Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 56, 2009, pp. 19–28

SOME NOTES ON τ-DISTANCE VERSIONS OF EKELAND'S VARIATIONAL PRINCIPLE

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Abstract

We give some notes on τ-distance versions of Ekeland's variational principle.

1. Introduction

Ekeland [4, 5] proved the following very useful existence theorem, which is called Ekeland's variational principle.

Theorem 1.1 (Ekeland [4, 5]). *Let (X, d) be a complete metric space. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Then for $u \in X$ and $\lambda > 0$, there exists $v \in X$ satisfying the following:*
(i) $f(v) \le f(u) - \lambda\, d(u, v)$.
(ii) $f(w) > f(v) - \lambda\, d(v, w)$ *for all* $w \in X \setminus \{v\}$.

Theorem 1.1 has many applications. Theorem 1.1 is equivalent to Caristi's fixed point theorem [2, 3], which is a generalization of the Banach contraction principle [1]. Also, Sullivan [11] proved that Theorem 1.1 characterizes the metric completeness of X. See also [9, 31].

In [13], Suzuki introduced the concept of τ-distances in order to generalize the results of Tataru [30], Zhong [32] and others.

Definition 1.2 ([13]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:
(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$.
(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.
(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_n p(w, x_n)$ for all $w \in X$.

Date: Received November 28, 2008.
2000 *Mathematics Subject Classification.* 54H25.
Key words and phrases. Ekeland's variational principle, τ-distance.
The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.
(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and properties are stated in [7, 8, 10, 12–29]. The following is a τ-distance version of Theorem 1.1.

THEOREM 1.3 ([13]). *Let (X, d) be a complete metric space with a τ-distance p. Let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Then the following* (A) *and* (B) *hold:*
(A) *For each $u \in X$, there exists $v \in X$ such that $f(v) \leq f(u)$ and $f(w) > f(v) - p(v, w)$ for all $w \in X \backslash \{v\}$.*
(B) *For each $\lambda > 0$ and $u \in X$ with $p(u, u) = 0$, there exists $v \in X$ such that $f(v) \leq f(u) - \lambda p(u, v)$ and $f(w) > f(v) - \lambda p(v, w)$ for all $w \in X \backslash \{v\}$.*

Motivated by the strong Ekeland's variational principle proved by Georgiev [6], Suzuki proved the following theorem.

THEOREM 1.4 ([20]). *Let (X, d) be a compact metric space with a τ-distance p. Assume that p is lower semicontinuous in its second variable. Let f be a function from X into $(-\infty, +\infty]$ which is proper and lower semicontinuous. Let $u \in X$ with $p(u, u) = 0$. Then for $\lambda > 0$, there exists $v \in X$ satisfying the following:*
(i) $f(v) \leq f(u) - \lambda p(u, v)$.
(ii) *$f(w) > f(v) - \lambda p(v, w)$ for all $w \in X \backslash \{v\}$.*
(iii) *If a sequence $\{x_n\}$ in X satisfies $\lim_n (f(x_n) + \lambda p(v, x_n)) = f(v)$, then $\{x_n\}$ is p-Cauchy, $\lim_n x_n = v$ and $p(v, v) = \lim_n p(v, x_n) = 0$.*

In this paper, we give some comments on Theorems 1.3 and 1.4.

2. Preliminaries

In this section, we give some preliminaries.

DEFINITION 2.1 ([13]). Let (X, d) be a metric space with a τ-distance p. Then a sequence $\{x_n\}$ in X is called *p-Cauchy* if there exist a function η from $X \times [0, \infty)$ into $[0, \infty)$ satisfying (τ2)–(τ5) and a sequence $\{z_n\}$ in X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$.

LEMMA 2.2 ([13]). *Let (X, d) be a metric space with a τ-distance p. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence in the usual sense.*

LEMMA 2.3 ([13]). *Let (X, d) be a metric space with a τ-distance p. If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is p-Cauchy.*

LEMMA 2.4 ([13]). *Let (X, d) be a metric space with a τ-distance p. If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is p-Cauchy. Moreover if* *a sequence* $\{y_n\}$ *in* X *satisfies* $\lim_n p(x_n, y_n) = 0$, *then* $\{y_n\}$ *is also p-Cauchy and* $\lim_n d(x_n, y_n) = 0$.

PROPOSITION 2.5 ([13, 15]). *Let p be a* τ*-distance on a metric space* (X, d). *Let c be a positive real number and let* α *be a function from* X *into* $[0, \infty)$. *Then two functions* q_1 *and* q_2 *from* $X \times X$ *into* $[0, \infty)$ *defined by*

(i) $q_1(x, y) = cp(x, y)$;

(ii) $q_2(x, y) = \max\{\alpha(x), p(x, y)\}$

are τ*-distances on* X.

3. A proof

In [13], using some fixed point theorem, we proved Theorem 1.3. In this section, we give a direct proof of Theorem 1.3. Concretely speaking, we give a direct proof of the following theorem, which Theorem 1.3 is an immediate consequence of.

THEOREM 3.1 ([13]). *Let* (X, d) *be a complete metric space with a* τ*-distance p, and let f be a function from* X *into* $(-\infty, +\infty]$ *which is proper lower semicontinuous and bounded from below. Define*

$$Mx = \{y \in X : f(y) + p(x, y) \le f(x)\}$$

for $x \in X$. *Then the following hold:*

(i) $y \in Mx$ *and* $z \in My$ *imply* $z \in Mx$.

(ii) *For every* $u \in X$ *with* $Mu \neq \emptyset$, *there exists* $v \in Mu$ *satisfying* $Mv \subset \{v\}$.

(iii) *There exists* $z \in X$ *such that* $Mz \subset \{z\}$.

(iv) $Mz = \{z\}$ *implies* $f(z) < \infty$ *and* $p(z, z) = 0$.

PROOF. (i) follows from

$$f(z) + p(x, z) \le f(z) + p(x, y) + p(y, z) \le f(y) + p(x, y) \le f(x).$$

(iii) follows from (ii). We can easily prove (iv). So we shall prove (ii). Fix $u \in X$ with $Mu \neq \emptyset$. Arguing by contradiction, we assume

- $Mx \backslash \{x\} \neq \emptyset$ for every $x \in Mu$.

We shall define inductively a sequence $\{u_n\}$ in X. Put $u_1 = u$. Suppose that $u_n \in X$ is known for some $n \in \mathbf{N}$. Since $Mu_n \backslash \{u_n\} \neq \emptyset$, we can choose $u_{n+1} \in Mu_n$ such that

$$u_{n+1} \neq u_n \qquad \text{and} \qquad f(u_{n+1}) \le \inf\{f(x) : x \in Mu_n\} + 1/n < \infty.$$

We have defined $\{u_n\}$. We note

$$f(u_{n+1}) \le f(u_n) \qquad \text{and} \qquad Mu_{n+1} \subset Mu_n$$

because $u_{n+1} \in Mu_n$. Since $\{f(u_n)\}$ is a nonincreasing sequence in $(-\infty, +\infty]$ and f is bounded from below, $\{f(u_n)\}$ converges to some real number. We have

$$\begin{aligned}
\lim_{n\to\infty} \sup_{m>n} p(u_n, u_m) &\le \lim_{n\to\infty} \sup_{m>n} \sum_{j=n}^{m-1} p(u_j, u_{j+1}) \\
&\le \lim_{n\to\infty} \sup_{m>n} \sum_{j=n}^{m-1} (f(u_j) - f(u_{j+1})) \\
&= \lim_{n\to\infty} \sup_{m>n} (f(u_n) - f(u_m)) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} (f(u_n) - f(u_m)) \\
&= 0.
\end{aligned}$$

By Lemma 2.4, $\{u_n\}$ is a p-Cauchy sequence. By Lemma 2.2, $\{u_n\}$ is a Cauchy sequence in the usual sense. Since X is complete, $\{u_n\}$ converges to some point $v \in X$. Using ($\tau 3$) and (i), we have

$$\begin{aligned}
f(v) + p(u_n, v) &\le \liminf_{m\to\infty} f(u_m) + \liminf_{m\to\infty} p(u_n, u_m) \\
&\le \liminf_{m\to\infty} (f(u_m) + p(u_n, u_m)) \\
&\le f(u_n)
\end{aligned}$$

for all $n \in \mathbf{N}$ and hence $v \in \bigcap_n Mu_n \subset Mu$. We choose $w \in Mv \backslash \{v\}$. Then $w \in \bigcap_n Mu_n$ by (i). We have

$$\begin{aligned}
\lim_{n\to\infty} p(u_n, w) &= \lim_{n\to\infty} p(u_{n+1}, w) \\
&\le \lim_{n\to\infty} (f(u_{n+1}) - f(w)) \\
&\le \lim_{n\to\infty} (f(u_{n+1}) - \inf\{f(x) : x \in Mu_n\}) \\
&\le \lim_{n\to\infty} 1/n \\
&= 0.
\end{aligned}$$

By Lemma 2.4, we have $\lim_n d(u_n, w) = 0$, which implies $v = w$. This is a contradiction. We have shown (ii). □

4. Strong Ekeland's theorem

In this section, we consider Theorem 1.4. We first show that the assumption "p is lower semicontinuous in its second variable" is needed in Theorem 1.4.

Proposition 4.1. *Let* (X, d) *be a metric space and let* c *be a positive real number. Let* p *be a function from* $X \times X$ *into* $[0, \infty)$. *Assume*

- $p(x, y) \le 2c$ *for all* $(x, y) \in X \times X$;
- $p(x, y) < c$ *implies* $x = y$.

Then p *is a* τ*-distance on* X.

Proof. Arguing by contradiction, we assume there exist $x, y, z \in X$ such that

$$p(x, z) > p(x, y) + p(y, z).$$

Then since $p(x, z) \le 2c$, either $p(x, y) < c$ or $p(y, z) < c$ holds. From the assumption, either $x = y$ or $y = z$ holds. In both cases,

$$p(x, z) \le p(x, y) + p(y, z)$$

holds. This is a contradiction. We have shown (τ1). Define a function η from $X \times [0, \infty)$ into $[0, \infty)$ by

$$\eta(x, t) = t \tag{1}$$

for $x \in X$ and $t \in [0, \infty)$. Then it is obvious that η satisfies (τ2). Assume that $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$. Since

$$\lim_{n\to\infty} \sup_{m \ge n} p(z_n, x_m) = 0,$$

there exists $\nu \in \mathbf{N}$ such that

$$\sup_{m \ge \nu} p(z_\nu, x_m) < c.$$

This implies $x_m = z_\nu$ for every $m \ge \nu$. Hence $x = z_\nu$. Therefore

$$p(w, x) = \liminf_{n\to\infty} p(w, x_n) \qquad \text{for all } w \in X.$$

This implies (τ3). By the definition of η, (τ4) clearly holds. We assume

$$\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0.$$

Then since $\lim_n p(z_n, x_n) = 0$ and $\lim_n p(z_n, y_n) = 0$, there exists $\nu \in \mathbf{N}$ such that

$$p(z_n, x_n) < c \qquad \text{and} \qquad p(z_n, y_n) < c$$

for all $n \ge \nu$. Thus, we have $z_n = x_n$ and $z_n = y_n$ for all $n \in \mathbf{N}$ with $n \ge \nu$. Therefore $\lim_n d(x_n, y_n) = 0$ holds. This implies (τ5). Therefore p is a τ-distance. □

Example 4.2. Define a compact metric space (X, d) by

$$X = [0, 1] \cup \{2\} \qquad \text{and} \qquad d(x, y) = |x - y|.$$

Define a τ-distance p on X by

$$p(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \in [0,1] \text{ and } y \neq x \\ 2 - y & \text{if } x = 2 \text{ and } y \in [0,1) \\ 2 & \text{if } x = 2 \text{ and } y = 1. \end{cases}$$

Define a continuous function f from X into $[0,\infty)$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 2 & \text{if } x = 2. \end{cases}$$

Put $u := 2 \in X$ and $\lambda := 1 > 0$. Then there does not exist $v \in X$ satisfying (i)–(iii) of Theorem 1.4.

PROOF. Using Proposition 4.1 with $c = 1$, p is a τ-distance on X. $p(u,u) = 0$ obviously holds. For $x \in [0,1]$, since $p(u,x) > 1$, we have

$$f(x) = 1 > 2 - p(u,x) = f(u) - \lambda p(u,x).$$

Thus $x \in [0,1]$ does not satisfy (i). Define a sequence $\{x_n\}$ in X by $x_n = 1 - 1/n$. Then we have

$$\lim_{n\to\infty}(f(x_n) + \lambda p(2,x_n)) = \lim_{n\to\infty}(2 + 1/n) = 2 = f(2)$$

and $\lim_n x_n = 1 \neq 2$. Thus 2 does not satisfy (iii). □

We can remove the lower semicontinuity of p when we assume another assumption.

THEOREM 4.3. *Let (X,d) be a compact metric space with a τ-distance p. Assume the following:*

(A) *For every sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is p-Cauchy.*

Let f be a function from X into $(-\infty,+\infty]$ which is proper and lower semicontinuous. Let $u \in X$ with $p(u,u) = 0$. Then for $\lambda > 0$, there exists $v \in X$ satisfying (i)–(iii) *of Theorem* 1.4.

PROOF. We first note that f is bounded from below because X is compact and f is lower semicontinuous. From Theorem 1.3 (B), there exists $v \in X$ satisfying (i) and (ii) of Theorem 1.4. We note $f(v) < \infty$ follows from (ii). We shall show such v satisfies (iii). Let $\{x_n\}$ be a sequence in X with

$$\lim_{n\to\infty}(f(x_n) + \lambda p(v,x_n)) = f(v).$$

Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$. From (A), there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ is p-Cauchy. We put $y_j = x_{n_{k_j}}$. By Lemma 2.2, $\{y_j\}$ is

Cauchy in the usual sense. Since X is compact, $\{y_j\}$ converges to some point $y \in X$. We have

$$f(y) + \lambda p(v, y) \leq \liminf_{j\to\infty} f(y_j) + \lambda \liminf_{j\to\infty} p(v, y_j) \tag{2}$$
$$\leq \liminf_{j\to\infty}(f(y_j) + \lambda p(v, y_j))$$
$$= \lim_{j\to\infty}(f(y_j) + \lambda p(v, y_j))$$
$$= f(v).$$

From this and (ii), $v = y$ holds. Since $\{x_{n_k}\}$ is arbitrary, we obtain $\{x_n\}$ converges to v. We also obtain $p(v, v) = 0$ from (2). We have

$$f(v) \leq \liminf_{n\to\infty} f(x_n)$$
$$\leq \limsup_{n\to\infty} f(x_n)$$
$$\leq \limsup_{n\to\infty}(f(x_n) + \lambda p(v, x_n))$$
$$= \lim_{n\to\infty}(f(x_n) + \lambda p(v, x_n))$$
$$= f(v)$$

and hence $\lim_n f(x_n) = f(v)$. Therefore $\lim_n p(v, x_n) = 0$. Thus $\{x_n\}$ is p-Cauchy by Lemma 2.3. □

In the above proof, we only use the completeness of X. However, X becomes compact from (A).

Proposition 4.4. *Let (X, d) be a complete metric space with a τ-distance p. Assume* (A) *of Theorem* 4.3. *Then X is compact.*

Proof. Let $\{x_n\}$ be a sequence in X. Then from the assumption, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is p-Cauchy. By Lemma 2.2, $\{x_{n_k}\}$ is Cauchy in the usual sense. Since X is complete, $\{x_{n_k}\}$ converges to some $y \in X$. Therefore every sequence $\{x_n\}$ in X has a subsequence which converges. Hence X is compact. □

We finally give the following example.

Example 4.5. Define a compact metric space (X, d) by

$$X = [0, 1] \qquad \text{and} \qquad d(x, y) = |x - y|.$$

Define a τ-distance p on X by

$$p(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ |x - y| & \text{if } x \neq 0. \end{cases}$$

Then p satisfies (A) of Theorem 4.3, but $p(0,0) > 0$ holds.

PROOF. By Proposition 2.5, p is a τ-distance. It is obvious that $p(0,0) > 0$. Define a function η from $X \times [0, \infty)$ by (1). Then η satisfies (τ2)–(τ5). In order to show (A) of Theorem 4.3, we let $\{x_n\}$ be an arbitrary sequence in X. Since X is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converging. Define a sequence $\{z_k\}$ in X by

$$z_k = \begin{cases} 1/k & \text{if } x_{n_k} = 0 \\ x_{n_k} & \text{if } x_{n_k} \neq 0. \end{cases}$$

Then we have

$$\begin{aligned} \lim_{k\to\infty} \sup_{j\geq k} \eta(z_k, p(z_k, x_{n_j})) &= \lim_{k\to\infty} \sup_{j\geq k} |z_k - x_{n_j}| \\ &\leq \lim_{k\to\infty} \sup_{j\geq k} (|z_k - x_{n_k}| + |x_{n_k} - x_{n_j}|) \\ &\leq \lim_{k\to\infty} \left(1/k + \sup_{j\geq k} |x_{n_k} - x_{n_j}| \right) \\ &= 0, \end{aligned}$$

thus, $\{x_{n_k}\}$ is p-Cauchy. □

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