{D} - D)^2} e^{-\sqrt{D}t} + \frac{2a_2 A_1 + (2\alpha - \rho - \sqrt{D})A_2}{\rho\sqrt{D} - D} \Omega_1 e^{-(\rho + \sqrt{D})t/2} \\ & + \frac{2a_2(2\alpha - \rho + \sqrt{D})\Omega_2^2}{(\rho\sqrt{D} + D)^2} e^{\sqrt{D}t} + \frac{2a_2 A_1 + (2\alpha - \rho + \sqrt{D})A_2}{\rho\sqrt{D} + D} \Omega_2 e^{-(\rho - \sqrt{D})t/2} - \frac{2\alpha - \rho + \sqrt{D}}{\rho D - D^2} \Omega_1 \Omega_2, \end{aligned}$$

$\lim_{t \rightarrow \infty} \pi x = 0$ requires $\Omega_2 = 0$ in the coefficients $e^{\sqrt{D}t}$ and $e^{-(\rho - \sqrt{D})t/2}$. Therefore, the optimal paths (6.28), (6.30) and (6.31) are reduced respectively to

$$(6.34) \quad x(t) = -\frac{(2\alpha - \rho - \sqrt{D})\Omega_1}{\rho\sqrt{D} - D} e^{(\rho - \sqrt{D})t/2} + A_1,$$

$$(6.35) \quad u^1(t) = n_1 \left(-\frac{2a_2\Omega_1}{\rho\sqrt{D} - D} e^{(\rho - \sqrt{D})t/2} + A_2 \right) + \frac{b_{12}b_2 - b_{22}b_1}{b_{11}b_{22} - b_{12}^2},$$

$$(6.36) \quad u^2(t) = n_2 \left(-\frac{2a_2\Omega_1}{\rho\sqrt{D} - D} e^{(\rho - \sqrt{D})t/2} + A_2 \right) + \frac{b_{12}b_1 - b_{11}b_2}{b_{11}b_{22} - b_{12}^2}.$$

Theorem 6.4. *In the case of infinite horizon $T = \infty$, there exist the feasible optimal paths of the forms (6.34), (6.35) and (6.36), which take respectively, if $\rho < \sqrt{D}$, the equilibrium values as $t \rightarrow \infty$: $\lim_{t \rightarrow \infty} x(t) = A_1$, $\lim_{t \rightarrow \infty} u^1(t) = n_1 A_2 + (b_{12}b_2 - b_{22}b_1)/(b_{11}b_{22} - b_{12}^2)$ and $\lim_{t \rightarrow \infty} u^2(t) = n_2 A_2 + (b_{12}b_1 - b_{11}b_2)/(b_{11}b_{22} - b_{12}^2)$, where A_1 and A_2 are the constants of (6.33).*

By $a_1 > 0$, $a_2 < 0$, $\alpha < 0$ and (6.32), the constant A_2 in (6.35) and (6.36) is negative if $b_1 = b_2 = 0$. Therefore it is concluded:

Theorem 6.5. *In the case of infinite horizon $T = \infty$, let $\rho < \sqrt{D}$ and $b_1 = b_2 = 0$. Then, it follows that*

$$\lim_{t \rightarrow \infty} u^1(t) \gtrless \lim_{t \rightarrow \infty} u^2(t) \quad \text{depending upon whether} \quad (b_{11} + b_{12})\beta_2 \gtrless (b_{22} + b_{12})\beta_1;$$

which leads, if $b_{11} = b_{22}$, to

$$\text{when } b_{11} + b_{12} > 0: \quad \lim_{t \rightarrow \infty} u^1(t) \gtrless \lim_{t \rightarrow \infty} u^2(t) \quad \text{depending upon whether} \quad \beta_1 \gtrless \beta_2,$$

$$\text{when } b_{11} + b_{12} < 0: \quad \lim_{t \rightarrow \infty} u^1(t) \lesseqgtr \lim_{t \rightarrow \infty} u^2(t) \quad \text{depending upon whether} \quad \beta_1 \lesseqgtr \beta_2.$$

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Remark 6.2. In the case of infinite horizon $T = \infty$ with $\rho < \sqrt{D}$ and $b_1 = b_2 = 0$, the maximizing problem provides the feasible optimal paths (6.34), (6.35) and (6.36) if and only if $\lim_{t \rightarrow \infty} u^1(t) \geq \lim_{t \rightarrow \infty} u^2(t)$ according to $(b_{11} + b_{12})\beta_2 \geq (b_{22} + b_{12})\beta_1$.

An example. The results can be applied to the model of Fershtman and Nitzan ([2], in which K , x_i and r are denoted here by x , u^i and ρ respectively), by setting the utility polynomial $U(x, u)$ and the constraint $f(x, u)$ respectively as

$$\begin{aligned} U &= 2\alpha(ax - bx^2) - \frac{1}{2}c(u^1)^2 - \frac{1}{2}c(u^2)^2, \\ f &= -\delta x + u^1 + u^2, \end{aligned}$$

which follow from (6.8) with $a_1 = 2a\alpha$, $a_2 = -4b\alpha$, $b_1 = b_2 = 0$, $b_{11} = b_{22} = -c$, $b_{12} = 0$, and from (6.10) with $\alpha = -\delta$, $\beta_1 = \beta_2 = 1$, respectively. Then, the conserved quantities (6.18) and (6.19) are reduced respectively to

$$\begin{aligned} \Omega_1 &= \dot{x}e^{(-\rho+\sqrt{D})t/2} - \frac{2\delta+\rho-\sqrt{D}}{8b\alpha}(\dot{\pi} + \rho\pi)e^{(\rho+\sqrt{D})t/2}, \\ \Omega_2 &= \dot{x}e^{-(\rho+\sqrt{D})t/2} - \frac{2\delta+\rho+\sqrt{D}}{8b\alpha}(\dot{\pi} + \rho\pi)e^{(\rho-\sqrt{D})t/2}, \end{aligned}$$

where $D = (2\delta + \rho)^2 + 32b\alpha/c$. Moreover, in the infinite horizon $T = \infty$, the optimal path (6.34) is also to

$$x(t) = \frac{(2\delta+\rho+\sqrt{D})\Omega_1}{\rho\sqrt{D}-D}e^{(\rho-\sqrt{D})t/2} + A_1,$$

where $A_1 = 4a\alpha/(c\delta(\delta + \rho) + 8b\alpha)$. So, it follows that

$$\lim_{t \rightarrow \infty} x(t) = \frac{4a\alpha}{c\delta(\delta + \rho) + 8b\alpha},$$

which is the quantity K^e appeared in the Pareto-efficient steady-state ([8], Theorem 1, in which n is given here as $n = 2$).

7 Open-loop Nash strategies

7.1 Introduction

The Noether theorem (Noether [37]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem. In contrast with the Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [32]) and applied it to various economic growth models (Mimura and Nôno [34]; Mimura, Fujiwara and Nôno [29], [30]; Fujiwara, Mimura and Nôno [11]-[17]) to discover new economic conservation laws including non-Noether ones.

The application can be extended to the n -sector differential game in which the sectors (the players) are not able to make binding commitments in advance of play on the strategies they will employ. Such strategies are called open-loop Nash strategies. In the game, each sector has his own objective functional to maximize under a constraint leaving the strategies of other sectors out of account, i.e., regarding the variables with respect to other sectors as constants. This fact put difficulties for the application of, as well as the Noether theorem, the new procedure in [32] to discover conservation laws in the open-loop Nash strategies.

Fershtman and Nitzan [8] introduced a model of the voluntary contributions to the provision of a collectively produced good in the dynamic framework of differential game. They compared the Parato-efficient and the level of the collective contributions in the corresponding steady states of the open-loop and the feedback Nash strategies. The model is interesting in the sense that the objective functional of each sector include both of the state variable (the stock of total contributions) and the control variable (the contribution rate of each sector). In this paper, the model is used with some generalization to formulate a way of discovering conservation laws in the n -sector open-loop Nash strategies (in which general derivation of conservation laws has been never discussed).

In 7.2, we set objective functional of each sector i ($i = 1, \dots, n$) whose maximizing problem will be discussed under a constraint. In the open-loop Nash strategies, we first show that the n functionals can be unified into a single objective functional. And then, introducing new variables, we give a composite maximizing problem (an extended maximizing problem in the usual variational principle), in which the optimal paths of the original variables are those in the maximizing problem of the single objective functional in the open-loop Nash strategies. In 7.3, by applying the new procedure in [32] to the composite maximizing problem in the usual variational principle, we find three conserved quantities for a model in two-sector

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open-loop Nash strategies. Finally in 7.4, through the obtained conserved quantities in 7.2, optimal paths are determined completely for finite horizon and then detailed for infinite horizon. The determined optimal paths will illustrate the level of the collective contribution at the stationary open-loop Nash equilibrium obtained in [8].

7.2 A composite maximizing problem

In the n -sector open-loop Nash strategies, we consider the following model which will be understood later as a generalization of the model of Fershtman and Nitzan [8]. Each sector i ($i = 1, \dots, n$) has a common state variable $x(t)$ and his own control variable $u^i(t)$. Leaving the behavior of $u^j(t)$ ($j \neq i$) out of account, sector i seeks to maximize the integration over finite ($T < \infty$) or infinite ($T = \infty$) period of time:

$$(7.1-i) \quad \int_0^T e^{-\rho t} (\varphi(x) + \psi_i(u^1, \dots, u^n)) dt$$

under a constraint

$$(7.2) \quad \dot{x} = \alpha x + \sum_{k=1}^n \beta_k u^k \quad (\alpha, \beta_k: \text{const.}, k = 1, \dots, n; \alpha < 0),$$

where φ and ψ_i are assumed to be a monotonically increasing function and a concave function with respect to his control variable u^i , respectively, i.e.,

$$(7.3-i) \quad \varphi' > 0, \quad \frac{\partial^2 \psi_i}{\partial u^i \partial u^i} < 0.$$

So that sector i has the following Lagrangian L_i with the multiplier π_i :

$$L_i = e^{-\rho t} (\varphi(x) + \psi_i(u^1, \dots, u^n)) + \pi_i \left(\dot{x} - \alpha x - \sum_{k=1}^n \beta_k u^k \right).$$

Here keep in mind that sector i regards u^j and π_j ($j \neq i$) as constants for the determination of his Euler-Lagrange equations which consist of (7.2) and

$$(7.4-i) \quad \dot{\pi}_i + \alpha \pi_i = e^{-\rho t} \varphi'(x),$$

$$(7.5-i) \quad \beta_i \pi_i = e^{-\rho t} \frac{\partial \psi_i(u^1, \dots, u^n)}{\partial u^i}.$$

Then n systems of Euler-Lagrange equations (7.2), (7.4-i) and (7.5-i) of sector i ($i = 1, \dots, n$) are called together in the space of all variables $x, u = (u^1, \dots, u^n)$ and $\pi = (\pi^1, \dots, \pi^n)$ to determine the optimal paths.

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Here assume that ψ_i in (7.1-i) ($i=1, \dots, n$) satisfy

$$(7.6) \quad \frac{\partial^2 \psi_i}{\partial u^j \partial u^i} = \frac{\partial^2 \psi_j}{\partial u^i \partial u^j} \quad (i, j = 1, \dots, n),$$

which guarantee an existence of function $\psi(u^1, \dots, u^n)$ such that

$$(7.7) \quad \frac{\partial \psi}{\partial u^i} = \frac{\partial \psi_i}{\partial u^i} \quad (i = 1, \dots, n).$$

Accordingly, in the open-loop Nash strategies, since the sector i regards u^j and π_j ($j \neq i$) as constants, the relating Euler-Lagrange equations of sector i remains unchanged even if the utilities $\varphi + \psi_i$ are replaced with $\varphi + \psi$, which satisfies the similar condition of (7.3-i):

$$(7.8) \quad \varphi' > 0, \quad \frac{\partial^2 \psi}{\partial u^i \partial u^i} < 0 \quad (i = 1, \dots, n).$$

Consequently, it follows:

Theorem 7.1. *Let $\varphi(x) + \psi_i(u^1, \dots, u^n)$ in (7.1-i) ($i=1, \dots, n$) satisfy (7.3-i) ($i=1, \dots, n$) and (7.6). Then, in the n -sector open-loop Nash strategies, the maximizing problem of (7.1-i) ($i=1, \dots, n$) under the constraint (7.2) is equivalent to that of*

$$(7.9) \quad \int_0^T e^{-\rho t} (\varphi(x) + \psi(u^1, \dots, u^n)) dt$$

under the constraint (7.2), where $\psi(u^1, \dots, u^n)$ is a function satisfying (7.7).

Remark 7.1. In the open-loop Nash strategies, since sector i regards u^j ($j \neq i$) as constants for the determination of his Euler-Lagrange equations, the equations (7.4-i) and (7.5-i) are unchanged even if $\sum_{k=1}^n \beta_k u^k$ is replaced with $\beta_i u^i$ in the constraint (7.2) of the maximizing problem of sector i .

Remark 7.2. In the model of [8] in the open-loop Nash strategies, the utility V_i of sector i ($i = 1, \dots, n$) is of the form $V_i = \gamma f(x) - C(u^i)$ (γ : const.). By virtue of the theorem 7.1, V_i can be unified into $V = \gamma f(x) - \sum_{k=1}^n C(u^k)$, where $f(x)$ and $C(u^k)$ were specified respectively as $f(x) = ax - bx^2$ (a, b : const.) and $C(u^k) = \frac{1}{2}c(u^k)^2$ (c : const.) for the tractability.

Now introduce new variables $y^\sigma(t)$ ($\sigma = 1, \dots, n-1$) and put

$$(7.10a) \quad z^1 = \frac{1}{n} \left(x + \sum_{\sigma=1}^{n-1} y^\sigma \right),$$

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$$(7.10b) \quad z^{\sigma+1} = \frac{1}{n}(x - y^\sigma) \quad (\sigma = 1, \dots, n-1).$$

The equations (7.10a) and all of (7.10b) for $\sigma = 1, \dots, n-1$ are added to see

$$(7.11) \quad \sum_{k=1}^n z^k = x,$$

so that the integration (7.9) is written as

$$(7.12) \quad \int_0^T e^{-\rho t} (\varphi(z^1 + \dots + z^n) + \psi(u^1, \dots, u^n)) dt.$$

So, in the usual variational principle, consider the maximizing problem of (7.12) under the reformed constraints

$$(7.13) \quad \dot{z}^i = \alpha z^i + \beta_i u^i \quad (i = 1, \dots, n).$$

Here denote the principal k -th minor of Hessian matrix of $\psi(u^1, \dots, u^n)$ by $D_k(\psi)$ ($k = 1, \dots, n$). Also, ψ in (7.12) is assumed to be a concave function, i.e., (see, e.g., [47])

$$(7.14) \quad (-1)^k D_k(\psi) > 0 \quad (k = 1, \dots, n).$$

Then the Lagrangian with the multipliers π_i ($i = 1, \dots, n$) is written as

$$(7.15) \quad L = e^{-\rho t} (\varphi(z^1 + \dots + z^n) + \psi(u^1, \dots, u^n)) + \sum_{k=1}^n \pi_k (\dot{z}^k - \alpha z^k - \beta_k u^k),$$

whose Euler-Lagrange equations consist of (7.13) and

$$(7.16) \quad \dot{\pi}_i + \alpha_i \pi_i = e^{-\rho t} \frac{\partial \varphi(z^1 + \dots + z^n)}{\partial z^i} \quad (i = 1, \dots, n),$$

$$(7.17) \quad \beta_i \pi_i = e^{-\rho t} \frac{\partial \psi(u^1, \dots, u^n)}{\partial u^i} \quad (i = 1, \dots, n).$$

All of (7.13) for $i = 1, \dots, n$ are added and then (7.11) is used to derive (7.2). The equation (7.16) is equivalent to the collection of (7.4-i) ($i = 1, \dots, n$) by $\partial \varphi(z^1 + \dots + z^n) / \partial z^i = \varphi'(x)$. The equation (7.17) is equivalent to the collection of (7.5-i) ($i = 1, \dots, n$) if (7.7) is satisfied. The equations (7.2), (7.16) and (7.17) are used to find optimal paths $x(t)$ and $u(t)$. The equation (7.10b) is substituted for (7.13) and then (7.2) is used to derive

$$(7.18) \quad \dot{y}^\sigma - \alpha y^\sigma = \sum_{k=1}^n \beta_k u^k - n \beta_{\sigma+1} u^{\sigma+1} \quad (\sigma = 1, \dots, n-1),$$

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whose right hand side is written as $\varphi^\sigma(t)$ on the optimal path $u(t) = (u^1(t), \dots, u^n(t))$. In the solution $y^\sigma = A^\sigma e^{\alpha t}$ (A^σ : const.) of the subsidiary equation $\dot{y}^\sigma - \alpha y^\sigma = 0$ of (7.18), the constant A^σ is replaced with an arbitrary function of t , and then determined as

$$A^\sigma(t) = \int \varphi^\sigma(t) e^{-\alpha t} dt.$$

Therefore, if the optimal paths $x(t)$ and $u(t)$ exist, the optimal path $y(t) = (y^1(t), \dots, y^{n-1}(t))$ exists also, where $y^\sigma(t) = A^\sigma(t) e^{\alpha t}$ ($\sigma = 1, \dots, n-1$). Therefore, it is deduced:

Theorem 7.2. *Let $\varphi(x) + \psi(u^1, \dots, u^n)$ in (7.9) satisfy (7.8) and $\varphi(z^1 + \dots + z^n) + \psi(u^1, \dots, u^n)$ in (7.12) satisfy $\varphi'(z^1 + \dots + z^n) > 0$ and (7.14). Then the maximizing problem of (7.9) under the constraints (7.2) in the n -sector open-loop Nash strategies is included in that of (7.12) under the constraints (7.13) in the usual variational principle, i.e., the former and the latter expect an existence of the same optimal paths $x(t)$ and $u(t)$.*

7.3 Conservation laws for a model in the open-loop Nash strategies

Particularly in the two-sector open-loop Nash strategies, we consider the problem that each sector i ($i = 1, 2$) seeks to maximize the integration

$$(7.1-i)' \quad \int_0^T e^{-\rho t} (\varphi(x) + \psi_i(u^1, u^2)) dt$$

under the constraints

$$(7.2)' \quad \dot{x} = \alpha x + \beta_1 u^1 + \beta_2 u^2 \quad (\alpha, \beta_1, \beta_2: \text{const.}, \alpha < 0);$$

where the utility $\varphi(x) + \psi_i(u^1, u^2)$ satisfying (7.3-i) and (7.6) is given by

$$(7.19a) \quad \varphi(x) = a_1 x + \frac{1}{2} a_2 x^2 \quad (x < -a_1/a_2; a_1, a_2: \text{const.}; a_1 > 0, a_2 < 0),$$

$$(7.19b) \quad \psi_i(u^1, u^2) = b_i u^i + \frac{1}{2} b_{ii} (u^i)^2 + b_{12} u^1 u^2 + g_i(u^j) \quad (b_i, b_{ij}: \text{const.}, i, j = 1, 2; b_{ii} < 0),$$

in which $g_i(u^j)$ is an arbitrary function of u^j ($j \neq i$). By the theorem 7.1, the functions $\varphi + \psi_i$ ($i = 1, 2$) are unified into $\varphi + \psi$ where

$$(7.20) \quad \psi(u^1, u^2) = b_1 u^1 + b_2 u^2 + \frac{1}{2} b_{11} (u^1)^2 + b_{12} u^1 u^2 + \frac{1}{2} b_{22} (u^2)^2,$$

which is also assumed to be provided with the concavity, i.e., $D_1(\psi) = b_{11} < 0$ and $D_2(\psi) = b_{11} b_{22} - b_{12}^2 > 0$. So it follows for $\varphi(x) + \psi(u^1, u^2)$ that

$$(7.21) \quad a_1 > 0, a_2 < 0, b_{11} < 0, b_{22} < 0, b_{11} b_{22} - b_{12}^2 > 0.$$

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Since $n = 2$, (7.10a) and (7.10b) are reduced respectively to

$$(7.10a)' \quad z^1 = \frac{1}{2}(x + y),$$

$$(7.10b)' \quad z^2 = \frac{1}{2}(x - y),$$

where $y \equiv y^1$ is the new variable. And the Lagrangian (7.15) is also to

$$(7.15)' \quad L = e^{-\rho t} (\varphi(z^1 + z^2) + \psi(u^1, u^2)) + \pi_1(\dot{z}^1 - \alpha z^1 - \beta_1 u^1) + \pi_2(\dot{z}^2 - \alpha z^2 - \beta_2 u^2),$$

where $\varphi(z^1 + z^2) + \psi(u^1, u^2)$ is given by (7.19a) and (7.20) with (7.21). The Euler-Lagrange equations for the Lagrangian (7.15)' consist of (7.13) with $i = 1, 2$ and (see (7.16) and (7.17))

$$(7.16)' \quad \dot{\pi}_i + \alpha \pi_i = e^{-\rho t} (a_1 + a_2(z^1 + z^2)) \quad (i = 1, 2),$$

$$(7.17)' \quad \beta_i \pi_i = e^{-\rho t} (b_i + b_{1i} u^1 + b_{2i} u^2) \quad (i = 1, 2).$$

In the situation, by the theorem 7.1 and the theorem 7.2, through the integration of $e^{-\rho t}(\varphi(x) + \psi(u^1, u^2))$, the maximizing problem of (7.1-i)' under the constraints (7.2)' in the two-sector open-loop Nash strategies is included in that of

$$(7.12)' \quad \int_0^T e^{-\rho t} (\varphi(z^1 + z^2) + \psi(u^1, u^2)) dt$$

under the constraints

$$(7.13)' \quad \dot{z}^i = \alpha z^i + \beta_i u^i \quad (i = 1, 2),$$

in the usual variational principle, where $\varphi(z^1 + z^2) + \psi(u^1, u^2)$ in (7.12)' is given by (7.19a) and (7.20) with (7.21).

Now recall the theorem 2.2 in 2.2 for the derivation of conservation laws. By putting $q = (z^1, z^2, u^1, u^2)$ and $\lambda = (\pi_1, \pi_2)$ respectively, it shows here the following result:

On the optimal paths for the maximizing problem of (7.12)' under the constraints (7.13)', let $\xi^i(\dot{q}, q, t)$ ($i = 1, 2, 3, 4$) and $\eta_a(\dot{q}, q, t)$ ($a = 1, 2$) satisfy the equations

$$(7.22a) \quad \xi^1 - \alpha \xi^1 = \beta_1 \xi^3,$$

$$(7.22b) \quad \xi^2 - \alpha \xi^2 = \beta_2 \xi^4,$$

$$(7.22c) \quad \dot{\eta}_1 + \alpha \eta_1 = a_2 e^{-\rho t} (\xi^1 + \xi^2),$$

$$(7.22d) \quad \dot{\eta}_2 + \alpha \eta_2 = a_2 e^{-\rho t} (\xi^1 + \xi^2),$$

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$$(7.22e) \quad \beta_1 \eta_1 = e^{-\rho t} (b_{11} \xi^3 + b_{12} \xi^4),$$

$$(7.22f) \quad \beta_2 \eta_2 = e^{-\rho t} (b_{12} \xi^3 + b_{22} \xi^4).$$

Then the following conserved quantity Ω in the problem is constructed:

$$(7.23) \quad \Omega = \dot{z}^1 \eta_1 + \dot{z}^2 \eta_2 - (\dot{\pi}_1 + \rho \pi_1) \xi^1 - (\dot{\pi}_2 + \rho \pi_2) \xi^2.$$

The solutions ξ^i ($i = 1, 2, 3, 4$) and η_a ($a = 1, 2$) can be determined as follows. The difference $\dot{\eta}_2 - \dot{\eta}_1 = -\alpha(\eta_2 - \eta_1)$ of (7.22c) and (7.22d) is integrated as

$$(7.24) \quad \eta_2 = \eta_1 + C_1 e^{-\alpha t} \quad (C_1: \text{const.}).$$

In view of $b_{11}b_{22} - b_{12}^2 \neq 0$ in (7.21), η_2 of (7.24) is substituted for the solutions ξ^3 and ξ^4 of (7.22e) and (7.22f) to see

$$(7.25) \quad \xi^3 = n_1 e^{\rho t} \eta_1 - \frac{b_{12}\beta_2}{B} C_1 e^{(\rho-\alpha)t},$$

$$(7.26) \quad \xi^4 = n_2 e^{\rho t} \eta_1 + \frac{b_{11}\beta_2}{B} C_1 e^{(\rho-\alpha)t},$$

where n_1 and n_2 are the constants:

$$n_1 = \frac{b_{22}\beta_1 - b_{12}\beta_2}{B}, \quad n_2 = \frac{b_{11}\beta_2 - b_{12}\beta_1}{B}, \quad (B \equiv b_{11}b_{22} - b_{12}^2).$$

The above appearances of (7.25) and (7.26) are used in the addition of (7.22a) and (7.22b) to derive

$$(7.27) \quad (\dot{\xi}^1 + \dot{\xi}^2) - \alpha(\xi^1 + \xi^2) = (n_1\beta_1 + n_2\beta_2)e^{\rho t}\eta_1 + n_2\beta_2 C_1 e^{(\rho-\alpha)t}.$$

Moreover the identity $\xi^1 + \xi^2 = \frac{1}{a_2} e^{\rho t} (\dot{\eta}_1 + \alpha \eta_1)$ from (7.22c) is substituted for (7.27) to have

$$(7.28) \quad \ddot{\eta}_1 + \rho \dot{\eta}_1 + (\alpha(\rho - \alpha) - a_2(n_1\beta_1 + n_2\beta_2))\eta_1 = a_2 n_2 \beta_2 C_1 e^{-\alpha t}.$$

By means of (7.21), which guarantees

$$N_\beta \equiv n_1\beta_1 + n_2\beta_2 = \frac{(b_{22}\beta_1 - b_{12}\beta_2)^2}{b_{22}(b_{11}b_{12} - b_{12}^2)} + \frac{\beta_2^2}{b_{22}} < 0,$$

the discriminant D of the subsidiary equation of (7.28) becomes positive:

$$D = (2\alpha - \rho)^2 + 4a_2 N_\beta > 0.$$

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So, the solution η_1 of (7.28) can be determined as

$$\eta_1 = A_1 e^{(-\rho+\sqrt{D})t/2} + A_2 e^{(-\rho-\sqrt{D})t/2} - \frac{n_2 \beta_2}{N_\beta} C_1 e^{-\alpha t} \quad (A_1, A_2, C_1: \text{const.}),$$

which is substituted for (7.24), (7.25) and (7.26) to obtain respectively

$$\eta_2 = A_1 e^{(-\rho+\sqrt{D})t/2} + A_2 e^{(-\rho-\sqrt{D})t/2} + \frac{n_1 \beta_1}{N_\beta} C_1 e^{-\alpha t},$$

$$\xi^3 = n_1 (A_1 e^{(\rho+\sqrt{D})t/2} + A_2 e^{(\rho-\sqrt{D})t/2}) - \frac{\beta_1 \beta_2^2}{B N_\beta} C_1 e^{(\rho-\alpha)t},$$

$$\xi^4 = n_2 (A_1 e^{(\rho+\sqrt{D})t/2} + A_2 e^{(\rho-\sqrt{D})t/2}) + \frac{\beta_1^2 \beta_2}{B N_\beta} C_1 e^{(\rho-\alpha)t}.$$

Moreover, after substituting the above ξ^3 for (7.22a) and ξ^4 for (7.22b), ξ^1 and ξ^2 are determined respectively as

$$\xi^1 = \frac{2n_1 \beta_1}{\rho-2\alpha+\sqrt{D}} A_1 e^{(\rho+\sqrt{D})t/2} + \frac{2n_1 \beta_1}{\rho-2\alpha-\sqrt{D}} A_2 e^{(\rho-\sqrt{D})t/2} - \frac{\beta_1^2 \beta_2^2}{B N_\beta (\rho-2\alpha)} C_1 e^{(\rho-\alpha)t} + C_2 e^{\alpha t},$$

$$\xi^2 = \frac{2n_2 \beta_2}{\rho-2\alpha+\sqrt{D}} A_1 e^{(\rho+\sqrt{D})t/2} + \frac{2n_2 \beta_2}{\rho-2\alpha-\sqrt{D}} A_2 e^{(\rho-\sqrt{D})t/2} + \frac{\beta_1^2 \beta_2^2}{B N_\beta (\rho-2\alpha)} C_1 e^{(\rho-\alpha)t} - C_2 e^{\alpha t},$$

$$(A_1, A_2, C_1, C_2: \text{const.}).$$

Since A_1, A_2, C_1 and C_2 are arbitrary constants, the conserved quantity Ω of the form (7.23) yields the following four conserved quantities:

$$\Omega_1 = (\dot{z}^1 + \dot{z}^2) e^{(-\rho+\sqrt{D})t/2} - \frac{2}{\rho-2\alpha+\sqrt{D}} (n_1 \beta_1 (\dot{\pi}_1 + \rho \pi_1) + n_2 \beta_2 (\dot{\pi}_2 + \rho \pi_2)) e^{(\rho+\sqrt{D})t/2},$$

$$\Omega_2 = (\dot{z}^1 + \dot{z}^2) e^{(-\rho-\sqrt{D})t/2} - \frac{2}{\rho-2\alpha-\sqrt{D}} (n_1 \beta_1 (\dot{\pi}_1 + \rho \pi_1) + n_2 \beta_2 (\dot{\pi}_2 + \rho \pi_2)) e^{(\rho-\sqrt{D})t/2},$$

$$\Omega_3 = (-n_2 \beta_2 \dot{z}^1 + n_1 \beta_1 \dot{z}^2) e^{-\alpha t} + \frac{\beta_1^2 \beta_2^2}{B(\rho-2\alpha)} ((\dot{\pi}_1 + \rho \pi_1) - (\dot{\pi}_2 + \rho \pi_2)) e^{(\rho-\alpha)t},$$

$$\Omega_4 = (\pi_1 - \pi_2) e^{\alpha t}.$$

Moreover, (7.10a)', (7.10b)' and (7.16)' imply $\dot{\pi}_i = e^{-\rho t} (a_1 + a_2 x) - \alpha \pi_i$ ($i = 1, 2$), which is substituted for $\dot{\pi}_i$ ($i = 1, 2$) in Ω_j ($j = 1, 2, 3$); $\dot{z}^1 + \dot{z}^2$ in Ω_j ($j = 1, 2$) is written by (7.2)', (7.10a)' and (7.10b)' as $\dot{z}^1 + \dot{z}^2 = \alpha x + \beta_1 u^1 + \beta_2 u^2$; (7.10a)' and (7.10b)' are substituted for \dot{z}^i ($i = 1, 2$) in Ω_3 and then, in the result, (7.2)' and $\dot{y} = \alpha y + \beta_1 u^1 + \beta_2 u^2$ (see (7.18)) are substituted for \dot{x} and \dot{y} ; and finally (7.17)' is used to eliminate u^i ($i = 1, 2$). Consequently, by

$$N_b = b_1 n_1 + b_2 n_2, \quad N_{\beta\pi} = n_1 \beta_1 \pi_1 + n_2 \beta_2 \pi_2,$$

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above conserved quantities are written respectively as

$$(7.29) \quad \Omega_1 = \left(\frac{\rho - \sqrt{D}}{2} x + \frac{a_1(\rho - 2\alpha - \sqrt{D})}{2a_2} - N_b \right) e^{(-\rho + \sqrt{D})t/2} - \frac{(\rho - \sqrt{D})N_{\beta\pi}}{\rho - 2\alpha + \sqrt{D}} e^{(\rho + \sqrt{D})t/2},$$

$$(7.30) \quad \Omega_2 = \left(\frac{\rho + \sqrt{D}}{2} x + \frac{a_1(\rho - 2\alpha + \sqrt{D})}{2a_2} - N_b \right) e^{(-\rho - \sqrt{D})t/2} - \frac{(\rho + \sqrt{D})N_{\beta\pi}}{\rho - 2\alpha - \sqrt{D}} e^{(\rho - \sqrt{D})t/2},$$

$$(7.31) \quad \Omega_3 = \left(\frac{\alpha(-n_1\beta_1 + n_2\beta_2)}{2} x + \frac{\alpha N_\beta}{2} y - \frac{\beta_1\beta_2(b_1\beta_2 - b_2\beta_1)}{B} \right) e^{-\alpha t} - \frac{\alpha\beta_1^2\beta_2^2}{(\rho - 2\alpha)B} (\pi_1 - \pi_2) e^{(\rho - \alpha)t},$$

$$(7.32) \quad \Omega_4 = (\pi_1 - \pi_2) e^{\alpha t}.$$

Thus, by virtue of the theorem 7.2, it is concluded (Ω_3 is the conserved quantity with the new variable y):

Theorem 7.3. *In the two-sector open-loop Nash strategies, let each sector i ($i = 1, 2$) seek to maximize:*

$$(7.1-i)'' \quad \int_0^T e^{-\rho t} (a_1 x + \frac{1}{2} a_2 x^2 + b_i u^i + \frac{1}{2} b_{ii} (u^i)^2 + b_{12} u^1 u^2 + g_i(u^j)) dt \quad (j \neq i)$$

($x < -a_1/a_2$; a_i, b_i, b_{ij} : const., $i, j = 1, 2$; $a_1 > 0, a_2 < 0, b_{ii} < 0$)

under a constraint (7.2)', where $g_i(u^j)$ is an arbitrary function of u^j ($j \neq i$). Then, there exist three conserved quantities (7.29), (7.30) and (7.32).

7.4 Optimal paths for a model in the open-loop Nash strategies

In the conserved quantities (7.29) and (7.30), the term $N_{\beta\pi}$ can be eliminated to obtain the optimal path $x(t)$:

$$(7.33) \quad x(t) = \Xi_1 e^{(\rho - \sqrt{D})t/2} + \Xi_2 e^{(\rho + \sqrt{D})t/2} + \frac{a_1 N_\beta + (\rho - \alpha) N_b}{\alpha(\rho - \alpha) - a_2 N_\beta},$$

where $\Xi_1 = \Omega_1/\sqrt{D}$ and $\Xi_2 = \Omega_2/\sqrt{D}$. Similarly, x in (7.29) and (7.30) is eliminated to have

$$N_{\beta\pi} = -2a_2 N_\beta \left(\frac{\Xi_1}{\rho - \sqrt{D}} e^{(-\rho + \sqrt{D})t/2} + \frac{\Xi_2}{\rho + \sqrt{D}} e^{(-\rho - \sqrt{D})t/2} \right) - \frac{N_\beta(a_1\alpha + a_2 N_b)}{\alpha(\rho - \alpha) - a_2 N_\beta} e^{-\rho t},$$

which and the conserved quantity (7.32) imply

$$(7.34) \quad \pi_1(t) = -\frac{2a_2\Xi_1}{\rho - \sqrt{D}} e^{(-\rho + \sqrt{D})t/2} + \frac{2a_2\Xi_2}{\rho + \sqrt{D}} e^{(-\rho - \sqrt{D})t/2} + n_2\beta_2\Xi_4 e^{-\alpha t} - \frac{a_1\alpha + a_2 N_b}{\alpha(\rho - \alpha) - a_2 N_\beta} e^{-\rho t},$$

$$(7.35) \quad \pi_2(t) = -\frac{2a_2\Xi_1}{\rho - \sqrt{D}} e^{(-\rho + \sqrt{D})t/2} + \frac{2a_2\Xi_2}{\rho + \sqrt{D}} e^{(-\rho - \sqrt{D})t/2} - n_1\beta_1\Xi_4 e^{-\alpha t} - \frac{a_1\alpha + a_2 N_b}{\alpha(\rho - \alpha) - a_2 N_\beta} e^{-\rho t},$$

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where $\Xi_4 = \Omega_4/N_\beta$. The optimal paths (7.34) and (7.35) are substituted for (7.17)' to obtain

$$(7.36) \quad u^1(t) = -\frac{2a_2n_1\Xi_1}{\rho-\sqrt{D}}e^{(\rho-\sqrt{D})t/2} + \frac{2a_2n_1\Xi_2}{\rho+\sqrt{D}}e^{(\rho+\sqrt{D})t/2} + \frac{\beta_1\beta_2^2\Xi_4}{B}e^{(\rho-\alpha)t} - \frac{n_1(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{22}b_1-b_{12}b_2}{B},$$

$$(7.37) \quad u^2(t) = -\frac{2a_2n_2\Xi_1}{\rho-\sqrt{D}}e^{(\rho-\sqrt{D})t/2} + \frac{2a_2n_2\Xi_2}{\rho+\sqrt{D}}e^{(\rho+\sqrt{D})t/2} - \frac{\beta_1^2\beta_2\Xi_4}{B}e^{(\rho-\alpha)t} - \frac{n_2(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{11}b_2-b_{12}b_1}{B}.$$

Finally, the optimal paths (7.33), (7.34) and (7.35) are used in (7.31) to have

$$(7.38) \quad \begin{aligned} y(t) = & \frac{n_1\beta_1-n_2\beta_2}{N_\beta}(\Xi_1e^{(\rho-\sqrt{D})t/2} + \Xi_2e^{(\rho+\sqrt{D})t/2}) \\ & + \Xi_3e^{\alpha t} + \frac{2\beta_1^2\beta_2((\rho-\alpha)(b_{12}n_1+b_{22}n_2)-(\rho-2\alpha)\beta_2)}{B\alpha(\rho-2\alpha)}\Xi_4e^{(\rho-\alpha)t} \\ & + \frac{(n_1\beta_1-n_2\beta_2)(a_1N_\beta+(\rho-\alpha)N_b)}{N_\beta(\alpha(\rho-\alpha)-a_2N_\beta)} + \frac{2\beta_1\beta_2(n_2(b_{22}b_1-b_{12}b_2)-n_1(b_{11}b_2-b_{12}b_1))}{BN_\beta\alpha}, \end{aligned}$$

where $\Xi_3 = 2\Omega_3/(N_\beta\alpha)$.

Theorem 7.4. *In the two-sector open-loop Nash strategies of finite horizon $T < \infty$, let each sector i ($i = 1, 2$) seek to maximize (7.1-i)'' under the constraint (7.2)'. Then the optimal paths $x(t)$, $u^1(t)$ and $u^2(t)$ are determined, completely as (7.33), (7.36) and (7.37).*

In the case of infinite horizon $T = \infty$, the optimal paths in the maximizing problem of (7.12)' under the constraints (7.13) with $i = 1, 2$ have to be feasible, i.e., they have to satisfy the transversality condition:

$$(7.39) \quad \lim_{t \rightarrow \infty} (\pi_1(t)z^1(t) + \pi_2(t)z^2(t)) = 0.$$

Such paths are called feasible optimal paths. By (7.10a)' and (7.10b)', the term $\pi_1(t)z^1(t) + \pi_2(t)z^2(t)$ is written as

$$\pi_1(t)z^1(t) + \pi_2(t)z^2(t) = \frac{1}{2}(\pi_1(t) + \pi_2(t))x(t) + \frac{1}{2}(\pi_1(t) - \pi_2(t))y(t),$$

for which the optimal paths (7.33), (7.34), (7.35) and (7.38) are substituted. Then, the result is a first order polynomial of $e^{(\rho-2\alpha-\sqrt{D})t/2}$, $e^{(\rho-2\alpha+\sqrt{D})t/2}$, $e^{(-\rho-\sqrt{D})t/2}$, $e^{(-\rho+\sqrt{D})t/2}$, $e^{\sqrt{D}t}$, $e^{-\sqrt{D}t}$, $e^{(\rho-2\alpha)t}$, $e^{-\alpha t}$ and $e^{-\rho t}$. Therefore, since $e^{(\rho-2\alpha-\sqrt{D})t/2}$, $e^{(-\rho-\sqrt{D})t/2}$, $e^{-\sqrt{D}t}$, $e^{-\alpha t}$ and $e^{-\rho t}$ go to zero as $t \rightarrow \infty$; the condition (7.39) requires that all of the coefficients of $e^{(\rho-2\alpha+\sqrt{D})t/2}$,

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$e^{(-\rho+\sqrt{D})t/2}$, $e^{\sqrt{D}t}$ and $e^{(\rho-2\alpha)t}$ (all of which go to ∞ as $t \rightarrow \infty$) and the constant term vanish:

$$\begin{aligned} (n_2\beta_2 - n_1\beta_1)\Xi_1\Xi_4 - \frac{n_2\beta_2 - n_1\beta_1}{N_\beta}\Xi_2 &= 0, \\ \left(a_1\alpha + a_2N_b + \frac{2a_1a_2(N_\beta + (\rho-\alpha)N_b)}{\rho+\sqrt{D}}\right)\Xi_2 &= 0, \\ \frac{4a_2}{\rho+\sqrt{D}}\Xi_2^2 &= 0, \\ \frac{2\beta_1^2\beta_2((\rho-\alpha)(b_{12}n_1+b_{22}n_2)-(\rho-2\alpha)\beta_2)}{B\alpha(\rho-2\alpha)}\Xi_4 &= 0, \\ \Xi_3 - \frac{2a_2\rho}{\alpha(\rho-\alpha)-a_2N_\beta}\Xi_1\Xi_2 &= 0, \end{aligned}$$

which imply that $\Xi_2 = \Xi_3 = \Xi_4 = 0$. Consequently, the optimal paths (7.33) (7.36) and (7.37) lead respectively to

$$(7.33)' \quad x(t) = \Xi_1 e^{(\rho-\sqrt{D})t/2} + \frac{a_1N_\beta + (\rho-\alpha)N_b}{\alpha(\rho-\alpha)-a_2N_\beta},$$

$$(7.36)' \quad u^1(t) = \frac{-2a_2n_1\Xi_1}{\rho-\sqrt{D}} e^{(\rho-\sqrt{D})t/2} - \frac{n_1(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{22}b_1-b_{12}b_2}{B},$$

$$(7.37)' \quad u^2(t) = -\frac{2a_2n_2\Xi_1}{\rho-\sqrt{D}} e^{(\rho-\sqrt{D})t/2} - \frac{n_2(a_1\alpha+a_2N_b)}{\alpha(\rho-\alpha)-a_2N_\beta} - \frac{b_{11}b_2-b_{12}b_1}{B}.$$

Theorem 7.5. *In the two-sector open-loop Nash strategies of infinite horizon $T = \infty$, let each sector i ($i = 1, 2$) seek to maximize (7.1-i)" under the constraints (7.2)'. Then there exist the feasible optimal paths of the form (7.33)', (7.36)' and (7.37)'.*

An Example. The results can be applied to the model of the voluntary contributions to the provision of a collectively produced good (Fershtman and Nitzan [8], in which the stock of total contributions K , the contribution rate of i -sector x_i , the discount rate r and the constant α are denoted here by x , u^i , ρ and γ respectively), by putting $a_1 = a\gamma$, $a_2 = -2b\gamma$, $b_1 = b_2 = 0$, $b_{11} = b_{22} = -c$, $b_{12} = 0$ and $g_1(u^2) = g_2(u^1) = 0$ in (7.1-i)", i.e.,

$$(7.1-i)''' \quad \int_0^T \left(\gamma(ax - bx^2) - \frac{1}{2}c(u^i)^2 \right) dt$$

$$(x < a/(2b), \ a, b, c, \gamma: \text{const.}, \ a > 0, \ b > 0, \ c > 0, \ \gamma > 0),$$

and by putting $\alpha = -\delta$ and $\beta_1 = \beta_2 = 1$ in the constraint (7.2)', i.e.,

$$(7.2)'' \quad \dot{x} = -\delta x + u^1 + u^2 \quad (\delta: \text{const.}, \ \delta > 0).$$

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Then, the conserved quantities (7.29)-(7.32) are reduced respectively to

$$\begin{aligned}\Omega_1 &= \left(\frac{\rho - \sqrt{D}}{2} x - \frac{a\gamma(\rho + 2\delta - \sqrt{D})}{4b\gamma} \right) e^{(-\rho + \sqrt{D})t/2} + \frac{\rho - \sqrt{D}}{c(\rho + 2\delta + \sqrt{D})} (\pi_1 + \pi_2) e^{(\rho + \sqrt{D})t/2}, \\ \Omega_2 &= \left(\frac{\rho + \sqrt{D}}{2} x - \frac{a\gamma(\rho + 2\delta + \sqrt{D})}{4b\gamma} \right) e^{-(\rho + \sqrt{D})t/2} + \frac{\rho + \sqrt{D}}{c(\rho + 2\delta - \sqrt{D})} (\pi_1 + \pi_2) e^{(\rho - \sqrt{D})t/2}, \\ \Omega_3 &= \frac{1}{c} (\delta x - \delta y - u^1 - u^2) e^{\delta t} - \frac{\rho + \delta}{(\rho + 2\delta)c^2} (\pi_1 - \pi_2) e^{(\rho + \delta)t}, \\ \Omega_4 &= (\pi_1 - \pi_2) e^{-\delta t},\end{aligned}$$

in which Ω_1 , Ω_2 and Ω_4 are the conserved quantities in the maximizing problem of (7.1-i)'' (i= 1, 2) under the constraint (7.2)'' in the two-sector open-loop Nash strategies. Moreover, the optimal path (7.33)' in the infinite horizon $T = \infty$ is also to

$$x(t) = \Xi_1 e^{(\rho - \sqrt{D})t/2} + \frac{2a\gamma}{\delta(\rho + \delta)c + 4b\gamma}.$$

So, it follows that

$$\lim_{t \rightarrow \infty} x(t) = \frac{2a\gamma}{\delta(\rho + \delta)c + 4b\gamma},$$

which is the level of the collective contribution at the stationary open-loop Nash equilibrium K^* appeared in ([8], Theorem 2, in which n is given here as $n = 2$).

Appendix: A Determination of Motions in the Central Force Problem

A.1 Introduction

As an illustration of the new operative method (Mimura and Nôno [32], Mimura, Ikeda and Fujiwara [31]) which grew up from the application of a suitable version of the Noether theorem [37] to the composite variational principle (Caviglia [3],[5]), it was derived a couple of independent conserved quantities (first integrals) for the motions of the following particle in the central force problem (see Whittaker [51], p.243):

A single particle moving in a plane under a central force directed towards a fixed center in a resisting medium, where the force is proportional to the particle's distance from the center and the medium imposes a retarding force equal to β times the velocity.

The origin is placed on the center of force and the position of particle at time t is defined by polar coordinates $(r(t), \varphi(t))$ to have the differential equations of the motion (e.g. Djukic [6]):

$$m(\ddot{r} - r\dot{\varphi}^2) + \beta\dot{r} + \sigma r = 0,$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) + \beta r\dot{\varphi} = 0,$$

where m ($m > 0$) is the mass of particle and σ ($\sigma > 0$) is the central force constant. So by putting

$$\mu = \frac{\beta}{2m}, \quad \omega^2 = \frac{\sigma}{m},$$

it follows that

$$(A.1) \quad \ddot{r} + 2\mu\dot{r} + \omega^2 r - r\dot{\varphi}^2 = 0,$$

$$(A.2) \quad r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + 2\mu r\dot{\varphi} = 0,$$

which are the Euler-Lagrange equations with the Lagrangian

$$L = \frac{1}{2}e^{2\mu t}(\dot{r}^2 + (r\dot{\varphi})^2 - (\omega r)^2).$$

The couple of conserved quantities of the equations (A.1) and (A.2) are [32, §4; 5, §6]

$$(A.3) \quad \Omega_1 = e^{2\mu t}(\frac{1}{2}\dot{r}^2 + \frac{1}{2}(r\dot{\varphi})^2 + \frac{1}{2}(\omega r)^2 + \mu r\dot{r}),$$

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$$(A.4) \quad \Omega_2 = e^{2\mu t} r^2 \dot{\varphi},$$

in which Ω_1 was obtained by Djukic [6] under the symmetry for the gauge-variant Lagrangians, while the equation (A.2) can be put as

$$\frac{d(r^2 \dot{\varphi})}{r^2 \dot{\varphi}} = -2\mu dt,$$

whose solution $r^2 \dot{\varphi} = \Omega_2 e^{-2\mu t}$ (Ω_2 : const.) leads to the appearance of Ω_2 of (A.4). In this paper, we show that the conserved quantities (A.3) and (A.4) contribute to determine completely the motions of the particle in the central force problem.

A.2 A determination of motions through the conserved quantities

In the couple of the conserved quantities Ω_1 and Ω_2 of the equations of the motion, $\dot{\varphi}$ can be eliminated to see

$$(r\dot{r})^2 + 2\mu r^3 \dot{r} + \omega^2 r^4 - 2\Omega_1 e^{-2\mu t} r^2 + (\Omega_2 e^{-2\mu t})^2 = 0,$$

which is transformed, by a change of variable $x = e^{\mu t} r$, into

$$(A.5) \quad (x\dot{x})^2 + (\omega^2 - \mu^2)x^4 - 2\Omega_1 x^2 + \Omega_2^2 = 0;$$

while (A.4) is also into

$$(A.6) \quad x^2 \dot{\varphi} = \Omega_2.$$

The motions of the particle in the considering central force problem can be determined completely by the equations (A.5) and (A.6).

1. We first settle the case with $\Omega_2 \neq 0$ which implies by (A.6) that $\dot{\varphi} \neq 0$, i.e., the particle is moving out of the straight. Then, Ω_1 and Ω_2 lie in the root which comes from the equation (A.5):

$$(A.7) \quad x\dot{x} = \pm \sqrt{-(\omega^2 - \mu^2)x^4 + 2\Omega_1 x^2 - \Omega_2^2},$$

satisfying the conditions: $\Omega_1 > 0$ if $\omega^2 - \mu^2 \geq 0$, and $\Omega_1^2 - (\omega^2 - \mu^2)\Omega_2^2 \geq 0$.

1.1. $\Omega_2 \neq 0$ and $\omega^2 - \mu^2 > 0$. By putting

$$a = \frac{\sqrt{\Omega_1^2 - (\omega^2 - \mu^2)\Omega_2^2}}{\omega^2 - \mu^2}, \quad b = \frac{\Omega_1}{\omega^2 - \mu^2},$$

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the equation (A.7) is written as

$$x\dot{x} = \pm\sqrt{\omega^2 - \mu^2}\sqrt{a^2 - (x^2 - b)^2}.$$

If $a = 0$, this equation has a solution $x^2 = b$; and if $a \neq 0$, by using a variable $y = x^2$, it leads to

$$\frac{dy}{\sqrt{a^2 - (y - b)^2}} = \pm 2\sqrt{\omega^2 - \mu^2} dt,$$

which is integrated:

$$\sin^{-1} \frac{y - b}{a} = \pm 2\sqrt{\omega^2 - \mu^2} t + \alpha \quad (\alpha: \text{const.}).$$

Consequently, together with $y = x^2 = b$ for $a = 0$, the solution y can be put as (replace $\pm\alpha$ with α)

$$y = \pm a \sin(2\sqrt{\omega^2 - \mu^2} t + \alpha) + b,$$

in which the minus sign is nonessential, since the constant α can be replaced with $\alpha + \pi$. Here note that the constants a and b ($b > 0$) satisfy $b^2 - a^2 = \Omega_2^2/(\omega^2 - \mu^2) > 0$, so that $y \geq b \pm a > 0$. For the solution, by a change of variable $\tau = 2\sqrt{\omega^2 - \mu^2} t + \alpha$, the equation (A.6) with $\Omega_2 = \pm\sqrt{(b^2 - a^2)(\omega^2 - \mu^2)}$ is transformed into

$$\frac{d\varphi}{d\tau} = \pm \frac{\sqrt{b^2 - a^2}}{2(a \sin \tau + b)} \quad (\Omega_2 \gtrless 0),$$

which is integrated:

$$\varphi = \pm \tan^{-1} \frac{b \tan(\frac{1}{2}\tau) + a}{\sqrt{b^2 - a^2}} + k \quad (k: \text{const.}).$$

In this way, the motion of the particle is determined:

$$\begin{aligned} r &= e^{-\mu t} \sqrt{a \sin(2\sqrt{\omega^2 - \mu^2} t + \alpha) + b}, \\ \varphi &= \pm \tan^{-1} \frac{b \tan(\sqrt{\omega^2 - \mu^2} t + \frac{1}{2}\alpha) + a}{\sqrt{b^2 - a^2}} + k \quad (\Omega_2 \gtrless 0). \end{aligned}$$

1.2. $\Omega_2 \neq 0$ and $\omega^2 - \mu^2 < 0$. By using the above b and

$$a = \frac{\sqrt{\Omega_1^2 - (\omega^2 - \mu^2)\Omega_2^2}}{\mu^2 - \omega^2},$$

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the equation (A.7) is written as

$$x\dot{x} = \pm\sqrt{\mu^2 - \omega^2}\sqrt{(x^2 - b)^2 - a^2};$$

which, by the variable $y = x^2$, leads to

$$\frac{dy}{\sqrt{(y - b)^2 - a^2}} = \pm 2\sqrt{\mu^2 - \omega^2}dt.$$

So that, through the integration:

$$\cosh^{-1} \frac{y - b}{a} = \pm 2\sqrt{\mu^2 - \omega^2}t + \alpha \quad (\alpha: \text{const.}),$$

the solution y can be put as (replace $\pm\alpha$ with α)

$$y = a \cosh(2\sqrt{\mu^2 - \omega^2}t + \alpha) + b,$$

where the constants a ($a > 0$) and b satisfy $a^2 - b^2 = \Omega_2^2/(\mu^2 - \omega^2) > 0$, so that $y \geq a + b > 0$. Accordingly, by the variable $\tau = 2\sqrt{\mu^2 - \omega^2}t + \alpha$, the equation (A.6) with $\Omega_2 = \pm\sqrt{(a^2 - b^2)(\mu^2 - \omega^2)}$ leads to

$$2\frac{d\varphi}{d\tau} = \pm \frac{\sqrt{a^2 - b^2}}{a \cosh \tau + b} \quad (\Omega_2 \geq 0).$$

Moreover, by a change of variable $\phi = \tanh(\frac{1}{2}\tau)$, this equation is transformed into

$$\frac{d\varphi}{d\phi} = \pm \frac{\sqrt{(a+b)/(a-b)}}{\phi^2 + (a+b)/(a-b)},$$

which is integrated:

$$\varphi = \pm \tan^{-1} \frac{(a-b)\phi}{\sqrt{a^2 - b^2}} + k \quad (k: \text{const.}).$$

Therefore the motion of the particle is determined:

$$r = e^{-\mu t} \sqrt{a \cosh(2\sqrt{\mu^2 - \omega^2}t + \alpha) + b},$$

$$\varphi = \pm \tan^{-1} \frac{(a-b) \tanh(\sqrt{\mu^2 - \omega^2}t + \frac{1}{2}\alpha)}{\sqrt{a^2 - b^2}} + k \quad (\Omega_2 \geq 0).$$

1.3. $\Omega_2 \neq 0$ and $\omega^2 - \mu^2 = 0$. By putting

$$a = \sqrt{2\Omega_1}, \quad b = \frac{\Omega_2}{a} = \frac{\Omega_2}{\sqrt{2\Omega_1}},$$

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the equation (A.7) is written as

$$\frac{xdx}{\sqrt{x^2 - b^2}} = \pm a dt.$$

Then, through the integration

$$\sqrt{x^2 - b^2} = \pm at + \alpha,$$

x^2 can be put as (replace $\pm\alpha$ with α)

$$x^2 = (at + \alpha)^2 + b^2.$$

Accordingly, by a change of variable $\tau = at + \alpha$, the equation (A.6) with $\Omega_2 = ab$ is transformed into

$$\frac{d\varphi}{d\tau} = \frac{b}{\tau^2 + b^2},$$

which is integrated:

$$\varphi = \tan^{-1} \frac{\tau}{b} + k \quad (k: \text{const.}).$$

Therefore the motion of the particle is determined:

$$r = e^{-\mu t} \sqrt{(at + \alpha)^2 + b^2},$$

$$\varphi = \tan^{-1} \frac{at + \alpha}{b} + k.$$

2. In the following case with $\Omega_2 = 0$, we leave the particular solution $r = 0$ of (A.1) and (A.2) out of consideration, since it means that the particle stays at the origin (center of force). Then (A.6) implies $\dot{\varphi} = 0$, i.e., the particle is moving straight towards the origin with coordinate $r(t)$ on the line. In this case, (A.1) leads to the equation of linearly damped one-dimensional harmonic oscillator. And the equation (A.7) is reduced to

$$(A.8) \quad \dot{x} = \pm \sqrt{-(\omega^2 - \mu^2)x^2 + 2\Omega_1}.$$

2.1. $\Omega_2 = 0$ and $\omega^2 - \mu^2 > 0$. Since $\Omega_1 > 0$ in the root of (A.8), by putting

$$a = \sqrt{\frac{2\Omega_1}{\omega^2 - \mu^2}},$$

the equation (A.9) leads to

$$\frac{dx}{\sqrt{a^2 - x^2}} = \pm \sqrt{\omega^2 - \mu^2} dt,$$

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whose solution

$$\sin^{-1} \frac{x}{a} = \pm \sqrt{\omega^2 - \mu^2} t + \alpha \quad (\alpha: \text{const.})$$

is arranged in $r = e^{-\mu t} x$ to obtain (the minus sign is omitted as remmaked in 1.1)

$$r = ae^{-\mu t} \sin(\sqrt{\omega^2 - \mu^2} t + \alpha).$$

2.2. $\Omega_2 = 0$ and $\omega^2 - \mu^2 < 0$. By putting

$$a = \sqrt{\frac{2|\Omega_1|}{\mu^2 - \omega^2}},$$

the equation (A.8) leads to

$$\frac{dx}{\sqrt{x^2 \pm a^2}} = \pm \sqrt{\mu^2 - \omega^2} dt,$$

in which $\pm a^2$ correspond respectively to $\Omega_1 \gtrless 0$, while $a = 0$ if $\Omega_1 = 0$. The respective integrations

$$\sinh^{-1} \frac{x}{a} = \pm \sqrt{\mu^2 - \omega^2} t + \alpha \quad (\alpha: \text{const.}), \quad \text{if } \Omega_1 > 0;$$

$$\cosh^{-1} \frac{x}{a} = \pm \sqrt{\mu^2 - \omega^2} t + \alpha \quad (\alpha: \text{const.}), \quad \text{if } \Omega_1 < 0;$$

$$x = \alpha e^{\pm \sqrt{\mu^2 - \omega^2} t} \quad (\alpha: \text{const.}), \quad \text{if } \Omega_1 = 0;$$

are arranged respectively in $r = e^{-\mu t} x$ to obtain

$$r = \pm ae^{-\mu t} \sinh(\sqrt{\mu^2 - \omega^2} t + \alpha), \quad \text{if } \Omega_1 > 0;$$

$$r = ae^{-\mu t} \cosh(\sqrt{\mu^2 - \omega^2} t + \alpha), \quad \text{if } \Omega_1 < 0;$$

$$r = \alpha e^{-\mu t} e^{\pm \sqrt{\mu^2 - \omega^2} t}, \quad \text{if } \Omega_1 = 0.$$

2.3. $\Omega_2 = 0$ and $\omega^2 - \mu^2 = 0$. In this case, from $\dot{x} = \pm \sqrt{2\Omega_1}$ immediately follows the solution

$$r = \pm e^{-\mu t} (\sqrt{2\Omega_1} t + \alpha) \quad (\alpha: \text{const}).$$

Thus the motions of the particle in the considering central force problem are determined completely. In conclusion, the results are summarized:

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Theorem. Let a single particle of mass m ($m > 0$) with polar coordinates $(r(t), \varphi(t))$ is moving under a central force σr (σ : const., $\sigma > 0$) directed towards the origin in a resisting medium which imposes a retarding force equal to β (β : const., $\beta > 0$) times the velocity.

When initial position (r_0, φ_0) and velocity $(\dot{r}_0, \dot{\varphi}_0)$ of the particle are given, the conserved quantities Ω_1 and Ω_2 can be evaluated by substituting the data for (A.3) and (A.4). And then the motions of the particle are determined completely as follows, where $K = (4m\sigma - \beta^2)/4m^2$ and $K \geq 0$ implies that $\Omega_1 > 0$.

In the case with $\Omega_2 \neq 0$, i.e., the particle is moving out of the straight, then two-dimensional motions are determined as

$$\begin{cases} r = e^{-\mu t} \sqrt{a \sin(2\sqrt{K}t + \alpha) + b}, \\ \varphi = \pm \tan^{-1} \frac{b \tan(\sqrt{K}t + \frac{1}{2}\alpha) + a}{\sqrt{b^2 - a^2}} + k \end{cases} \quad \text{if } K > 0; \quad (\Omega_2 \geq 0),$$

$$\begin{cases} r = e^{-\mu t} \sqrt{a \cosh(2\sqrt{-K}t + \alpha) + b}, \\ \varphi = \pm \tan^{-1} \frac{(a - b) \tanh(\sqrt{-K}t + \frac{1}{2}\alpha)}{\sqrt{a^2 - b^2}} + k \end{cases} \quad \text{if } K < 0; \quad (\Omega_2 \geq 0),$$

$$\begin{cases} r = e^{-\mu t} \sqrt{(at + \alpha)^2 + b^2}, \\ \varphi = \tan^{-1} \frac{at + \alpha}{b} + k, \end{cases} \quad \text{if } K = 0;$$

where a and b are the constants:

$$a = \frac{\sqrt{\Omega_1^2 - K\Omega_2^2}}{K}, \quad b = \frac{\Omega_1}{K}, \quad \text{if } K > 0;$$

$$a = \frac{\sqrt{\Omega_1^2 - K\Omega_2^2}}{-K}, \quad b = \frac{\Omega_1}{K}, \quad \text{if } K < 0;$$

$$a = \sqrt{2\Omega_1}, \quad b = \frac{\Omega_2}{\sqrt{2\Omega_1}}, \quad \text{if } K = 0.$$

respectively; while the constants k and α are specified by the initial data.

Particularly in the case with $\Omega_2 = 0$, i.e., the particle is moving straight towards the origin with a coordinate $r(t)$ on the line, then one-dimensional motions are determined as

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$$\begin{aligned}
 r &= ae^{-\mu t} \sin(\sqrt{K}t + \alpha), & \text{if } K > 0; \\
 \begin{cases} r = \pm ae^{-\mu t} \sinh(\sqrt{-K}t + \alpha) & (\Omega_1 > 0), \\ r = ae^{-\mu t} \cosh(\sqrt{-K}t + \alpha) & (\Omega_1 < 0), \\ r = \alpha e^{-\mu t} e^{\pm\sqrt{-K}t} & (\Omega_1 = 0), \end{cases} & \text{if } K < 0; \\
 r &= \pm e^{-\mu t}(at + \alpha), & \text{if } K = 0;
 \end{aligned}$$

where a is the constant:

$$\begin{aligned}
 a &= \sqrt{\frac{2\Omega_1}{K}}, & \text{if } K > 0; \\
 a &= \sqrt{\mp \frac{2\Omega_1}{K}} \quad (\Omega_1 \geq 0), & \text{if } K < 0; \\
 a &= \sqrt{2\Omega_1}, & \text{if } K = 0;
 \end{aligned}$$

respectively, while the constant α is specified by the initial data.

Remark A.1. In the case of $K < 0$ with $\Omega_2 \neq 0$, by replacing the constant α with $\pm \log(2\alpha^2/a)$ ($\Omega_2 \geq 0$), we have the other appearance of r :

$$r = e^{-\mu t} \sqrt{\alpha^2 e^{\pm 2\sqrt{-K}t} + (a/2\alpha)^2 e^{\mp 2\sqrt{-K}t} + b} \quad (\Omega_2 \geq 0).$$

Remark A.2. Let $\Omega_2 \rightarrow 0$ in the case with $\Omega_2 \neq 0$. Then, $a \rightarrow \Omega_1/K = b$ if $K > 0$, $a \rightarrow \mp \Omega_1/K = \mp b$ ($\Omega_1 \geq 0$) if $K < 0$ and $b \rightarrow 0$ if $K = 0$; accordingly the angle φ in each case of K converges to a constant. And, in view of that for $K > 0$ with $a = b = \Omega_1/K$:

$$a \sin(2\sqrt{K}t + \alpha) + b = \frac{2\Omega_1}{K} \sin^2(\sqrt{K}t + \frac{1}{2}\alpha + \frac{1}{4}\pi);$$

and for $K < 0$ with $a = \mp b = \mp \Omega_1/K$ ($\Omega_1 \geq 0$):

$$\begin{aligned}
 a \cosh(2\sqrt{-K}t + \alpha) + b &= -\frac{2\Omega_1}{K} \sinh^2(\sqrt{-K}t + \frac{1}{2}\alpha) & (\Omega_1 > 0), \\
 a \cosh(2\sqrt{-K}t + \alpha) + b &= \frac{2\Omega_1}{K} \cosh^2(\sqrt{-K}t + \frac{1}{2}\alpha) & (\Omega_1 < 0),
 \end{aligned}$$

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the particle's distance r in each case converges respectively to that (up to the sign) in each case with $\Omega_2 = 0$ except the case of $K < 0$ with $\Omega_1 = 0$. However we can avoid the peculiarity by means of the appearance of r in the remark 1 (the case of $K < 0$ with $\Omega_2 \neq 0$). In fact, for $a = \mp b = \mp \Omega_1/K$ ($\Omega_1 \geq 0$), the terms in the root lead to

$$\alpha^2 e^{\pm 2\sqrt{-K}t} + \frac{a^2}{4\alpha^2} e^{\mp 2\sqrt{-K}t} + b = \left(\alpha e^{\pm \sqrt{-K}t} + \frac{\Omega_1}{2\alpha K} e^{\mp \sqrt{-K}t} \right)^2 \quad (\Omega_2 \geq 0),$$

which turns into

$$a^2 \sinh^2(\sqrt{-K}t + \gamma), \quad \text{if } \Omega_1 > 0,$$

$$a^2 \cosh^2(\sqrt{-K}t + \gamma), \quad \text{if } \Omega_1 < 0,$$

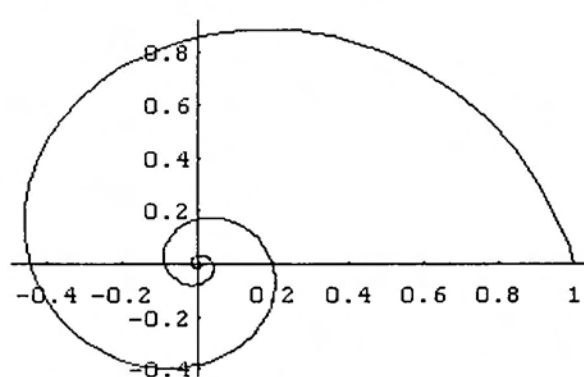
where $a = \sqrt{\mp(2\Omega_1/k)}$ ($\Omega_1 \geq 0$) and $\gamma = \pm \log(2\alpha/a)$ ($\Omega_2 \geq 0$). Thus the respective motions with $\Omega_2 = 0$ can be regarded as the limiting case: $\Omega_2 \rightarrow 0$ of that with $\Omega_2 \neq 0$.

Let a single particle of mass $m = 1$, moving against a medium with retarding force constant $\beta = 2$, have the initial position $(r_0, \varphi_0) = (1, 0)$ and velocity $(\dot{r}_0, \dot{\varphi}_0) = (-1, 5)$. Then $\Omega_2 = 5$, and μ, ω, K, Ω_1 are determined according to the following central force constants σ :

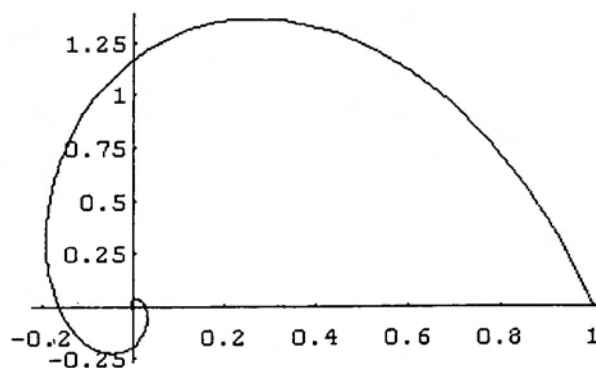
σ	β	μ	ω	K	Ω_1
16	2	1	4	15	20
4	2	1	2	3	14
2	2	1	$\sqrt{2}$	1	13
1	2	1	1	0	12.5
0.2	2	1	$1/\sqrt{5}$	-0.8	12.1
0	2	1	0	-1	12

For the values of σ and $\beta = 2$, the trajectories of the particle determined in the theorem are as follows.

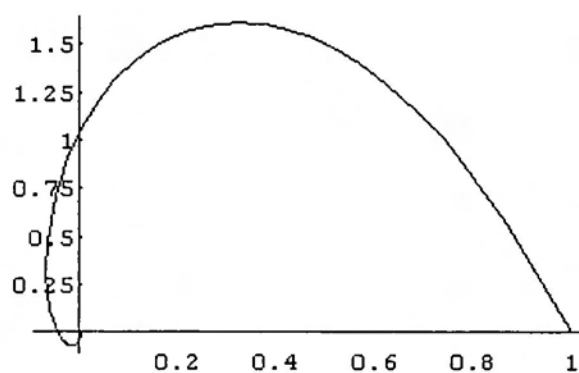
Appendix: A Determination of Motions in the Central Force Problem



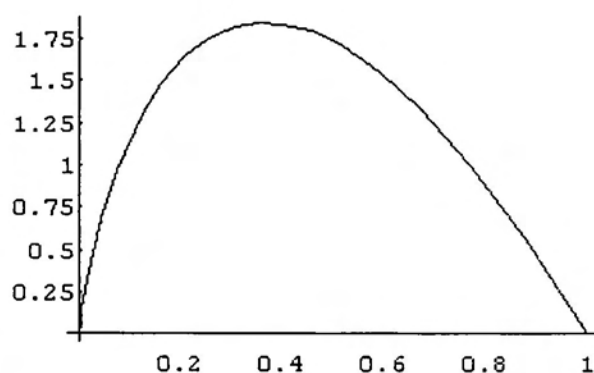
$$\sigma = 16 \quad \beta = 2$$



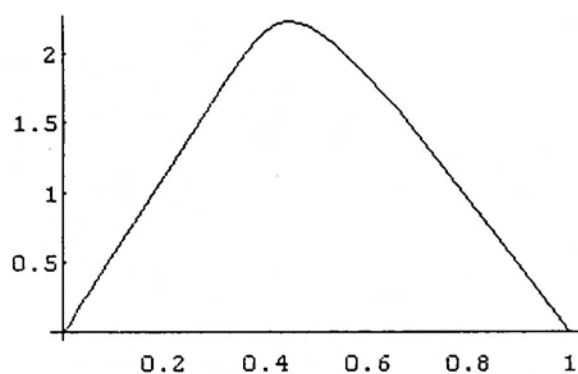
$$\sigma = 4 \quad \beta = 2$$



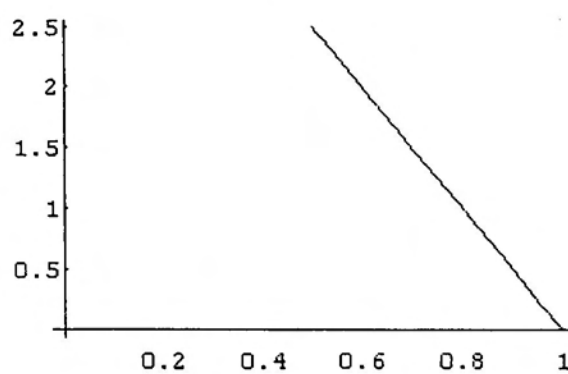
$$\sigma = 2 \quad \beta = 2$$



$$\sigma = 1 \quad \beta = 2$$



$$\sigma = 0.2 \quad \beta = 2$$



$$\sigma = 0 \quad \beta = 2$$

Conclusion

In this paper we have formulated new procedure for the derivation of conserved quantities within the context of exterior differential calculus. And then the procedure has been realized and detailed in the applications to the models below in optimal control problems.

We have generalized the traditional models to the best of our ability for the derivation of conserved quantities. The results indicate that the method formulated in this paper represents a useful way of discovering conserved quantities. The new approach enables us to find various conserved quantities including non-Noether ones. As seen below, it has contributed to the dynamic analysis of considering models. In what follows, special mentions are made in the results.

Growth model of von Neumann type. In the growth model of von Neumann type, Samuelson put the objective functional as an integration of the capital formation $U = \dot{K}^1$ to derive the invariance of the income-wealth ratio. We have shown that the invariance still exists even if U is replaced with $U = \dot{K}^1 \partial G / \partial \dot{K}^1 + \dot{K}^2 \partial G / \partial \dot{K}^2$ ($G = G(K)$ is first degree homogeneous with respect to \dot{K}^1 and \dot{K}^2) and also the first degree homogeneous transformation function $F(\dot{K}, K) = 0$ with arbitrary degree homogeneous one.

Neoclassical growth models. We have discovered some conserved quantities in the generalization of the neoclassical growth models including the model to which Sato [44] gave a negative answer for the existence of global conserved quantities from Noether theorem. Optimal paths has been determined for some models, e.g., the model with Cobb-Douglas production function.

A model in the intergenerational problem. By replacing with the constant consumption with exponentially growing consumption, we have generalized the Hotelling rule and the Hartwick IRR (investment resource rents) rule for the model in the intergenerational problem with the variables: extracted amount of exhaustible resources with stocks, reproducible capital and consumption. A way of determining optimal paths are given with the aid of two generalized rules. It is realized for a model with accounting relation in which production function is given by the Cobb-Douglas one.

A Model in the q -theory of investment. Generalizing the Hayashi model for the Tobin's q -theory of investment, we have derived the conserved quantities and then determined the optimal paths, through which, given every detail of the theory.

Multi-sector growth models of Ramsey type. We have introduced some generalized growth models of Ramsey type in one, two or three sectors. One more model has been given by replacing the non-homogeneous utility function of the control variables in the generalized models with non-homogeneous second order polynomial of the state variable and

the control variables. For the models, through derived conserved quantities, we have determined the optimal paths including those which illustrated the Parete-efficient steady-state in (Fershtman and Nitzan [8]).

Model in multi-sector open-loop Nash strategies. For the open-loop Nash strategies, extending the original variables, we have found the composite maximizing problem (new way of discovering conserved quantities which have been never derivable from the usual variational principle). By applying the new procedure in 1 to the composite maximizing problem, we have found conserved quantities and then determined optimal paths for the model in two-sector open-loop Nash strategies. The optimal paths illustrate the level of the collective contribution of public goods at the stationary open-loop Nash equilibrium in (Fershtman and Nitzan [8]).

A model in the central force problem. As final application, we have discussed a single particle moving under a central force directed towards a fixed center in a resisting medium, where the force is proportional to the particle's distance from the center and the medium imposes a retarding force proportional to the velocity. The new procedure in 1 have enabled us to find two independent conserved quantities, through which we have determined the equation of orbit of the particle.

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Acknowledgement

The author would like to express her deep thanks to Professor Fumitake Mimura of Kyushu Institute of Technology for his constant guides and encouragements throughout this work.

A special gratitude goes to Professor Shoji Harada, Professor Fujio Kubo, Professor Michihiro Nishi and Professor Akimichi Okuma of Kyushu Institute of Technology for their many helpful discussions and suggestions.

Finally, the author must express her great thanks to her colleagues, especially Miss Naoko Sumi for her helpful assistances.

List of Published Papers

- [1] F. Mimura, T. Ikeda and F. Fujiwara, A geometric derivation of new conservation laws, *Bull. Kyushu Inst. Tech. Math. Natur. Sci.*, **43** (1996), 37-55.
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- [14] F. Fujiwara, F. Mimura and T. Nôno, New derivation of conservation laws in three-sector growth model, *Reports of the Japan Institute of Industrial Science*, **10** (2000), 1-10.
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