

A new characterization of a value of the generalized game

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Abstract

A new characterization of a solution concept for the generalized game, a cooperative defined on the set system is proposed. This solution corresponds to the Shapley value for classical cooperative game. We give a new understandable axiomatization of a value of the generalized game in the sense that is related closely to the one of the original Shapley value.

1 Preliminaries

Let $N = \{1, 2, \dots, n\}$ and v be a set function defined on 2^N . Now we consider N is a set consisting of n players. Then a subset of N is called a coalition, and the set function $v : 2^N \rightarrow \mathbb{R}$ means the profit by any coalition of n players and is called a *cooperative game* or a *game* simply. The solution is the function from the whole set of the game v to n -dimension real which measures each player's contribution or share-out. The Shapley value is the most important concept as a solution of the game, and they are characterized by natural axiomatizations [8]. We have generalized the Shapley value for applying games defined on $\mathfrak{N} \subseteq 2^N$ which satisfies a kind of regularity, not only defined on 2^N [5], [7]. Our generalization make the Shapley value applicable to more general games, for example multi-choice game [4], bi-capacity [3], games on anti-matroid [1]. We have also shown an axiom system which consists of six axioms characterizing this generalized Shapley value. In this paper we shall show another axiom system which is a generalization of axiom system for original Shapley value, in other words, this new axiom system is more natural and understandable than the last one as a solution of the game.

2 Preliminary

Throughout this paper, we consider a finite universal set $N = \{1, 2, \dots, n\}$, $n \geq 1$, and 2^N denotes the power set of N . $S(\subseteq N)$ is called a *coalition* and a real valued function $v : 2^N \rightarrow \mathbb{R}$ with $v(\emptyset) = 0$ is called a *characteristic function*. Moreover we call the pair (N, v) *cooperative game* or simply *game*.

Definition 1 (set system) \mathfrak{N} a subset of 2^N which contains \emptyset and N . Then we call (N, \mathfrak{N}) (or simply \mathfrak{N} if no confusion occurs) a *set system*.

Let $A, B \in \mathfrak{N}$. We say that A is *covered* by B , and write $A \prec B$ or $B \succ A$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ together with $C \in \mathfrak{N}$ imply $C = A$.

Definition 2 (maximal chain) Let \mathfrak{N} be a set system. We call \mathcal{C} a *maximal chain* of \mathfrak{N} if $\mathcal{C} = (C_0, C_1, \dots, C_m)$ satisfies $\emptyset = C_0 \prec C_1 \prec \dots \prec C_m = N, C_i \in \mathfrak{N}, i = 0, \dots, m$.

The length of the maximal chain $\mathcal{C} = (C_0, C_1, \dots, C_m)$ is m . We denote the set of all maximal chains of \mathfrak{N} by $\mathcal{M}(\mathfrak{N})$.

Definition 3 (chain, totally ordered set system) We say that (N, \mathfrak{N}) is a *chain* or *totally ordered set system* if for any $A, B \in \mathfrak{N}$, either $A \subseteq B$ or $A \supseteq B$.

If (N, \mathfrak{N}) is a chain, then it has only one maximal chain which length is n .

Definition 4 (regular set system) We say that (N, \mathfrak{N}) is a *regular set system* if for any $A, B \in \mathfrak{N}$ satisfying $A \succ B$, $|A \setminus B| = 1$ holds.

Definition 5 (generalized cooperative game) Let (N, \mathfrak{N}) be a set system and v a real valued function on \mathfrak{N} . Then we call (N, \mathfrak{N}, v) *generalized cooperative game* or simply *game*.

The classical cooperative game is defined on $(N, 2^N)$, i.e., $(N, 2^N, v)$ is a classical cooperative game.

Definition 6 (Shapley value of game on $(N, 2^N)$ [8]) Let $(N, 2^N, v)$ be a game. The *Shapley value* of $(N, 2^N, v)$, $\Phi_{\text{sh}}(v) = (\phi^1(v_{\text{sh}}), \dots, \phi^n(v_{\text{sh}})) \in [0, 1]^n$ is defined by

$$\phi_{\text{sh}}^i(v) := \sum_{E \subseteq N \setminus \{i\}} \gamma_{|E|}^n (v(E \cup \{i\}) - v(E)), \quad i = 1, \dots, n,$$

where

$$\gamma_k^n := \frac{(n-k-1)! k!}{n!}.$$

Remark that we denote simply $\Phi(v)$ or $\phi^i(v)$ instead of $\Phi(N, \mathfrak{N}, v)$ and $\phi^i(v)$ as far as no confusion occurs.

It is known that the Shapley value is represented with maximal chains as follows.

$$\phi_{\text{sh}}^i(v) = \frac{1}{|\mathcal{M}(2^N)|} \sum_{C \in \mathcal{M}(2^N)} (v(C_*(i) \cup \{i\}) - v(C_*(i))), \quad i = 1, \dots, n,$$

where $C_*(i) := \sup\{C \in \mathcal{C} \mid C \not\ni i\}$.

Faigle and Kern had defined a value for applying the generalized cooperative game using the concept of the maximal chain. We extend their value for applying to more general cases.

Definition 7 (Generalization of Shapley value) Let (N, \mathfrak{N}) be a regular set system and (N, \mathfrak{N}, v) be a game. The solution of (N, \mathfrak{N}, v) , $\Psi(v) = (\psi^1(v), \dots, \psi^n(v)) \in \mathbb{R}^n$ is defined by

$$\psi^i(v) := \frac{1}{|\mathcal{M}(\mathfrak{N})|} \sum_{C \in \mathcal{M}(\mathfrak{N})} (v(C_*(i) \cup \{i\}) - v(C_*(i))), \quad i = 1, \dots, n. \quad (1)$$

We discuss the domains of Ψ . Let $N := \{1, 2, \dots, n\}$ and Σ_n be the set of all regular set systems (N, \mathfrak{N}) . And let (N, \mathfrak{N}, v) be a game. We denote the set of all games defined on (N, \mathfrak{N}) by $\mathcal{G}(N, \mathfrak{N})$. The domain of Φ is $\mathcal{G}_{\mathfrak{N}} := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{N} \in \Sigma_n} \mathcal{G}(N, \mathfrak{N})$, and Ψ is a function defined on $\mathcal{G}_{\mathfrak{N}}$ to \mathbb{R}^n .

3 Characterization of the original Shapley value

We denote $\mathcal{G}_{\mathcal{P}} := \bigcup_{n=1}^{\infty} \mathcal{G}(N, 2^N)$. The original Shapley value is characterized by several reasonable axioms.

Axiom 1 (efficiency). For any $v \in \mathcal{G}_{\mathcal{P}}$,

$$\sum_{i \in N} \phi^i(v) = v(N).$$

Axiom 2 (0-valued for null player). Fix $v \in \mathcal{G}_{\mathcal{P}}$. For any null player $i \in N$, i.e., $v(A \cup \{i\}) = v(A)$ for any $A \in N \setminus \{i\}$,

$$\phi^i(v) = 0.$$

Axiom 3 (symmetry). If $i, j \in N$ are symmetry of $v \in \mathcal{G}_{\mathcal{P}}$, i.e., $v(A \cup \{i\}) = v(A \cup \{j\})$ for any $A \in N \setminus \{i, j\}$,

$$\phi^i(v) = \phi^j(v)$$

holds.

Axiom 4 (linearity). For any $v, w \in \mathcal{G}_{\mathcal{P}}$,

$$\Phi^i(v + w) = \Phi^i(v) + \Phi^i(w)$$

holds.

Theorem 1 Let $(N, 2^N, v)$ be a game. Then there exists the unique function $\Phi : \mathcal{G}_{\mathcal{P}} \rightarrow \mathbb{R}^n$ satisfying Axioms 1, 2, 3 and 4, and it is given by Φ_{sh} .

For yielding to our generalized value, we generalize axioms the above and add one more axiom. We denote the set of all game defined on n -length chain (N, \mathfrak{N}) and define $\mathcal{G}_{\mathcal{C}} := \bigcup_{n=1}^{\infty} \mathcal{G}(N, \mathcal{C})$.

Axiom 1' (efficiency). For any $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$

$$\sum_{i \in N} \phi^i(v) = v(N).$$

Definition 8 (null-player of v on the set system) If $i \in N$ satisfies that $v(A \cup \{i\}) = v(A)$ whenever $A \in \mathfrak{N}, A \cup \{i\} \in \mathfrak{N}$,

$$\phi^i(v) = 0.$$

then i is called *null-player*.

Axiom 2' (0-valued for null-player). Fix $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$. For any null-player $i \in N$

$$\phi^i(v) = 0.$$

Axiom 3' (symmetry). Let σ be a permutation on N . For any $v \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$,

$$\begin{aligned} \Phi^i(N, 2^N, v) &= \phi^{\sigma(i)}(N, 2^N, \sigma \circ v) \\ \Phi^i(N, \mathcal{C}, v) &= \phi^{\sigma(i)}(N, \sigma(\mathcal{C}), \sigma \circ v), \end{aligned}$$

where $\sigma(\mathcal{C}) := \{\sigma(A) \mid A \in \mathcal{C}\}, \sigma \circ v(S) := v(\sigma^{-1}(S)), S \in \sigma(\mathcal{C})$.

Axiom 4' (linearity). For any $v, w \in \mathcal{G}_{\mathcal{P}} \cup \mathcal{G}_{\mathcal{C}}$,

$$\Phi^i(v + w) = \Phi^i(v) + \Phi^i(w).$$

For yielding to our generalized value, we generalize axioms the above and add one more axiom.

Axiom 5 (convexity). Let $(N, \mathfrak{N}), (N, \mathfrak{N}_1), (N, \mathfrak{N}_2), \dots, (N, \mathfrak{N}_m)$ be regular set systems satisfying $\mathcal{M}(\mathfrak{N}) = \mathcal{M}(\mathfrak{N}_1) \cup \dots \cup \mathcal{M}(\mathfrak{N}_m)$ and $\mathcal{M}(\mathfrak{N}_1) \cap \dots \cap \mathcal{M}(\mathfrak{N}_m) = \emptyset$ and v be a game on \mathfrak{N} . Then there exists $\alpha_1, \dots, \alpha_m \in]0, 1[$, with $\sum_{k=1}^m \alpha_k = 1$ such that for every game $v \in \mathcal{G}(N, \mathfrak{N})$,

$$\Phi(v) = \alpha_1 \Phi(v|_{\mathfrak{N}_1}) + \dots + \alpha_m \Phi(v|_{\mathfrak{N}_m}).$$

Theorem 2 Let (N, \mathfrak{N}) be a regular set system and (N, \mathfrak{N}, v) a game. Then there exists the unique function satisfying Axioms 1', 2', 3', 4' and 5, and it is given by Ψ .

Our new axioms 1', 2', 3' and 4' include axioms 1, 2, 3 and 4, respectively, and assertion of the theorem 2 includes the theorem 1, so that our new system of axioms is suitable for a solution of the cooperative game.

We treat cooperative games defined on regular set systems. Most cooperative games which appear in applications can be regarded as games on regular set systems using by a kind of translations (See [5]).

4 Proof of the theorem

Proof of theorem 2 Assume that Φ satisfies axioms 1', 2', 3', 4' and 5.

(i) Case \mathfrak{N} is a chain. Define $v_j \in \mathcal{G}_{\mathcal{C}}$ as

$$v_j(S) := \begin{cases} 1, & |S| \geq j, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

By axioms 1' and 2', for any $\lambda \in \mathbb{R}$, it holds that

$$\Phi^i(\lambda v_j) = \begin{cases} \lambda, & \{i\} = C_j \setminus C_{j-1} \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$\Phi^i(\lambda v_j) = \lambda v_j(\mathcal{C}_*(i) \cup \{i\}) - \lambda v_j(\mathcal{C}_*(i)).$$

$v \in \mathcal{G}_{\mathcal{C}}$ is identified by $v(C_1), \dots, v(C_n)$, so that we can regard $v \in \mathcal{G}_{\mathcal{C}}$ as the element of n -dimension vector space. Since the linear independency of $v_j, j = 1, \dots, n$, for $v \in \mathcal{G}_{\mathcal{C}}$, there exist $\lambda_j, j = 1, \dots, n$ with

$$v = \sum_{j=1}^n \lambda_j v_j$$

such that

$$v(C_j) = \sum_{j=i}^n \lambda_j.$$

By axiom 4', for any $v \in \mathcal{G}_{\mathcal{C}}$, it holds that

$$\begin{aligned} \Phi^i(v) &= \Phi^i\left(\sum_{j=1}^n \lambda_j v_j\right) = \sum_{j=1}^n \Phi^i(\lambda_j v_j) = \sum_{j=1}^n \{\lambda_j v_j(\mathcal{C}_*(i) \cup \{i\}) - \lambda_j v_j(\mathcal{C}_*(i))\} \\ &= \sum_{j=i}^n \{\lambda_j v_j(\mathcal{C}_*(i) \cup \{i\}) - \lambda_j v_j(\mathcal{C}_*(i))\} = v(\mathcal{C}_*(i) \cup \{i\}) - v(\mathcal{C}_*(i)) \end{aligned}$$

and $\Phi = \Psi$.

(ii) Case $\mathfrak{N} = 2^N$. By axiom 5, there exist $\alpha_{\mathcal{C}} \in [0, 1], \mathcal{C} \in \mathcal{M}(2^N)$ with $\sum_{\mathcal{C} \in \mathcal{M}(2^N)} \alpha_{\mathcal{C}} = 1$ such that for any $v \in \mathcal{G}_{\mathcal{P}}$

$$\Phi^i(v) = \sum_{\mathcal{C} \in \mathcal{M}(2^N)} \alpha_{\mathcal{C}} \Phi^i(v|_{\mathcal{C}}) \quad (3)$$

holds.

Define $\mathfrak{N}_{\{i\}} := \{S \in 2^N \mid S \subseteq \{i\} \text{ or } S \supseteq \{i\}\}, i = 1, \dots, n$, then by axiom 5 there exist $\beta_{1,1}, \dots, \beta_{1,n}$ with $\beta_{1,1} + \dots + \beta_{1,n} = 1$ such that for any $v \in \mathcal{G}_{\mathcal{P}}$,

$$\Phi^i(v) = \beta_{1,1} \Phi^i(v|_{\mathfrak{N}_{\{1\}}}) + \dots + \beta_{1,n} \Phi^i(v|_{\mathfrak{N}_{\{n\}}})$$

holds.

Assume that $\beta_{1,1}, \dots, \beta_{1,n}$ are not unique and there exists another representation

$$\Phi^i(v) = \beta'_{1,1} \Phi^i(v|_{\mathfrak{N}_{\{1\}}}) + \dots + \beta'_{1,n} \Phi^i(v|_{\mathfrak{N}_{\{n\}}}).$$

Defining $v_j \in \mathcal{G}_{\mathcal{P}}$ as

$$v_j(S) := \begin{cases} 1, & |S| \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

players except i are null-players of $v_1|_{\mathfrak{N}_{\{i\}}}$, so that for any $j \neq i$, $\Phi^i(v|_{\mathfrak{N}_{\{j\}}}) = 0$ holds and

$$\begin{aligned} \Phi^i(v_1) &= \beta_{1,i} \Phi^1(v_1|_{\mathfrak{N}_{\{i\}}}) \\ \Phi^i(v_1) &= \beta'_{1,i} \Phi^1(v_1|_{\mathfrak{N}_{\{i\}}}). \end{aligned}$$

Hence $\beta_{1,i} = \beta'_{1,i}$, so that $\beta_{1,i}$ is unique. Moreover let σ be a permutation of players i_1 and i_2 , then by axiom 3' for any $i_1, i_2 \in N$,

$$\begin{aligned} \Phi^{i_1}(N, \mathfrak{N}_{\{i_1\}}, v_1|_{\{i_1\}}) &= \Phi^{\sigma(i_1)}(N, \sigma(\mathfrak{N}_{\{i_1\}}), \sigma \circ v_1|_{\{i_1\}}) \\ &= \Phi^{i_2}(N, \mathfrak{N}_{\{i_2\}}, v_1|_{\{i_2\}}) \end{aligned}$$

and since any players i_1 and $i_2 \in N$ are symmetry of v_1 , for any $i = 1, \dots, n$,

$$\Phi^i(v_1) = \beta_{1,i} \Phi^1(v_1|_{\mathfrak{N}_{\{1\}}}) = \Phi^1(v_1),$$

so that we obtain $\beta_{1,1} = \dots = \beta_{1,n} = 1/n$. Next define $\mathfrak{N}_{\{1,i\}} := \{S \in \mathfrak{N}_{\{1\}} \mid S \subseteq \{1, i\} \text{ or } S \supseteq \{1, i\}\}$, $i = 2, \dots, n$, then by axiom 5 there exist $\beta_{2,2}, \dots, \beta_{2,n}$ with $\beta_{2,2} + \dots + \beta_{2,n} = 1$, such that for any $v|_{\mathfrak{N}_{\{1\}}} \in \mathcal{G}(N, \mathfrak{N}_{\{1\}})$, it holds that

$$\Phi^i(v|_{\mathfrak{N}_{\{1\}}}) = \beta_{2,2} \Phi^i(v|_{\mathfrak{N}_{\{1,2\}}}) + \dots + \beta_{2,n} \Phi^i(v|_{\mathfrak{N}_{\{1,n\}}}).$$

Players except i are null-players of $v_2|_{\mathfrak{N}_{\{2,i\}}}$, so that for any $j \neq i$, $\Phi^i(v|_{\mathfrak{N}_{\{1,j\}}}) = 0$ holds and since for $i = 2, \dots, n$, we have

$$\Phi^i(v_2|_{\mathfrak{N}_{\{1\}}}) = \beta_{2,i} \Phi^i(v_2|_{\mathfrak{N}_{\{1,i\}}}),$$

$\beta_{2,i}$ is unique, and by axiom 3' we have $\beta_{2,2} = \dots = \beta_{2,n} = 1/(n-1)$. Similarly, define $\mathfrak{N}_{\{1, \dots, k, i\}} := \{S \in \mathfrak{N}_{\{1, \dots, k\}} \mid S \subseteq \{1, \dots, k, i\} \text{ or } S \supseteq \{1, \dots, k, i\}\}$, then we have generally for $i = k+1, \dots, n$, by axiom 5 there exist $\beta_{k+1, k+1}, \dots, \beta_{k+1, n}$ with $\beta_{k+1, k+1} + \dots + \beta_{k+1, n} = 1$ such that for any $v|_{\mathfrak{N}_{\{1, \dots, k\}}} \in \mathcal{G}(N, \mathfrak{N}_{\{1, \dots, k\}})$,

$$\Phi^i(v|_{\mathfrak{N}_{\{1, \dots, k\}}}) = \beta_{k+1, k+1} \Phi^i(v|_{\mathfrak{N}_{\{1, \dots, k, k+1\}}}) + \dots + \beta_{k+1, n} \Phi^i(v|_{\mathfrak{N}_{\{1, \dots, k, n\}}})$$

holds.

Players except i are null-players of $v_{k+1}|_{\{1, \dots, k, i\}}$, so that for any $j \neq i$, $\Phi^i(v|_{\mathfrak{N}_{\{1, \dots, k, j\}}}) = 0$ holds and since for $i = k+1, \dots, n$, we have

$$\begin{aligned} \Phi^i(v_{k+1}|_{\mathfrak{N}_{\{1, \dots, k\}}}) &= \beta_{k+1, k+1} \Phi^i(v_{k+1}|_{\mathfrak{N}_{\{1, \dots, k, k+1\}}}) + \dots + \beta_{k+1, n} \Phi^i(v_{k+1}|_{\mathfrak{N}_{\{1, \dots, k, n\}}}) \\ &= \beta_{k+1, i} \Phi^i(v_{k+1}|_{\mathfrak{N}_{\{1, \dots, k, i\}}}), \end{aligned}$$

$\beta_{k+1, i}$ is unique, and by axiom 3' we have

$$\beta_{k+1, k+1} = \dots = \beta_{k+1, n} = \frac{1}{n-k}.$$

For other terms, decomposing the set system to chains, we obtain

$$\begin{aligned}
\Phi^i(v) &= \beta_{1,1}\Phi^i(v|\mathfrak{N}_{\{1\}}) + \cdots + \beta_{1,n}\Phi^i(v|\mathfrak{N}_{\{n\}}) \\
&= \beta_{1,1}[\beta_{2,2}\Phi^i(v|\mathfrak{N}_{\{1,2\}}) + \cdots + \beta_{2,n}\Phi^i(v|\mathfrak{N}_{\{1,n\}})] + \beta_{1,2}\Phi^i(v|\mathfrak{N}_{\{2\}}) + \cdots + \beta_{1,n}\Phi^i(v|\mathfrak{N}_{\{n\}}) \\
&\vdots \\
&= \beta_{1,1} \cdots \beta_{n-1,n-1}\Phi^i(v|\mathfrak{N}_{\{1,\dots,n-1\}}) + \beta_{1,1} \cdots \beta_{n-1,n-2}\beta_{n-1,n}\Phi^i(v|\mathfrak{N}_{\{1,\dots,n-2,n\}}) \\
&\quad + \sum_{\mathcal{C} \in \mathcal{M}(2^N) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)} \beta\Phi^i(v|\mathcal{C}) \\
&= \frac{1}{n!}\Phi^i(v|\mathcal{C}_1) + \frac{1}{n!}\Phi^i(v|\mathcal{C}_2) + \sum_{\mathcal{C} \in \mathcal{M}(2^N) \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)} \beta\Phi^i(v|\mathcal{C}),
\end{aligned}$$

where $\mathcal{C}_1 := \{\emptyset, \{1\}, \{1, 2\}, \dots, N\}$, $\mathcal{C}_2 = \{\emptyset, \{1\}, \{1, 2\}, \dots, \{1, \dots, n-2\}, \{1, \dots, n-2, n\}, N\}$.

Therefore for (2) $\alpha_{\mathcal{C}_1} = \alpha_{\mathcal{C}_2} = 1/n!$ are obtained. Similary $\alpha_{\mathcal{C}}, \mathcal{C} \in \mathcal{M}(2^N) \setminus \{\mathcal{C}_1, \mathcal{C}_2\}$ is obtained uniquely as $1/n!$.

Consequently, it holds that

$$\Phi^i(v) = \frac{1}{|\mathcal{M}(2^N)|} \Phi^i(v|\mathcal{C}), \quad (4)$$

and $\Phi = \Psi$.

(iii) Case \mathfrak{N} is a general set ssystem. Let $\mathfrak{N} \subsetneq 2^N$. We regarding

$$\mathcal{M}(2^N) = \mathcal{M}(\mathfrak{N}) \cup \bigcup_{\mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \mathcal{C},$$

by axiom 5 there exist $\alpha, \beta_{\mathcal{C}}, \mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})$ with

$$\alpha + \sum_{\mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \beta_{\mathcal{C}} = 1 \quad (5)$$

such that for any $v \in \mathcal{G}_{\mathcal{P}}$

$$\begin{aligned}
\Phi^i(v) &= \alpha\Phi^i(v|\mathfrak{N}) + \sum_{\substack{\mathcal{C} \in \\ \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})}} \beta_{\mathcal{C}}\Phi^i(v|\mathcal{C}) \\
&= \alpha \left(\sum_{\mathcal{C} \in \mathcal{M}(\mathfrak{N})} \alpha_{\mathcal{C}}\Phi^i(v|\mathcal{C}) \right) + \sum_{\mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \beta_{\mathcal{C}}\Phi^i(v|\mathcal{C}) = \sum_{\mathcal{C} \in \mathcal{M}(2^N)} \beta_{\mathcal{C}}\Phi^i(v|\mathcal{C}),
\end{aligned} \quad (6)$$

where for $\mathcal{C} \in \mathcal{M}(\mathfrak{N})$ $\beta_{\mathcal{C}} := \alpha\alpha_{\mathcal{C}}$, holds. By (4) for $\mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})$ $\beta_{\mathcal{C}} = 1/n!$ and for $\mathcal{C} \in \mathcal{M}(\mathfrak{N})$ $\beta_{\mathcal{C}} = \alpha\alpha_{\mathcal{C}} = 1/n!$, and since (5)

$$\alpha = 1 - \frac{1}{n!} \cdot |\mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})| = 1 - \frac{n! - |\mathcal{M}(\mathfrak{N})|}{n!} = \frac{|\mathcal{M}(\mathfrak{N})|}{n!}, \quad (7)$$

substituting (7) for (6), we obtain

$$\Phi^i(v|\mathfrak{N}) = \frac{1}{\alpha} \left(\Phi^i(v) - \sum_{\mathcal{C} \in \mathcal{M}(2^N) \setminus \mathcal{M}(\mathfrak{N})} \frac{1}{n!}\Phi^i(v|\mathcal{C}) \right) = \frac{1}{|\mathcal{M}(\mathfrak{N})|} \sum_{\mathcal{C} \in \mathcal{M}(\mathfrak{N})} \Phi^i(v|\mathcal{C}).$$

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