Isomorphic factorization, the Kronecker product and the line digraph

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Abstract

In this paper, we investigate isomorphic factorizations of the Kronecker product graphs. Using these relations, it is shown that (1) the Kronecker product of the d-out-regular digraph and the complete symmetric digraph is factorized into the line digraph, (2) the Kronecker product of the Kautz digraph and the de Bruijn digraph is factorized into the Kautz digraph, (3) the Kronecker product of binary generalized de Bruijn digraphs is factorized into the binary generalized de Bruijn digraph.

Key words: interconnection networks, isomorphic factorization, Kronecker product, line digraphs.

1 Introduction

The de Bruijn digraph and the Kautz digraph have been noted as a model for interconnection networks for massively parallel computers because of their good properties such as small diameter, high connectivity and easy routing (see [2]).

Isomorphic factorizations of graphs or digraphs has been extensively studied by Harary et al. [4–6]. In their studies, while directed graphs were pointed out to be objects of research, only very simple digraphs have been treated [5]. In [8,9], it was shown that de Bruijn digraphs and Kautz digraphs were nice objects for isomorphic factorization of directed graphs. In [8], they investigated several relations among the Kronecker product, line digraphs, and isomorphic factorizations, which offer tools for algebraic manipulation of digraphs. In this paper, based on such tools, we show several results on isomorphic factorizations. First, the Kronecker product of the d-out-regular digraph and the complete symmetric digraph is factorized into the line digraph of d-out-regular digraph. Secondly, the Kronecker product of the Kautz digraph and the de Bruijn digraph is factorized into the Kautz digraph. Thirdly, the Kronecker product of binary generalized de Bruijn digraphs is factorized into the binary generalized de Bruijn digraph.

The paper is organized as follows. In Section 2, we give terminology and known results. Section 3 is devoted to some results on isomorphic factorizations of Kronecker product of graphs. Section 3 is constructed by three subsections, Subsection 3.1 shows the result on the line digraph of the d-out-regular digraph. In subsection 3.2 and 3.3, we consider the isomorphic factorization of specific graph class, namely, the Kautz digraph and the binary generalized de Bruijn digraph. Final remarks in Section 4 conclude the paper.

2 Preliminaries

Let G be a digraph. The vertex set of G and the arc set of G are denoted by $V(G)$ and $A(G)$, respectively. For $H \subseteq A(G)$, the *edge induced subdigraph* $\langle H \rangle$ is a subdigraph of G whose vertex set consists of those vertices of G incident with at least one edge of H and whose edge set is H. The *complete digraph* of order n is a digraph such that every vertex is adjacent to all vertices (including itself). We denote this graph by K_n^+. The *complete symmetric digraph* K_n^* of order n is obtained from K_n^+ by removing all selfloops. The set of vertices of G adjacent from v is denoted by $\Gamma^{\mathrm{O}}(v)$, and the arc set $A^{\mathrm{O}}(v)$ is defined by $A^{\mathrm{O}}(v) = \{(v,w) | w \in \Gamma^{\mathrm{O}}(v)\}$. For all vertices v, G is *d-out-regular* if $|\Gamma^{\mathrm{O}}(v)| = d$. A *factor* of G is a spanning subdigraph of G, that is, a subdigraph of G whose vertex set is equal to $V(G)$. If $H \cong H_0 \cong H_1 \cong \cdots \cong H_n$ are pairwise arc-disjoint factors of G such that $A(G) = \bigcup_{i=0}^{n} A(H_i)$, then G has an *isomorphic factorization* into H and H is said to divide G and we denote it by $H \mid G$. Let G_1, G_2 be digraphs. A digraph G_1 is isomorphic to a digraph G_2 if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ to $V(G_2)$ such that ϕ preserves adjacency and nonadjacency; that is, $(u,v) \in A(G_1)$ if and only if $(\phi(u), \phi(v)) \in A(G_2)$. An *automorphism* of a digraph G is a permutation α on $V(G)$ such that (u,v) is an arc of G if and only if $(\alpha(u), \alpha(v))$ is an arc of G. Let G, H be digraphs. The Kronecker product of G and H, denoted by $G \otimes H$, is a digraph with the vertex set $V(G) \times V(H)$ and the arc set $\{((u,v),(w,x)) | (u,w) \in A(G) \,\text{and}\, (v,x) \in A(H)\}$. Let G be a digraph. The *line digraph* $L(G)$ of G is a digraph whose vertex set $V(L(G))$ is the arc set $A(G)$ of G. In $L(G)$, a vertex (u,v) is adjacent to (x,y) if and only if $v = x$. We can say that $L(G)$ is obtained from G by applying the *line digraph operation*

L. Then $L^m(G)$ denotes the digraph obtained from G by applying the line digraph operation m times.

Lemma 1 *If G is a d-out-regular digraph, then $L(G)$ is a d-out-regular digraph.*

PROOF. Let $(u,v) \in A(G)$ and $\Gamma^{O}(v) = \{v_0, v_1, \ldots, v_{d-1}\}$. Then, $(u,v) \in V(L(G))$ is adjacent to the vertices $\{(v,v_0), (v,v_1), \ldots, (v,v_{d-1})\}$. Since $|\Gamma^{O}((u,v))| = d$, $L(G)$ is d-out-regular. $\square$

Shibata et al. [8] showed useful relationships among the Kronecker product, the line digraph operation and the isomorphic factorization.

Proposition 2 *[8] Let G, G', H and H' be digraphs such that $G' \mid G$ and $H' \mid H$. Then,*

$$G' \otimes H' \mid G \otimes H.$$

Proposition 3 *[8] Let G, H be digraphs. Then*

$$L(G \otimes H) \cong L(G) \otimes L(H).$$

Let $\mathbf{Z}_d$ be a set of integer $\{0, 1, \ldots, d-1\}$. Symbols $\ominus_d$ and $\oplus_d$ are used to indicate the modulo d subtraction and addition, respectively.The d-ary n-dimensional de Bruijn digraph $B(d,n)$ is a digraph with the vertex set $\{x_0 x_1 \cdots x_{n-1} | x_i \in \mathbf{Z}_d \,, 0 \le i \le n-1\}$ and the arc set defined as follows: there exists an arc from vertex $v_0 v_1 \cdots v_{n-1}$ to $u_0 u_1 \cdots u_{n-1}$ if and only if $v_{i+1} = u_i$, $0 \le i \le n-2$. The de Bruijn digraph $B(d,n)$ can be defined by using the line digraph operations as follows:

$$B(d,n) = L^{n-1}(K_d^+).$$

The d-ary n-dimensional Kautz digraph $K(d,n)$ is a digraph with the vertex set $\{x_0 x_1 \cdots x_{n-1} | x_i \in \mathbf{Z}_{d+1} \,, x_i \ne x_{i+1} \,, 0 \le i \le n-1\}$ and the arc set defined as follows: there exists an arc from vertex $v_0 v_1 \cdots v_{n-1}$ to $u_0 u_1 \cdots u_{n-1}$ if and only if $v_{i+1} = u_i$, $0 \le i \le n-2$. The Kautz digraph $K(d,n)$ can be defined by using the line digraph operations as follows:

$$K(d,n) = L^{n-1}(K_{d+1}^*).$$

The d-ary n-dimensional generalized de Bruijn digraph $G_B(n,d)$ is a digraph with the vertex set $\mathbf{Z}_n$. The vertex x is adjacent to the vertices $y \equiv dx + a \pmod{n}$ where $0 \le a \le d-1$. The d-ary n-dimensional generalized Kautz digraph $G_I(n,d)$ is a digraph with the vertex set $\mathbf{Z}_n$. The vertex x is adjacent to the vertices $y \equiv -dx - a \pmod{n}$ where $1 \le a \le d$.

On other terminology and notation, we refer to [3].

In [8] and [9], several results on isomorphic factorization of de Bruijn digraphs have been shown.

Lemma 4 *On isomorphic factorizations related to the de Bruijn digraph, following statements hold.*

(1) [9] $B(d, D_1 + D_2) \,|\, B(d, D_1) \otimes B(d, D_2)$.
(2) [8] $B(d_1, D) \otimes B(d_2, D) \cong B(d_1 d_2, D)$.

In section 3, we show several further results on isomorphic factorizations of digraphs.

3 Isomorphic factorizations of Kronecker product of digraphs

3.1 Isomorphic factorization into line digraphs

In this section, we treat results on line digraphs of d-out-regular digraphs.

Theorem 5 *Let d be a positive integer and G a d-out-regular digraph. Then*

$$L(G) \,|\, G \otimes K_d^+.$$

PROOF. Let $A^{\mathrm{O}}(u) = \{u_0, u_1, \ldots, u_{d-1}\}$. For a vertex $u \in V(G)$, if $u_i = (u, v) \in A(G)$, then $(u_i, v_j) \in A(L(G))$ for any $j \in \mathbf{Z}_d$. Let $V(K_d^+)$ be $\mathbf{Z}_d$ and $(u, v) \in A(G)$. Then $((u, x), (v, y)) \in A(G \otimes K_d^+)$ for any $x, y \in \mathbf{Z}_d$. For $s \in \mathbf{Z}_d$, a mapping $f_s : V(L(G)) \mapsto V(G \otimes K_d^+)$ is defined as follows:

$$f_s(u_i) = (u, i \oplus_d s).$$

For any s, f_s is bijective. Moreover if $(u_i, v_j) \in A(L(G))$, then $(f_s(u_i), f_s(v_j)) = ((u, i \oplus_d s), (v, j \oplus_d s)) \in A(G \otimes K_d^+)$. Next, let the subset A_s of $A(G \otimes K_d^+)$ be

$$A_s = \{(f_s(u_i), f_s(v_j)) \,|\, (u_i, v_j) \in A(L(G)), i, j \in \mathbf{Z}_d\}.$$

By the definition, each $\langle A_i \rangle$ is a spanning subdigraph. We show that for any $s \neq t$, $A_s \cap A_t = \emptyset$. Let $u_i = (u, v) \in A(G)$. Suppose to the contrary, we assume that there exists an edge $((u, i), (v, j)) \in A_s \cap A_t$ for some s, t. Then, there also exists an edge $((u, x), (v, y)) \notin A_r$ for any $r \in \mathbf{Z}_d$. It is for this reason that in each subset A_s, there are exactly d edges that can be represented as $((u, *), (v, *))$, while $A(G \otimes K_d^+)$ has d^2 edges. If no subset A_s includes an edge $((u, x), (v, y))$, then there is no pair i, j such that $i + s \equiv x \pmod{d}$ and

$j + s \equiv y \pmod d$. Nevertheless $u_i = (u, v) \in A(G)$ and $(u_i, v_j) \in A(L(G))$ for any $j \in \mathbf{Z}_d$, we have

$$(f_{x\ominus_d i}(u_i), f_{x\ominus_d i}(v_{y\ominus_d x\oplus_d i})) = ((u, x), (v, y)) \in A_{x\ominus_d i}.$$

This is contradictory to the assumption that $((u, x), (v, y))$ does not belong to any subset A_r. Therefore, $A_0 \cup A_1 \cup \cdots \cup A_{d-1} = A(G \otimes K_d^+)$ and for any $i \neq j$, $A_i \cap A_j = \emptyset$. □

We obtain the following corollary by applying Theorem5 repeatedly.

Corollary 6 *Let $n \geq m \geq 0$ and $d \geq 1$ be integers. If a digraph G is d-out-regular, then*

$$L^n(G) \mid L^{n-m}(G) \otimes K_{d^m}^+.$$

PROOF. Since digraph G is d-out-regular, from Lemma 1, $L(G), L^2(G), \ldots, L^{n-1}(G)$ are all d-out-regular digraphs. From Theorem 5, $L^n(G)|L^{n-1}(G) \otimes K_d^+$ and $L^{n-1}(G)|L^{n-2}(G) \otimes K_d^+$. From Proposition 2,

$$\begin{aligned} L^{n-1}(G) \otimes K_d^+ &\mid L^{n-2}(G) \otimes K_d^+ \otimes K_d^+ = L^{n-2}(G) \otimes K_{d^2}^+, \\ L^{n-2}(G) \otimes K_{d^2}^+ &\mid L^{n-3}(G) \otimes K_{d^3}^+, \\ &\vdots \\ L^{n-m+1}(G) \otimes K_{d^{m-1}}^+ &\mid L^{n-m}(G) \otimes K_{d^m}^+. \end{aligned}$$

Therefore, $L^n(G)|L^{n-1}(G) \otimes K_d^+|L^{n-2}(G) \otimes K_{d^2}^+| \cdots |L^{n-m}(G) \otimes K_{d^m}^+$. □

3.2 Isomorphic factorization into the Kautz digraph

In the previous section, we have treated the line digraph of a d-out-regular digraph. The Kautz digraph $K(d, s+1)$ is a line digraph of $K(d, s)$, that is, $K(d, s+1) \cong L(K(d, s))$. Therefore we have $K(d, s+1) \mid K(d, s) \otimes K_d^+$ as a direct consequence from Theorem5. In this section, we show that the Kautz digraph $K(d, D)$ divides the Kronecker product of the Kautz digraph and the de Bruijn digraph.

At first, we give the following description derived from the vertex labeling of the Kautz digraph in [10].

Proposition 7 *[10] The Kautz digraph $K(d, n)$ is isomorphic to the digraph defined as follows: The vertex set is $\mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$. A vertex $(x, u_0u_1 \cdots u_{n-2})$ is* *adjacent to* $(y, v_0v_1\cdots v_{n-2})$ *if and only if* $y = x \oplus_{d+1} (u_0 \oplus_d 1)$ *and* $v_i = u_{i+1}$ *for* $0 \le i \le n-3$.

PROOF. Let ϕ be a function from $\mathbf{Z}_{d+1}^2$ onto $\mathbf{Z}_d$ defined by $\phi(xy) = (y \ominus_{d+1} x) \ominus_d 1$ for some $x, y \in \mathbf{Z}_{d+1}$. A mapping $f : V(K(d,n)) \to \mathbf{Z}_{d+1} \times \mathbf{Z}_d^{n-1}$ is defined as follows:

$$f(x_0x_1\cdots x_{n-1}) = (x_0, \phi(x_0x_1)\phi(x_1x_2)\cdots\phi(x_{n-2}x_{n-1})).$$

It is easy to verify that f is an isomorphism. $\square$

Theorem 8 *Let* $d \ge 2$ *and* s, t *be integers. Then,*

$$K(d, s+t) \,|\, K(d,s) \otimes B(d,t).$$

PROOF. For $i \in \mathbf{Z}_d$, a mapping $f_i : V(K(d,s+t)) \to V(K(d,s) \otimes B(d,t))$ is defined as follows:

$$f_i((w, v_0v_1\cdots v_{s+t-2})) = ((w, v_0v_1\cdots v_{s-2}), (v_{s-1}\ominus_d i)(v_s \ominus_d i)\cdots(v_{s+t-2}\ominus_d i)).$$

For a vertex $x = ((w, v_0v_1\cdots v_{s-2}), v_{s-1}v_s\cdots v_{s+t-2})$ in $V(K(d,s)\otimes B(d,t))$, we consider an image $f_i^{-1}(x)$ of x. Then,

$$f_i^{-1}(x) = (w, v_0v_1\cdots v_{s-2}(v_{s-1}\oplus_d i)(v_s \oplus_d i)\cdots(v_{s+t-2}\oplus_d i)).$$

A vertex x has an image therefore f_i is a surjection. Moreover $|V(K(d,s+t))| = |K(d,s)\otimes B(d,t)|$, hence f_i is a bijection.

Next, we show that f_i preserves adjacency. For a vertex $x = (w, v_0v_1\cdots v_{s+t-2})$ in $K(d,s+t)$, vertices adjacent from x can be represented as $y = (w \oplus_{d+1} (v_0 \oplus_d 1), v_1v_2\cdots v_{s+t-2}\alpha)$, where $\alpha \in \mathbf{Z}_d$ and $f_i(x) = ((w, v_0v_1\cdots v_{s-2}), (v_{s-1}\ominus_d i)(v_s\ominus_d i)\cdots(v_{s+t-2}\ominus_d i))$, $f_i(y) = ((w\oplus_{d+1}(v_0\oplus_d 1), v_1v_2\cdots v_{s-1}), (v_s\ominus_d i)\cdots(v_{s+t-2}\ominus_d i)(\alpha\ominus_d i))$. Thus a mapping f_i preserves adjacency.

A subset A_i of $A(K(d,s)\otimes B(d,t))$ is defined by

$$A_i = \{(f_i(u), f_i(v)) | (u,v) \in A(K(d,s+t))\},$$

where $i \in \mathbf{Z}_d$. Clearly, each $\langle A_i\rangle$ is isomorphic to $K(d,s+t)$, and is a spanning subdigraph of $K(d,s)\otimes B(d,t)$. We show that for any $i \ne j, A_i \cap A_j = \emptyset$. Suppose, to the contrary, there exists an edge (x,y) such that $(x,y) \in A_i \cap A_j$ for some i, j. Then, there exists vertices u, u', v, v' such that $f_i(u) = f_j(u') = x$ and $f_i(v) = f_j(v') = y$. Let $x = ((w, x_0x_1\cdots x_{s-2}), x_{s-1}x_s\cdots x_{s+t-2})$ and $y = ((w\oplus_{d+1}(x_0\oplus_d 1), x_1\cdots x_{s-2}(x_{s-1}\oplus_d \gamma)), x_sx_{s+1}\cdots x_{s+t-2}\delta)$ for some $\gamma, \delta \in \mathbf{Z}_d$. Then,

$$
\begin{aligned}
u &= (w, x_0x_1 \cdots x_{s-2}(x_{s-1} \oplus_d i)(x_s \oplus_d i) \cdots (x_{s+t-2} \oplus_d i)),\\
v &= (w \oplus_{d+1} (x_0 \oplus_d 1), x_1 \cdots x_{s-2}(x_{s-1} \oplus_d \gamma)(x_s \oplus_d i) \cdots (x_{s+t-2} \oplus_d i)(\delta \oplus_d i)),\\
u' &= (w, x_0x_1 \cdots x_{s-2}(x_{s-1} \oplus_d j)(x_s \oplus_d j) \cdots (x_{s+t-2} \oplus_d j)),\\
v' &= (w \oplus_{d+1} (x_0 \oplus_d 1), x_1 \cdots x_{s-2}(x_{s-1} \oplus_d \gamma)(x_s \oplus_d j) \cdots (x_{s+t-2} \oplus_d j)(\delta \oplus_d j)).
\end{aligned}
$$

We assume that $(u, v) \in A(K(d, s+t))$ and $(u', v') \in A(K(d, s+t))$, then we obtain $\gamma = i$ and $\gamma = j$, which produce a contradiction. □

Since $K(d, D) \cong L^{D-1}(K^*_{d+1})$ and $B(d, D) \cong L^{D-1}(K^+_d)$, we obtain the following corollary.

Corollary 9 *On isomorphic factorizations related to the Kautz digraph, following statements hold.*

(1) $K(d, D) \mid K^*_{d+1} \otimes K^+_{d^{D-1}}$.
(2) $K(d, 2s) \mid L^{s-1}(K^*_{d+1} \otimes K^+_d)$.
(3) $K(d, s+t) \otimes B(d', t) \mid K(d, s) \otimes B(dd', t)$.
(4) $K(d, D_1 + D_2 + \cdots + D_k) \mid K(d, D_1) \otimes \left(\bigotimes_{i=2}^{k} B(d, D_i)\right)$.
(5) $K(d, kD) \mid K(d, D) \otimes B(d^{(k-1)}, D)$.

PROOF. (1): From Corollary 6. (2): Since $L^{s-1}(K^*_{d+1}) \cong K(d, s)$ and $L^{s-1}(K^+_d) \cong B(d, s)$, we apply Proposition 3 and Corollary 6. (3): From Theorem 8, Proposition 2, and Lemma 4 (1). (4): From Theorem 8 and Lemma 4 (1) repeatedly. (5): $D_1 = D_2 = \cdots D_k = D$ in Corollary 9 (4). Then apply Lemma 4 (2). □

Note that $K^*_{d+1} \otimes K^+_{d^{D-1}}$ is a complete $d+1$-partite digraph with each partite set having d^{D-1} vertices. In [9], it was shown that the de Bruijn digraph divides the complete digraph, namely, $B(d, D) \mid K^+_{d^D}$. A similar result for the Kautz digraph is obtained by slightly modifying the right-hand side and we have shown that the Kautz digraph $K(d, n)$ divides some kind of complete digraph $K^*_{d+1} \otimes K^+_{d^{D-1}}$.

3.3 Isomorphic factorization into the binary generalized de Bruijn digraph

In this section, we will consider factorizations on generalized de Bruijn digraphs. Since $L(G_B(n, d)) \cong G_B(dn, d)$ has been shown in [7], we have $G_B(dn, d) \mid G_B(n, d) \otimes K^+_d$ as a direct consequence from Theorem 5. Now, we want to generalize this formula by replacing K^+_d to $G_B(m, d)$.

For real x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Lemma 10 *For positive integers m, n and x,*

$$\left\lfloor \frac{x \bmod mn}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor \bmod m.$$

PROOF. We remember that $x \bmod n = x - n\lfloor x/n \rfloor$. Then,

$$\begin{aligned}
\left\lfloor \frac{x \bmod mn}{n} \right\rfloor &= \left\lfloor \frac{x - mn\lfloor x/mn \rfloor}{n} \right\rfloor \\
&= \left\lfloor \frac{x}{n} - m \left\lfloor \frac{x}{mn} \right\rfloor \right\rfloor \\
&= \left\lfloor \frac{x}{n} \right\rfloor - m \left\lfloor \frac{x}{mn} \right\rfloor \\
&= \left\lfloor \frac{x}{n} \right\rfloor \bmod m.
\end{aligned}$$

□

Lemma 11 *Let $m, n \geq d \geq 2$ be integers. Then*

$$G_B(mn, d) \subseteq G_B(m, d) \otimes G_B(n, d).$$

PROOF. We define a bijection σ from $\mathbf{Z}_{mn}$ to $\mathbf{Z}_m \times \mathbf{Z}_n$ so that

$$\sigma : x \mapsto \left(\left\lfloor \frac{x}{n} \right\rfloor, x \bmod n \right).$$

Note that $\sigma^{-1}\big((a, b)\big) = na + b$. Let H be a digraph with the vertex set $V(G_B(m, d) \otimes G_B(n, d))$ and the arc set $\{(\sigma(x), \sigma(y)) | (x, y) \in A(G_B(mn, d))\}$. Then, a vertex (a, b) of H is adjacent to vertices

$$\begin{aligned}
\sigma\big((d(na + b) + r) \bmod mn\big) &= \left(\left\lfloor \frac{(d(na + b) + r) \bmod mn}{n} \right\rfloor, (db + r) \bmod n \right) \\
&= \left(\left(da + \left\lfloor \frac{db + r}{n} \right\rfloor \right) \bmod m, (db + r) \bmod n \right),
\end{aligned}$$

where $0 \leq r \leq d - 1$. Since $db + r \leq d(n - 1) + (d - 1)$, $\lfloor (db + r)/n \rfloor \leq \lfloor (dn - 1)/n \rfloor = d - 1$. That is, H is a factor of $G_B(m, d) \otimes G_B(n, d)$. □

We obtained that $G_B(mn, d)$ is a factor of $G_B(m, d) \otimes G_B(n, d)$ for any $d \geq 2$, but the next question is what is the structure of $G_B(m, d) \otimes G_B(n, d)$ related to $G_B(mn, d)$. We investigate the case for $d = 2$, the binary generalized de Bruijn digraphs.

Lemma 12 *A function $\alpha : x \mapsto n-1-x$ is an automorphism of the digraph $G_B(n,d)$. Moreover, if $d=2$, it swaps the two sets of arcs (x,dx) and $(x,dx+1)$, $x \in \mathbf{Z}_n$.*

PROOF. For $x \in \mathbf{Z}_n$ and $r \in \mathbf{Z}_d$, an arc $(x,(dx+r) \bmod n)$ of the digraph $G_B(n,d)$ is mapped to $(n-1-x,(-dx-r-1) \bmod n)$ by α. The vertex $n-1-x$ of $G_B(n,d)$ is adjacent to vertices $(-dx-d+r') \bmod n$, $r' \in \mathbf{Z}_d$. Since $-d \leq -r-1 \leq -1$ and $-d \leq -d+r' \leq -1$, the function α is an automorphism of $G_B(n,d)$. If $d=2$, the set of arcs $\{(x, 2x \bmod n) \,|\, x \in \mathbf{Z}_n\}$ is mapped to the set of arcs $\{(n-1-x, n-1-(2x \bmod n)) \,|\, x \in \mathbf{Z}_n\}$ which is equal to $\{(x,(2x+1) \bmod n) \,|\, x \in \mathbf{Z}_n\}$. $\square$

Theorem 13 *Let $m \geq 2$ and $n \geq 2$ be integers. Then,*

$$G_B(mn,2) \,|\, G_B(m,2) \otimes G_B(n,2).$$

PROOF. We define τ as a permutation on $\mathbf{Z}_m \times \mathbf{Z}_n$ so that $\tau : (x,y) \mapsto (n-1-x,y)$. It is easy to see that τ is an automorphism of $G_B(m,2) \otimes G_B(n,2)$. From Lemma 11, the digraph $G_B(mn,2)$ can be embedded into the digraph $G_B(m,2) \otimes G_B(n,2)$ by the function σ. Let H_1 be a digraph with the vertex set $V(G_B(m,2)) \otimes V(G_B(n,2))$ and the arc set $\{(\sigma(x),\sigma(y)) \mid (x,y) \in A(G_B(mn,d))\}$. We notice that a vertex (a,b) of H_1 is adjacent to two vertices $(2a+s,2b)$ and $(2a+s,2b+1)$, where s is either zero or one. Let H_2 be a digraph obtained from H_1 by applying the permutation τ. From Lemma 12, a vertex (a,b) of H_2 is adjacent to vertices $(2a+s',2b)$ and $(2a+s',2b+1)$, where $s' = (s+1) \bmod 2$. Thus, H_1 and H_2 do not have a common arc. $\square$

Applying Proposition 2 to Theorem 13, we have the next corollary. This corollary shows a result about factorization into the digraph $G_B(n,d)$ with respect to the prime factorization of the order n, while Shibata et al. [8] investigated the factorization into the de Bruijn digraph $B(d,D)$ with respect to the prime factorization of the degree d.

Corollary 14 *Let N be an integer. Assume that N is factorized into $n_1 n_2 \cdots n_k$ for $n_i \geq 2$ $(1 \leq i \leq k)$. Then*

$$G_B(N,2) \,|\, \bigotimes_{1 \leq i \leq k} G_B(n_i,2).$$

We may have the similar result on the generalized Kautz digraph using the same embedding functions.

Corollary 15 *Let N be an integer. Assume that N is factorized into $n_1 n_2 \cdots n_k$ for $n_i \geq 2$ $(1 \leq i \leq k)$. Then*

$$G_I(N,2) \,|\, \bigotimes_{1 \leq i \leq k} G_I(n_i, 2).$$

4 Concluding remarks

In this paper, we investigated several relations between the Kronecker product and the line digraph. We showed that the Kronecker product of some d-out-regular digraph and complete symmetric digraph K_d^* is factorized into the line digraph of d-out-regular digraph. In addition, we showed that the Kronecker product of the Kautz digraph and the de Bruijn digraph is factorized into the Kautz digraph, the Kronecker product of binary generalized de Bruijn digraphs is factorized into the binary generalized de Bruijn digraph.

Hamilton decomposition is one of the most simple isomorphic factorization. The existence of a Hamilton decomposition allows the message traffic evenly distributed across the network. Hamilton decomposition of graphs has been studied in [1], etc. In [8], isomorphic factorization problem is applied to fault-tolerance of interconnection networks. For reasons mentioned above, it is expected that isomorphic factorization is effective to construct parallel algorithms, especially message passing, on interconnection networks.

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