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FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS VERSUS THAT FOR NONEXPANSIVE SEMIGROUPS

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Abstract. The main result of this paper is that a closed convex subset of a Banach space has the fixed point property for nonexpansive mappings if and only if it has the fixed point property for nonexpansive semigroups.

1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. In 1965, Browder proved that $F(T)$ is nonempty provided E is a Hilbert space and C is bounded [3]. See also [1, 4, 8–12] and others.

A family of mappings $\{S(t) : t \geq 0\}$ is called a *nonexpansive semigroup* on C if the following are satisfied:

(NS1) for each $t \geq 0$, $S(t)$ is a nonexpansive mapping on C;
(NS2) $S(s+t) = S(s) \circ S(t)$ for all $s, t \geq 0$;
(NS3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $[0, \infty)$ into C is strongly continuous.

$\{S(t) : t \geq 0\}$ is called a *nonexpansive semigroup with identity* if (NS1)–(NS3) and the following hold:

(NS0) $S(0) = I$, where I is the identity mapping on C.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\{S(t) : t \geq 0\}$, i.e.,

$$F(\mathcal{S}) = \bigcap \big\{ F\big(S(t)\big) : t \geq 0 \big\}.$$

Fixed point theorems for families of nonexpansive mappings are proved in [2, 4, 5, 7, 14] and others.

C is said to have the *fixed point property for nonexpansive mappings* if every nonexpansive mapping on C has a fixed point. Also, C is said to have the *fixed point property for nonexpansive semigroups (with identity, resp.)* if every nonexpansive semigroup (with identity, resp.) on C has a common fixed point. Very recently, Suzuki [16] proved the following. See also [15].

Theorem 1 ([16]). *Let $\{S(t) : t \geq 0\}$ be a nonexpansive semigroup on a closed convex subset C of a Banach space E. Assume that C has the fixed point property for nonexpansive mappings. Then $\{S(t) : t \geq 0\}$ has a common fixed point.*

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That is, the fixed point property for nonexpansive mappings is stronger than that for nonexpansive semigroups. It is a very natural and significant question of whether or not we can replace 'stronger' with 'strictly stronger'.

In this paper, we give the answer to the problem. Our answer is 'no'. That is, both properties are equivalent.

2. Preliminaries

In this section, we give some preliminaries.

Let A be a set-valued mapping on a Banach space E. We put $D(A) = \{x \in E : Ax \neq \varnothing\}$ and $R(A) = \bigcup\{Ax : x \in E\}$. A is said to be *accretive* if

$$\|x - y\| \leq \|x + \lambda\, u - y - \lambda\, v\|$$

for all $x, y \in E$, $u \in Ax$, $v \in Ay$ and $\lambda > 0$. For $\lambda > 0$, define a set-valued mapping $I + \lambda\, A$ on E by

$$x \mapsto \{x + \lambda\, u : u \in Ax\}.$$

A is said to satisfy the *range condition* if $\mathrm{cl}D(A) \subset \bigcap\{R(I + \lambda\, A) : \lambda > 0\}$, where $\mathrm{cl}D(A)$ is the closure of $D(A)$. We know the following. See [13, 17] and others.

Lemma 1. *Let A be an accretive operator and let $\lambda > 0$. Then we can define a mapping J_λ from $R(I + \lambda\, A)$ onto $D(A)$ satisfying $x \in \{J_\lambda x + \lambda\, u : u \in AJ_\lambda x\}$, that is, the following hold*:

- (i) *For $x \in R(I + \lambda\, A)$, there exists a unique $y \in D(A)$ such that $x \in \{y + \lambda\, v : v \in Ay\}$.*
- (ii) *For $y \in D(A)$, there exists $x \in R(I + \lambda\, A)$ such that $J_\lambda x = y$.*

Lemma 2. *Let C be a closed convex subset of a Banach space E and let T be a nonexpansive mapping on C. Define a set-valued mapping A on E by $A = I - T$, that is,*

$$Ax = \begin{cases} \{x - Tx\} & \text{if } x \in C, \\ \varnothing & \text{if } x \notin C. \end{cases} \tag{1}$$

Then A is accretive and satisfies the range condition.

Theorem 2 (Grandall and Liggett [6])**.** *Let A be an accretive operator satisfying the range condition. Put*

$$S(t)x := \lim_{n\to\infty} {J_{t/n}}^n x \tag{2}$$

for $x \in \mathrm{cl}D(A)$ and $t \in [0, \infty)$. Then the following hold:

- (i) *For each $x \in \mathrm{cl}D(A)$ and $t \in [0, \infty)$, the sequence $\{{J_{t/n}}^n x\}$ in $\mathrm{cl}D(A)$ converges uniformly in every bounded interval of $[0, \infty)$, that is, for $x \in \mathrm{cl}D(A)$ and $\tau > 0$, there exists $\nu \in \mathbb{N}$ such that $\|S(t)x - {J_{t/n}}^n x\| < \varepsilon$ for all $t \in [0, \tau]$ and $n \in \mathbb{N}$ with $n \geq \nu$.*
- (ii) *The family $\{S(t) : t \geq 0\}$ of mappings on $\mathrm{cl}D(A)$ is a nonexpansive semigroup with identity.*

3. Main Result

In this section, we prove our main result. The following theorem plays a very important role in this paper.

Theorem 3. *Let C be a closed convex subset of a Banach space E and let T be a nonexpansive mapping on C. Define A by* (1) *and a nonexpansive semigroup $\{S(t) : t \geq 0\}$ with identity by* (2)*. Then $F(T) = F(\mathcal{S})$ holds.*

Proof. We note that for $x \in C$ and $\lambda > 0$, $J_\lambda x$ is a unique element of C satisfying

$$J_\lambda x = \frac{1}{1+\lambda}\, x + \frac{\lambda}{1+\lambda}\, T J_\lambda x. \tag{3}$$

We first prove that $F(T) \subset F(\mathcal{S})$. Assume that $x \in F(T)$. Then it follow from (3) and

$$x = \frac{1}{1+\lambda}\, x + \frac{\lambda}{1+\lambda}\, Tx$$

that $J_\lambda x = x$ for all $\lambda > 0$. From the definition of $S(t)$, we obtain $S(t)x = x$ for all $t \geq 0$. That is, $x \in F(\mathcal{S})$. We have shown that $F(T) \subset F(\mathcal{S})$. Conversely, we shall prove that $F(T) \supset F(\mathcal{S})$. Assume that $x \in F(\mathcal{S})$ and fix $\varepsilon > 0$. Then by Theorem 2, there exists $\nu \in \mathbb{N}$ such that

$$\|x - {J_{t/n}}^n x\| \leq \|S(t)x - {J_{t/n}}^n x\| < \varepsilon$$

for all $t \in [1, 2]$ and $n \in \mathbb{N}$ with $n \geq \nu$. In particular, the following holds:

$$\|x - {J_{1/n}}^k x\| < \varepsilon \tag{4}$$

for $k, n \in \mathbb{N}$ with $n \geq \nu$ and $n \leq k \leq 2n$ because $1/n = (k/n)/k$ and $k/n \in [1, 2]$. For $n \geq \nu$, we have by (3) and (4)

$$\begin{aligned}
&\|Tx - {J_{2/(2n)}}^{2n} x\| \\
&= \|Tx - {J_{1/n}}^{2n} x\| \\
&= \left\| Tx - \frac{1}{1+1/n}\, {J_{1/n}}^{2n-1} x - \frac{1/n}{1+1/n}\, T {J_{1/n}}^{2n} x \right\| \\
&\leq \frac{1}{1+1/n}\, \|Tx - {J_{1/n}}^{2n-1} x\| + \frac{1/n}{1+1/n}\, \|Tx - T{J_{1/n}}^{2n} x\| \\
&\leq \frac{1}{1+1/n}\, \|Tx - {J_{1/n}}^{2n-1} x\| + \frac{1/n}{1+1/n}\, \|x - {J_{1/n}}^{2n} x\| \\
&\leq \frac{1}{1+1/n}\, \|Tx - {J_{1/n}}^{2n-1} x\| + \left(1 - \frac{1}{1+1/n}\right) \varepsilon \\
&= \frac{1}{1+1/n} \left\| Tx - \frac{1}{1+1/n}\, {J_{1/n}}^{2n-2} x - \frac{1/n}{1+1/n}\, T{J_{1/n}}^{2n-1} x \right\| + \left(1 - \frac{1}{1+1/n}\right) \varepsilon \\
&\leq \frac{1}{(1+1/n)^2}\, \|Tx - {J_{1/n}}^{2n-2} x\| + \frac{1/n}{(1+1/n)^2}\, \|Tx - T{J_{1/n}}^{2n-1} x\| + \left(1 - \frac{1}{1+1/n}\right) \varepsilon \\
&\leq \frac{1}{(1+1/n)^2}\, \|Tx - {J_{1/n}}^{2n-2} x\| + \frac{1/n}{(1+1/n)^2}\, \|x - {J_{1/n}}^{2n-1} x\| + \left(1 - \frac{1}{1+1/n}\right) \varepsilon \\
&\leq \frac{1}{(1+1/n)^2}\, \|Tx - {J_{1/n}}^{2n-2} x\| + \left(1 - \frac{1}{(1+1/n)^2}\right) \varepsilon
\end{aligned}$$

$$\leq \frac{1}{(1+1/n)^3} \|Tx - {J_{1/n}}^{2n-3}x\| + \Bigl(1 - \frac{1}{(1+1/n)^3}\Bigr)\varepsilon$$
$$\leq \frac{1}{(1+1/n)^4} \|Tx - {J_{1/n}}^{2n-4}x\| + \Bigl(1 - \frac{1}{(1+1/n)^4}\Bigr)\varepsilon$$
$$\vdots$$
$$\leq \frac{1}{(1+1/n)^n} \|Tx - {J_{1/n}}^{n}x\| + \Bigl(1 - \frac{1}{(1+1/n)^n}\Bigr)\varepsilon$$

and hence

$$\begin{aligned}\|Tx - x\| &= \|Tx - S(2)x\| \\ &\leq \exp(-1)\,\|Tx - S(1)x\| + \bigl(1 - \exp(-1)\bigr)\,\varepsilon \\ &= \exp(-1)\,\|Tx - x\| + \bigl(1 - \exp(-1)\bigr)\,\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\|Tx - x\| \leq \exp(-1)\,\|Tx - x\|,$$

which implies $Tx = x$. That is, $x \in F(T)$. Therefore $F(T) \supset F(\mathcal{S})$ holds. □

Now we can prove our main result.

Theorem 4. *Let C be a closed convex subset of a Banach space E. Then the following are equivalent*:

(i) *C has the fixed point property for nonexpansive mappings.*
(ii) *C has the fixed point property for nonexpansive semigroups.*
(iii) *C has the fixed point property for nonexpansive semigroups with identity.*

Proof. We have already proved that (i) implies (ii); see Theorem 1 above. It is obvious that (ii) implies (iii). Let us prove that (iii) implies (i). We assume that C has the fixed point property for nonexpansive semigroups with identity. Let T be a nonexpansive mapping on C. Define A by (1) and a nonexpansive semigroup $\{S(t) : t \geq 0\}$ with identity by (2). Then $F(T) = F(\mathcal{S})$ holds by Theorem 3. From the assumption, $F(\mathcal{S})$ is nonempty, which implies $F(T)$ is nonempty. □

Remark. Consequently, we complete the study for the existence of common fixed points of nonexpansive semigroups. That is, we can consider Theorem 1 as the final fixed point theorems for nonexpansive semigroups.

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