Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 56, 2009, pp. 11–18

SOME NOTES ON FIXED POINT THEOREMS WITH CONSTANTS

Misako Kikkawa and Tomonari Suzuki

Abstract

We give some notes on recent fixed point theorems with constants which are generalizations of the Banach contraction principle. We also discuss nonexpansive semigroups with constants.

1. Introduction

The following fixed point theorem is proved in [8]. This theorem is a generalization of the famous Banach contraction principle [1].

Theorem 1 ([8]). *Define a function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2 \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2} \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases} \tag{1}$$

Let (X,d) be a complete metric space and let T be a mapping on X. Assume that there exists $r \in [0,1)$ such that

$$\theta(r) d(x,Tx) \le d(x,y) \quad \text{implies} \quad d(Tx,Ty) \le r\, d(x,y) \tag{2}$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Remark. For every $r \in [0,1)$, $\theta(r)$ is the best constant.

While the Banach contraction principle does not characterize the metric completeness of X (see [2]), Theorem 1 does characterize the metric completeness as follows.

Theorem 2 ([8]). *For a metric space (X,d), the following are equivalent:*
(i) *X is complete.*
(ii) *Every mapping T on X satisfying the following has a fixed point:*
- *There exists $r \in [0,1)$ such that (2) for all $x, y \in X$.*

Date: Received November 28, 2008.
2000 *Mathematics Subject Classification*. 54H25, 47H20.
Key words and phrases. The Banach contraction principle, nonexpansive semigroup.
The second author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

(iii) *There exists $r \in (0,1)$ such that every mapping T on X satisfying the following has a fixed point:*
- $\frac{1}{10000} d(x, Tx) \le d(x, y)$ *implies* $d(Tx, Ty) \le r\, d(x, y)$ *for all* $x, y \in X$.

In recent years we have proven some fixed point theorems related to Theorem 1. See [3, 5, 6, 9–11]. In this paper, we give some notes on the results in [5, 9, 11]. In Section 2 we prove a fixed point theorem for multivalued mappings, and in Section 3, we give an alternative proof of a common fixed point theorem for commuting mappings. Then in Section 4, we give a comment for nonexpansive semigroups.

2. A Nadler-type theorem

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ the family of all nonempty closed bounded subsets of X. Let $H(\cdot,\cdot)$ be the *Hausdorff metric*, i.e.,

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\} \qquad \text{for } A, B \in \mathrm{CB}(X),$$

where $\delta(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$.

The following is a multivalued version of Theorem 1 and also a generalization of Nadler's fixed point theorem [7].

THEOREM 3 ([5, 11]). *Define a function η from $[0,1)$ into $(1/2,1]$ by*

$$\eta(r) = \begin{cases} 1 & \text{if } 0 \le r < 1/2 \\ (1+r)^{-1} & \text{if } 1/2 \le r < 1. \end{cases} \tag{3}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists $r \in [0,1)$ such that

$$\eta(r) d(x, Tx) \le d(x, y) \qquad \text{implies} \qquad H(Tx, Ty) \le r\, d(x, y) \tag{4}$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

We note that x and y do not play the same role in (4). Motivated by this fact, we give a slight generalization of Theorem 3.

THEOREM 4. *Define a function η by (3). Let (X, d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists $r \in [0,1)$ such that*

$$\eta(r) d(x, Tx) \le d(x, y) \qquad \text{implies} \qquad \delta(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

PROOF. We first show the conclusion in the case where $r \in [0, 1/2)$. Take a real number r_1 with $r < r_1 < 1/2$. Let $u_1 \in X$ and $u_2 \in Tu_1$. Since $\eta(r) d(u_1, Tu_1) \le \eta(r) d(u_1, u_2) \le d(u_1, u_2)$, we have

$$d(u_2, Tu_2) \le \delta(Tu_1, Tu_2) \le r\, d(u_1, u_2).$$

So, there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \le r_1\, d(u_1, u_2)$. Thus, we have a sequence $\{u_n\} \subset X$ such that $u_{n+1} \in Tu_n$ and $d(u_{n+1}, u_{n+2}) \le r_1\, d(u_n, u_{n+1})$. We have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \le \sum_{n=1}^{\infty} {r_1}^{n-1}\, d(u_1, u_2) < \infty$$

and hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

Next we show that

$$d(z, Tx) \le r\, d(z, x) \tag{5}$$

holds for $x \in X$ with $x \ne z$. Since $\{u_n\}$ converges and $u_{n+1} \in Tu_n$, $\eta(r)d(u_n, Tu_n) \le d(u_n, x)$ holds for sufficiently large $n \in \mathbf{N}$. Hence $\delta(Tu_n, Tx) \le r\, d(u_n, x)$, which implies $d(u_{n+1}, Tx) \le r\, d(u_n, x)$. Letting n tend to ∞, we have (5).

Arguing by contradiction, we assume $z \notin Tz$. Since Tz is a closed, $d(z, Tz) > 0$. We fix $\varepsilon > 0$ with $2r_1(d(z, Tz) + \varepsilon) < d(z, Tz)$. Furthermore take $a \in Tz$ with $d(z, a) \le d(z, Tz) + \varepsilon$. Since $a \ne z$, from (5), we have $d(z, Ta) \le r\, d(z, a)$. So there exists $b \in Ta$ such that $d(z, b) \le r_1\, d(z, a)$. On the other hand, since $a \in Tz$ and $\delta(Tz, Ta) \le r\, d(z, a)$, there exists $b' \in Ta$ such that $d(a, b') \le r_1\, d(z, a)$. So we have

$$\eta(r)d(a, Ta) = d(a, Ta) \le d(a, b') \le r_1\, d(z, a) \le d(z, a).$$

Hence $\delta(Ta, Tz) \le r\, d(z, a)$ holds. So we can choose $a' \in Tz$ with $d(b, a') \le r_1\, d(z, a)$. Therefore we obtain

$$\begin{aligned} d(z, Tz) &\le d(z, a') \le d(z, b) + d(b, a') \le 2r_1\, d(z, a) \\ &\le 2r_1(d(z, Tz) + \varepsilon) < d(z, Tz). \end{aligned}$$

This is a contradiction. So we obtain $z \in Tz$.

In the case where $r \in [1/2, 1)$, we take a real number r_1 with $r < r_1 < 1$. Then as in the case where $r \in [0, 1/2)$, there exists a sequence $\{u_n\} \subset X$ such that $u_{n+1} \in Tu_n$ and $\{u_n\}$ converges to some point $z \in X$. Furthermore we obtain

$$d(z, Tx) \le r\, d(z, x)$$

for $x \in X$ with $x \ne z$. Next we show that $\delta(Tx, Tz) \le r\, d(x, z)$ for $x \in X$. This is obvious in the case where $x = z$. In the case where $x \ne z$, there exists a sequence $\{y_n\} \subset Tx$ such that $d(z, y_n) \le d(z, Tx) + \frac{1}{n}\, d(x, z)$ for $n \in \mathbf{N}$. Since

$$\begin{aligned} d(x, Tx) &\le d(x, y_n) \le d(x, z) + d(z, y_n) \le d(x, z) + d(z, Tx) + \frac{1}{n}\, d(x, z) \\ &\le d(x, z) + r\, d(x, z) + \frac{1}{n}\, d(x, z) = \left(1 + r + \frac{1}{n}\right) d(x, z), \end{aligned}$$

$(1/(1+r))d(x, Tx) \le d(x, z)$ holds. From the assumption, we have $\delta(Tx, Tz) \le r\, d(x, z)$. Hence

$$d(z, Tz) = \lim_{n\to\infty} d(u_{n+1}, Tz) \le \lim_{n\to\infty} \delta(Tu_n, Tz) \le \lim_{n\to\infty} r\, d(u_n, z) = 0.$$

Since Tz is closed, we obtain $z \in Tz$. This completes the proof. □

REMARK. We do not know whether Theorem 4 is a strict generalization of Theorem 3.

3. A Jungck-type theorem

In [5], we generalized Theorem 1 as Jungck [4] generalized the Banach contraction principle. The proof given in [5] is a little complicated. So, in this section, we give a simpler proof.

THEOREM 5 ([5]). *Define a function θ by (1). Let (X, d) be a complete metric space. Let S and T be mappings on X satisfying the following:*
(a) *S is continuous.*
(b) *$T(X) \subset S(X)$.*
(c) *S and T commute.*
Suppose that there exists $r \in [0, 1)$ such that

$$\theta(r)d(Sx, Tx) \le d(Sx, Sy) \qquad \textit{implies} \qquad d(Tx, Ty) \le r\, d(Sx, Sy)$$

for all $x, y \in X$. Then there exists a unique common fixed point of S and T.

REMARK. $\theta(r)$ is the best constant for every r.

PROOF. From (b), we can define a mapping I on X with $SIx = Tx$ for $x \in X$. Since $\theta(r) \le 1$, $\theta(r)d(Sx, Tx) = \theta(r)d(Sx, SIx) \le d(Sx, SIx)$ holds. Hence from the assumption, we have

$$d(SIx, SIIx) = d(Tx, TIx) \le r\, d(Sx, SIx) \tag{6}$$

for all $x \in X$. Let $u \in X$. Put $u_0 = u$ and $u_n = I^n u$ for $n \in \mathbf{N}$. Then $Su_{n+1} = Tu_n$ obviously holds. By (6), we have

$$\begin{aligned} d(Su_n, Su_{n+1}) &= d(SIu_{n-1}, SIIu_{n-1}) \le r\, d(Su_{n-1}, SIu_{n-1}) \\ &= r\, d(Su_{n-1}, Su_n) \le \cdots \le r^n\, d(Su_0, Su_1) \end{aligned}$$

for $n \in \mathbf{N}$. So we have $\sum_{n=0}^{\infty} d(Su_n, Su_{n+1}) < \infty$ and hence $\{Su_n\}$ is a Cauchy sequence. Since X is complete, $\{S^n u\}$ converges to some point $z \in X$.

Next we show

$$d(Tx, z) \le r\, d(Sx, z) \tag{7}$$

holds for $x \in X$ with $Sx \neq z$. Since $Su_n \to z$ we have $\theta(r)d(Su_n, Tu_n) \leq d(Su_n, Sx)$ for sufficiently large $n \in \mathbf{N}$. Hence we have $d(Tu_n, Tx) \leq r\, d(Su_n, Sx)$. Letting n tend to ∞, we have (7).

Let us prove that z is a fixed point of S. Arguing by contradiction, we assume $z \neq Sz$. We have

$$\lim_{n\to\infty} \theta(r)d(Su_n, Tu_n) = 0 < d(z, Sz) = \lim_{n\to\infty} d(Su_n, SSu_n).$$

So, $d(Tu_n, TSu_n) \leq r\, d(Su_n, SSu_n)$ holds for sufficiently large $n \in \mathbf{N}$. Then we have

$$\begin{aligned} d(z, Sz) &= \lim_{n\to\infty} d(Su_{n+1}, SSu_{n+1}) = \lim_{n\to\infty} d(Tu_n, STu_n) \\ &= \lim_{n\to\infty} d(Tu_n, TSu_n) \leq \lim_{n\to\infty} r\, d(Su_n, SSu_n) = r\, d(z, Sz). \end{aligned}$$

This is a contradiction. Therefore we obtain $z = Sz$.

We shall prove that z is a fixed point of T, dividing the following three cases:

- $0 \leq r < \frac{1}{\sqrt{2}}$
- $\frac{1}{\sqrt{2}} \leq r < 1$ and $\#\{n : Su_n \neq z\} = \infty$
- $\frac{1}{\sqrt{2}} \leq r < 1$ and $\#\{n : Su_n \neq z\} < \infty$

In the case where $0 \leq r < 1/\sqrt{2}$, we note that $\theta(r) \leq (1-r)r^{-2}$. Arguing by contradiction, we assume $SIz = Tz \neq z$. We note $SI^2z \neq z$ because

$$d(SIz, SI^2z) \leq r\, d(Sz, SIz) = r\, d(z, SIz).$$

Since

$$d(z, SIz) \leq d(z, SI^2z) + d(SI^2z, SIz) \leq d(z, SI^2z) + r\, d(z, SIz),$$

we have $(1-r)d(z, SIz) \leq d(z, SI^2z)$ and hence

$$\begin{aligned} \theta(r)d(SI^2z, SI^3z) &\leq (1-r)r^{-2}\, d(SI^2z, SI^3z) \\ &\leq (1-r)d(z, SIz) \\ &\leq d(z, SI^2z). \end{aligned}$$

By the assumption, we have $d(SI^3z, SIz) \leq r\, d(SI^2z, z)$. Since

$$d(SI^3z, z) \leq r\, d(SI^2z, z) \leq r^2\, d(SIz, z),$$

we have

$$\begin{aligned} d(z, SIz) &\leq d(z, SI^3z) + d(SI^3z, SIz) \leq d(z, SI^3z) + r\, d(SI^2z, z) \\ &\leq 2r^2\, d(SIz, z) < d(SIz, z). \end{aligned}$$

This is a contradiction. Therefore we obtain $z = Tz$.

In the case where $1/\sqrt{2} \le r < 1$ and $\#\{n : Su_n \ne z\} = \infty$, then there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $Su_{n_j} \ne z$. By (7), we have

$$\begin{aligned}\theta(r)d(Su_{n_j}, Tu_{n_j}) &\le \theta(r)(d(Su_{n_j}, z) + d(Tu_{n_j}, z)) \\ &\le \theta(r)(d(Su_{n_j}, z) + r\, d(Su_{n_j}, z)) \\ &= d(Su_{n_j}, z).\end{aligned}$$

From the assumption we have $d(Tu_{n_j}, Tz) \le r\, d(Su_{n_j}, z)$ and hence

$$d(z, Tz) = \lim_{j\to\infty} d(Su_{n_j+1}, Tz) = \lim_{j\to\infty} d(Tu_{n_j}, Tz) \le \lim_{j\to\infty} r\, d(Su_{n_j}, z) = 0.$$

Therefore we obtain $Tz = z$.

In the case where $1/\sqrt{2} \le r < 1$ and $\#\{n : Su_n \ne z\} < \infty$, there exists $\nu \in \mathbf{N}$ such that $Su_n = z$ for $n \ge \nu$. In particular, $Su_\nu = Su_{\nu+1} = z$, which implies

$$Tz = TSu_\nu = STu_\nu = SSu_{\nu+1} = Sz = z.$$

We have shown that z is a common fixed point of S and T in all the cases.

We conclude the proof by showing that the common fixed point is unique. Suppose that y is a common fixed point of S and T. Since $\theta(r)d(Sz, Tz) = 0 \le d(Sz, Sy)$, we have

$$d(z, y) = d(Tz, Ty) \le r\, d(Sz, Sy) = r\, d(z, y).$$

Therefore we obtain $z = y$. □

4. Nonexpansive semigroups

Let T be a mapping on a subset C of a Banach space E. In [9], we considered the following condition:

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \qquad \text{implies} \qquad \|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. And we proved fixed point theorems for such a mapping. In this section, we shall show that we cannot consider the semigroup version of this condition.

Proposition 1. *Let $\{T(t) : t \ge 0\}$ be a family of mappings on a subset C of a Banach space E. Assume that*

- *$T(0)$ is the identity mapping on C;*
- *$T(s + t) = T(s) \circ T(t)$ for $s, t \ge 0$;*
- *$t \mapsto T(t)x$ is continuous for $x \in C$.*

Then the following are equivalent:

(i) *$\|T(t)x - T(t)y\| \le \|x - y\|$ for $t \ge 0$ and $x, y \in C$.*

(ii) *There exists* $\beta \in (0, \infty)$ *such that*

$$\beta \|x - T(t)x\| \le \|x - y\| \quad \textit{implies} \quad \|T(t)x - T(t)y\| \le \|x - y\|$$

for $t \ge 0$ *and* $x, y \in C$.

Proof. It is obvious that (i) implies (ii). We shall show that (ii) implies (i). We assume (ii). Arguing by contradiction, we assume that (i) does not hold, that is, there exist $\tau \in [0, \infty)$ and $x, y \in C$ such that

$$\|T(\tau)x - T(\tau)y\| > \|x - y\|. \tag{8}$$

Since $\|T(0)x - T(0)y\| = \|x - y\|$, we have $\tau > 0$. We put

$$M = \min\{\|T(t)x - T(t)y\| : 0 \le t \le \tau\}$$

and

$$\sigma = \max\{t \in [0, \tau] : M = \|T(t)x - T(t)y\|\}.$$

By (8), we have $\sigma < \tau$. If $M = 0$, then we have $T(\sigma)x = T(\sigma)y$ and hence

$$T(\tau)x = T(\tau - \sigma) \circ T(\sigma)x = T(\tau - \sigma) \circ T(\sigma)y = T(\tau)y,$$

which contradicts (8). Therefore $M > 0$. So we can choose δ satisfying

- $0 < \delta < \tau - \sigma$
- $\beta \|T(\sigma)x - T(\delta) \circ T(\sigma)x\| \le \|T(\sigma)x - T(\sigma)y\|$.

It follows from (ii), $\delta + \sigma < \tau$ and the definition of σ that

$$\begin{aligned} M &< \|T(\delta + \sigma)x - T(\delta + \sigma)y\| \\ &= \|T(\delta) \circ T(\sigma)x - T(\delta) \circ T(\sigma)y\| \\ &\le \|T(\sigma)x - T(\sigma)y\| \\ &= M. \end{aligned}$$

This is a contradiction. Therefore (i) holds. □

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math., **3** (1922), 133–181.

[2] E. H. Connell, Properties of fixed point spaces, Proc. Amer. Math. Soc., **10** (1959), 974–979. MR0110093

[3] Y. Enjouji, M. Nakanishi and T. Suzuki, A generalization of Kannan's fixed point theorem, submitted.

[4] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly, **83** (1976), 261–263. MR0400196

[5] M. Kikkawa and T. Suzuki, Three fixed point theorems for generalized contractions with constants in complete metric spaces, Nonlinear Anal., **69** (2008), 2942–2949.
[6] ———, Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl., **2008** (2008), Article ID 649749, 1–8. MR2395313
[7] S. B. Nadler, Jr., Multi-valued contraction mappings, Pacific J. Math., **30** (1969), 475–488. MR0254828
[8] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., **136** (2008), 1861–1869. MR2373618
[9] ———, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl., **340** (2008), 1088–1095. MR2390912
[10] ———, A new type of fixed point theorem in metric spaces, submitted.
[11] T. Suzuki and M. Kikkawa, Some remarks on a recent generalization of the Banach contraction principle, in Proceedings of the Eighth International Conference on Fixed Point Theory and its Applications (S. Dhompongsa, K. Goebel, W. A. Kirk, S. Plubtieng, B. Sims, and S. Suantai Eds.), pp. 151–161, Yokohama Publishers (2008).

(M. Kikkawa)
Department of Mathematics
Faculty of Science
Saitama University
Sakura, Saitama 338-8570, *Japan*
E-mail address: mi-sa-ko-kikkawa@jupiter.sannet.ne.jp

(T. Suzuki)
Department of Basic Sciences
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550, *Japan*
E-mail address: suzuki-t@mns.kyutech.ac.jp