More Accurate Breakdown Voltage Estimation for the New Step-up Test Method in the Gumbel Distribution Model

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Abstract

The estimation problems for the conventional step-up method (the observed breakdown voltages are not given at all) and the new step-up method (some of the observed breakdown voltages are given) are analyzed when the underlying probability distribution (of breakdown voltage level) is assumed to be gumbel distributions for minima and maxima. The new step-up test method has advantages compared to the conventional method: (1) the confidence intervals of the estimates become smaller and (2) the estimates can be obtained with higher probability. In some case of real step-up breakdown voltage test, a fit of the gumbel distribution to the data case is found to be superior to that of the normal distribution, which suggests the usefulness of the gumbel distribution for the underlying distribution in the step-up breakdown voltage test.

Key Words : Impulse breakdown voltage, step-up test, up-and-down test, optimal test, electrical insulation, gumbel distribution for minima and maxima, epidemiology.

1. Introduction

To estimate the impulse breakdown voltage (or impulse flashover voltage) for electrical insulation which has a self-restoring property such as air and SF_6 gas, *the up-and-down test method* by Dixon and Mood [3] is often used, where the underlying distribution is assumed to be the normal distribution; this method is adopted in the insulation test standards such as IEC Pub. 60-1 [10], JEC-0202 [11] because the observed breakdown voltage is considered to follow the normal distribution when the insulation has a self-restoring property. In addition, to know the mean value of the breakdown voltage is our concern in designing the power network insulation system. The up-and-down test method is as follows: (1) the initial voltage is set around the mean breakdown voltage level, say v_0, and (2) if the insulation is not broken at stress level v_0, then the stress level is increased by d amount, to a higher level $v_1 = v_0 + d$, otherwise the stress level will be set to a lower level, $v_{-1} = v_0 - d$, and this procedure continues for the prescribed number of times. This technique has applicability to epidemiological data (e.g., in [1]), where at each "level" of exposure to a toxin or a new medicine, some people die (or perhaps get better and leave the study) and this would correspond to the insulation being broken.

When the electrical insulation does not have a self-restoring property such as epoxy resin, the insulation will not be able to be used when it is broken; then, the impulse test by increasing voltage is used. This is *the step-up test method* (see [5] in the normal distribution case). The step-up test method is as follows: (1) the initial voltage is set to a sufficiently low stress level (e.g., v_0) where the insulation would not be broken and (2) the stress level is increased by d amount, to a higher level $v_1 = v_0 + d$. The unit is tested m times at the new stress level. If the insulation is not broken in m impulse tests, the voltage is again increased by d amount and the procedure continues until the insulation is finally broken. This test procedure may be favorable when we want to know the lower extreme value of the breakdown voltage rather than to know the mean value because we observe the much of data below the mean.

We have conventionally used only the two-valued information of breakdown and non-breakdown in the step-up test method. However, because of recent improvements of the high speed voltage measuring instrument, we can now measure the accurate value of the breakdown voltage during the very short term of voltage increasing. If the breakdown voltage measured just at the moment that the insulation is broken, the test method is called *the new step-up method*, while it is not obtained, the test method is called *the* *(conventional) step-up method* (Hirose [6]). This new technique might have applicability to epidemiological data, if the response of death or recovery can be instantaneously observed during toxin or medicine injection increasing with a restriction of an upper limit.

When the impulse breakdown voltage follows a normal distribution, $N(\mu, \sigma)$, with mean, μ, and standard deviation, σ, Hirose [6] first recommends the use of the parameters of the underlying probability distribution rather than the use of the nominal breakdown voltage, and second to use the new step-up method if the observed breakdown voltage itself rather than the two-valued information of breakdown and non-breakdown is available, from a viewpoint of stable and accurate parameter estimation. This paper deals with a similar problem to [6], but the underlying distribution is assumed to be the two-parameter gumbel distribution. As is well known, the insulation which does not have a self-restoring property is observed to have the Weibull type breakdown probability distribution, which leads us to investigate the case that the breakdown voltage follows the Weibull distribution (see [7]). The gumbel distribution is the limiting distribution of the Weibull distribution when the Weibull shape parameter goes to infinity (see [2,4,9], e.g.), it is natural to use the gumbel distribution as the underlying distribution function. It is known that there are two kinds of distributions in the Weibull distributions; one is for minima and the other is maxima. Thus, we consider the two distributions in the gumbel distribution here. This paper describes the comparison between the conventional and new step-up methods in the gumbel distribution model. Discussions for usefulness of the gumbel distribution for the underlying distribution (of breakdown voltage level) are made using a real data case.

2. Step-up Test with Gumbel Distribution Model

Similarly to the normal distribution assumption for the breakdown voltage, we assume that the underlying probability distribution for the breakdown voltage follows the two-parameter gumbel distribution. That is, the random variable V which is the observed breakdown value when the voltage is increasingly applied to the insulation follows the gumbel distribution. The gumbel distribution has two types of the cumulative distribution function; one is for minima and the other is maxima, expressed as,

$$p_{min} = P(V \leq v) = F_{min}(v; \alpha, \beta) = 1 - \exp\Big\{-\exp\Big(\frac{v-\beta}{\alpha}\Big)\Big\}, \tag{1a}$$

$$p_{max} = P(V \leq v) = F_{max}(v; \alpha, \beta) = \exp\Big\{-\exp\Big(-\frac{v-\beta}{\alpha}\Big)\Big\}, \tag{1b}$$

where α and β are scale and location parameters, respectively. The corresponding density functions are,

$$f_{min}(V) = \Big(\frac{1}{\alpha}\Big)\exp\Big(\frac{V-\beta}{\alpha}\Big)\exp\Big\{-\exp\Big(\frac{V-\beta}{\alpha}\Big)\Big\}, \tag{2a}$$

$$f_{max}(V) = \Big(\frac{1}{\alpha}\Big)\exp\Big(-\frac{V-\beta}{\alpha}\Big)\exp\Big\{-\exp\Big(-\frac{V-\beta}{\alpha}\Big)\Big\}. \tag{2b}$$

We abbreviate F_{min} and F_{max} to F for simplicity, hereafter; similarly, f_{min} and f_{max} are abbreviated to f. Then, mean, μ, and standard deviation, σ, are,

$$\mu = \beta \mp \alpha \cdot \gamma, \quad \sigma = \frac{\alpha\pi}{\sqrt{6}}, \tag{3}$$

where, $\gamma = 0.57722...$ is Euler's constant, and compound expressions in sign are written in order for minima and maxima. The impulse breakdown test by the step-up method starts at a very low stress level v_0 and continues until the insulation is broken at some stress level $v_i = v_0 + id$. If each test piece is numbered as $1, 2, \cdots, n$, we obtain n sampled values of $v_i(k)$, $(k = 1, \cdots, n)$. We note here for clarity that F_{min} and F_{max} are not referred to the cumulative probability distribution for minimum and maximum value of the observed breakdown voltages; they are just the two kinds of the distribution to be fitted to the breakdown voltage data when the voltage is increasingly applied to the insulation.

3. Estimation Method

Suppose first that the breakdown voltage test is done by the conventional step-up method. Then, the likelihood function for the test sequence is denoted as

$$L^F = \prod_{k=1}^{n} l_k^F, \tag{4}$$

and

$$l_k^F = F(v_{i(k)})\{1 - F(v_{i(k)})\}^{m(k)-1} \prod_{j=0}^{i(k)-1} \{1 - F(v_j)\}^m, \tag{5}$$

where $m(k)$ denotes the number of strikes until the insulation is broken at the final stage $i(k)$ for sample k. The expression $l_{i(k)}^F$ can be considered as the probability of an extended geometric distribution that the insulation is first broken at stress level $v_{i(k)}$. We can see that we are tracking the probability of each piece surviving the m trials in each of the $i(k)$ steps until it gets to the $m(k)$th one. Here, the notation F is used both to F_{min} and

F_{max}, and we can use one of these two distribution functions for the underlying probability distribution.

Suppose next that the breakdown voltage test is done by the new step-up method. Then, the likelihood function for the test sequence is denoted as

$$L^f = \prod_{k=1}^{n} l_k^f, \tag{6}$$

where

$$l_k^f = f(V_{i(k)})\{1 - F(v_{i(k)})\}^{m(k)-1} \prod_{j=0}^{i(k)-1} \{1 - F(v_j)\}^m, \tag{7}$$

The random variable $V_{i(k)}$ is obtained under the condition that $V_{i(k)} \leq v_{i(k)}$. The difference between (6) and (7) is just to use $F(v_{i(k)})$ or $f(V_{i(k)})$; $v_{i(k)}$ is the preset value and the $V_{i(k)}$ is the random variable less than $v_{i(k)}$.

The estimates, $\hat{\alpha}$ and $\hat{\beta}$ for the conventional step-up method, can be obtained by solving the log-likelihood equations,

$$\partial \log L^F / \partial \alpha = 0, \quad \partial \log L^F / \partial \beta = 0. \tag{8}$$

Some iterative methods, e.g., the Newton method, can be used to obtain the estimates of the parameters. Their confidence intervals are computed using the observed Fisher information matrix. However, (8) may not have solutions in a mathematical sense when *B-level* < 2 (see [5]) because the maximum likelihood method is ill posed (indefinite).

For the new step-up method, the solution can be obtained by solving the log-likelihood equations

$$\partial \log L^f / \partial \alpha = 0, \quad \partial \log L^f / \partial \beta = 0. \tag{9}$$

It should be noted that (9) has the solutions with probability 1, unlike the log-likelihood equations in the conventional step-up method. This beneficial property in the new step-up test procedure is also true, similarly to the case that the underlying distribution is a normal type.

Example 1

Suppose that the breakdown voltages are obtained as shown in Table 1. The starting stress level is 500; the step-up stress is 50, and $m = 1$. For example, test piece 1 is not broken at level 500, then the preset value is raised to 550. The insulation is not broken also at this level. This procedure is continued until level 2650. With preset value, 2650, of the impulse application, test piece 1 is first broken at the voltage of 2184 during voltage increasing. Test piece 1 is broken after $44(=\{(2650-500)/50\}+1)$ impulse strikes.

Then, the maximum likelihood estimates for the conventional step-up method are such that $\hat{\alpha}=287.5$ and $\hat{\beta}=3110$ in the gumbel distribution for minima. The approximate standard errors for $\hat{\alpha}$ and $\hat{\beta}$ using the observed Fisher information matrix are 59.13 and 157.0. In the case of the new step-up method in the gumbel distribution for minima, the maximum likelihood estimates are $\hat{\alpha}=262.6$ and $\hat{\beta}=3045$ and their approximate standard errors are 41.94 and 116.6. The standard errors for the estimates in the new step-up method seem to be smaller than those in the conventional step-up method. This tendency is generally true as will be shown in the next section. This is the second beneficial property in the new step-up method.

(INSERT TABLE 1 ABOUT HERE.)

3. Optimal Test Procedure

Let us define, the asymptotic errors, $s(\alpha)$ and $s(\beta)$, for α and β by the square root of each diagonal element of the inverse matrix of I, where

$$I=-\begin{pmatrix} E(\frac{\partial^2\log L}{\partial\alpha^2}) & E(\frac{\partial^2\log L}{\partial\beta\partial\alpha}) \\ E(\frac{\partial^2\log L}{\partial\alpha\partial\beta}) & E(\frac{\partial^2\log L}{\partial\beta^2}) \end{pmatrix}. \tag{10}$$

If the insulation is not broken at level i, the expectation of surviving at level i for 1 test piece is

$$E(w_i^0)=\prod_{j=0}^{i}\{1-F(v_j)\}^m. \tag{11}$$

If the insulation is broken at level i, the expectation of failure at level i for 1 test piece is

$$E(w_i^1)=F(v_i)\{1-F(v_i)\}^{m(i)-1}\prod_{j=0}^{i-1}\{1-F(v_j)\}^m. \tag{12}$$

Therefore, each element of I for the conventional step-up test is expressed as

$$\begin{aligned} E\left(\frac{\partial^2\log L^F}{\partial\theta_a\partial\theta_b}\right) &= \sum_i E(w_i^0)E\left(\frac{\partial^2\log(1-F(v_i))}{\partial\theta_a\partial\theta_b}\right) \\ &\quad+\sum_i E(w_i^1)E\left(\frac{\partial^2\log F(v_i)}{\partial\theta_a\partial\theta_b}\right), \end{aligned} \tag{13}$$

and that for the new step-up test is expressed as

$$
\begin{aligned}
E\left(\frac{\partial^2 \log L^f}{\partial\theta_a\partial\theta_b}\right) &= \sum_i E(w_i^0)E\left(\frac{\partial^2 \log(1-F(v_i))}{\partial\theta_a\partial\theta_b}\right) \\
&+ \sum_i E(w_i^1)E\left(\frac{\partial^2 \log f(V_i)}{\partial\theta_a\partial\theta_b}\Big| V_i < v_i\right),
\end{aligned} \tag{14}
$$

where θ_a or θ_b denotes α or β for brevity. More specifically,

$$
\begin{aligned}
E\left(\frac{\partial^2 \log F(v_i)}{\partial\alpha^2}\right) &= -\frac{1}{p_i^2}\left(\mp\frac{1}{\alpha}y_i z_i \exp(-z_i)\right)^2 \\
&\pm \frac{1}{p_i}\left[\frac{1}{\alpha^2}y_i z_i\{y_i(1-z_i)+2\}\exp(-z_i)\right], \\
E\left(\frac{\partial^2 \log F(v_i)}{\partial\alpha\partial\beta}\right) &= -\frac{1}{p_i^2}\left(\mp\frac{1}{\alpha}y_i z_i \exp(-z_i)\right)\left(-\frac{1}{\alpha}z_i \exp(-z_i)\right) \\
&+ \frac{1}{p_i}\left[\frac{1}{\alpha^2}z_i\{y_i(1-z_i)+1\}\exp(-z_i)\right], \\
E\left(\frac{\partial^2 \log F(v_i)}{\partial\beta^2}\right) &= -\frac{1}{p_i^2}\left(-1\frac{1}{\alpha}z_i \exp(-z_i)\right)^2 \\
&\pm \frac{1}{p_i}\left[\frac{1}{\alpha^2}z_i(1-z_i)\exp(-z_i)\right],
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 \log(1-F(v_i))}{\partial\alpha^2}\right) &= -\frac{1}{q_i^2}\left(\pm\frac{1}{\alpha}y_i z_i \exp(-z_i)\right)^2 \\
&\mp \frac{1}{q_i}\left[\frac{1}{\alpha^2}y_i z_i\{y_i(1-z_i)+2\}\exp(-z_i)\right], \\
E\left(\frac{\partial^2 \log(1-F(v_i))}{\partial\alpha\partial\beta}\right) &= -\frac{1}{q_i^2}\left(\pm\frac{1}{\alpha}y_i z_i \exp(-z_i)\right)\left(-\frac{1}{\alpha}z_i \exp(-z_i)\right) \\
&- \frac{1}{q_i}\left[\frac{1}{\alpha^2}z_i\{y_i(1-z_i)+1\}\exp(-z_i)\right], \\
E\left(\frac{\partial^2 \log(1-F(v_i))}{\partial\beta^2}\right) &= -\frac{1}{q_i^2}\left(-\frac{1}{\alpha}z_i \exp(-z_i)\right)^2 \\
&\mp \frac{1}{q_i}\left[\frac{1}{\alpha^2}z_i(1-z_i)\exp(-z_i)\right],
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
E\left(\frac{\partial^2 \log f(V_i)}{\partial\alpha^2}\Big| V_i < v_i\right) &= \frac{1}{p_i}\int_{-\infty}^{v_i}\frac{\partial^2 \log f(v)}{\partial\alpha^2}f(v)dv, \\
E\left(\frac{\partial^2 \log f(V_i)}{\partial\alpha\partial\beta}\Big| V_i < v_i\right) &= \frac{1}{p_i}\int_{-\infty}^{v_i}\frac{\partial^2 \log f(v)}{\partial\alpha\partial\beta}f(v)dv, \\
E\left(\frac{\partial^2 \log f(V_i)}{\partial\beta^2}\Big| V_i < v_i\right) &= \frac{1}{p_i}\int_{-\infty}^{v_i}\frac{\partial^2 \log f(v)}{\partial\beta^2}f(v)dv,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\frac{\partial^2 \log f(v)}{\partial \alpha^2} &= \frac{1}{\alpha^2}\left\{1 + 2y_i - y_i z_i (y_i + 2)\right\}, \\
\frac{\partial^2 \log f(v)}{\partial \alpha \partial \beta} &= \pm \frac{1}{\alpha^2}\left\{1 - z_i (y_i + 1)\right\}, \\
\frac{\partial^2 \log f(v)}{\partial \beta^2} &= -\frac{1}{\alpha^2} z_i,
\end{aligned} \tag{18}
$$

where $y_i = (v_i - \beta)/\alpha$ for minima, $y_i = -(v_i - \beta)/\alpha$ for maxima, $z_i = \exp(y_i)$, and $p_i = F(v_i) = 1 - q_i$. Compound expressions are written in order for minima and maxima.

Here, we define the asymptotic unit errors, $e(\alpha)$ and $e(\beta)$ as,

$$
e(\alpha) = s(\alpha)/\alpha, \quad e(\beta) = s(\beta)/\alpha \tag{19}
$$

Figures 1 and 2 show $e(\alpha)$ and $e(\beta)$ against d/σ in the gumbel distribution for minima, and Figures 3 and 4 for maxima, when $m = 1$ for both the conventional and new step-up methods; the thick line expresses the case when some level v_j is equal to μ, and the thin line expresses the case when μ is located in the middle of v_{j-1} and v_j; the dotted line expresses the case of the conventional step-up method; the solid line expresses the case of the new step-up method. Why we use d/σ and not use other parameters as the horizontal axis is that this index is considered to be general to all the (unimodal) probability distributions; it is originally used in Dixon and Mood [3]. Also, we may compare the results here to those in other distribution cases. The following is suggested from Figures 1 and 2.

(1) The asymptotic unit errors in the new step-up method, $e_n(\alpha)$ and $e_n(\beta)$, become smaller than those in the conventional step-up method, $e_c(\alpha)$ and $e_c(\beta)$.

(2) The optimal test for $e_c(\beta)$ can be realized around $0.5 \leq d/\sigma \leq 1.5$, but that for $e_n(\beta)$ can be realized for larger d/σ.

(3) The asymptotic unit error in the new step-up method, $e_n(\alpha)$, becomes considerably smaller than that in the conventional step-up method, $e_c(\alpha)$. For example, the difference between $e_n(\alpha) = 0.6477$ and $e_c(\alpha) = 0.8749$ means that the samples in the conventional step-up method requires samples about twice as large as in the new step-up method to obtain the equivalent confidence interval, when some level v_j is equal to μ and $d/\sigma = 1.0$.

(4) The optimal test for $e_n(\alpha)$ does not depend on d/σ, while the optimal test for $e_c(\alpha)$ can be realized at smaller d/σ.

(5) The asymptotic unit errors, $e_n(\alpha)$, $e_n(\beta)$, $e_c(\alpha)$, and $e_c(\beta)$, do not depend on the starting point x_0, when $d/\sigma < 1.0$.

Also, the following is suggested from Figures 3 and 4.

(6) The asymptotic unit errors in the new step-up method, $e_n(\alpha)$ and $e_n(\beta)$, become smaller than those in the conventional step-up method, $e_c(\alpha)$ and $e_c(\beta)$.

(7) The optimal test for $e_c(\beta)$ can be realized around $0.5 \leq d/\sigma \leq 1.5$, but that for $e_n(\beta)$ can be realized for wider range of d/σ.

(8) The asymptotic unit error in the new step-up method, $e_n(\alpha)$, becomes considerably smaller than that in the conventional step-up method, $e_c(\alpha)$. For example, the difference between $e_n(\alpha) = 0.7263$ and $e_c(\alpha) = 1.157$ means that the samples in the conventional step-up method requires samples about twice as large as in the new step-up method to obtain the equivalent confidence interval, when some level v_j is equal to μ and $d/\sigma = 1.0$.

(9) The optimal test for $e_n(\alpha)$ and $e_n(\beta)$ do not depend on d/σ, while the optimal test for $e_c(\alpha)$ and $e_c(\beta)$ can be realized at smaller d/σ.

(10) The asymptotic unit errors, $e_n(\alpha)$, $e_n(\beta)$, $e_c(\alpha)$, and $e_c(\beta)$, do not depend on the starting point x_0, when $d/\sigma < 1.0$.

In short, the new step-up method markedly improves the reliability of the estimates of α and β compared to the conventional step-up method. The larger the d/σ, the smaller the asymptotic unit errors as long as $d/\sigma \leq 2.0$ in the distribution for minima, and the asymptotic unit errors show almost constant values in the distribution for maxima. For the estimate of α, about a half of the sample size in the new step-up method is sufficient for obtaining the equivalent reliability to the conventional method. This is the result for the case of $m = 1$, but this tendency is also true for m is 2 or 3; such a value is often used in the field.

(INSERT FIGURES 1 TO 4 ABOUT HERE.)

5. Monte Carlo Simulation

A Monte Carlo simulation study is done in order to investigate the asymptotic properties of the estimates for the conventional and new step-up methods. The simulation conditions are as follows:

(1) The very first stress step, x_0, is set to around the point that satisfies $F(x_0) = 10^{-15}$, and some stress level is set just to $\alpha = v_j$ because the errors are not affected by the starting point as long as $d/\sigma < 2$.

(2) The number of samples, n, is 100, 50, 20, 10.

(3) The step-up distance to σ, d/σ, is 0.1, 0.2, 0.5, 1.0.

(4) The parameter values are $\alpha = 1$ and $\beta = 0$.

(5) The number of repetition times of strikes at the same stage is $m = 1$.

(6) The number of trial times is 1000.

Here, we define the biases and standardized unit errors as,

$$\bar{\theta} = \left(\frac{1}{M} \sum_{i=1}^{M} \hat{\theta}_i \right), \quad bias(\hat{\theta}) = \bar{\theta} - \theta, \tag{20}$$

$$S(\hat{\theta}) = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \left(\hat{\theta}_i - \theta \right)^2}, \qquad sue(\hat{\alpha}) = \frac{\sqrt{n} S(\hat{\alpha})}{\alpha}, \quad sue(\hat{\beta}) = \frac{\sqrt{n} S(\hat{\beta})}{\alpha}, \tag{21}$$

where, M denotes the number of successful estimation cases. Table 2 and 3 show the $bias(\hat{\theta})$ and $sue(\hat{\theta})$ for the estimates $\hat{\alpha}$ and $\hat{\beta}$ in the gumbel distribution for minima, and Table 4 and 5 for maxima. Comparing the unit asymptotic errors in Figures 1 to 4 with the simulation results, mean and standardized error in the simulation agree well with the asymptotic values as long as n is larger than 20. It can be seen that the cases in which we cannot obtain the estimates (not numerically, but mathematically) are not rare in the conventional step-up method when n is small such as $n \leq 10$.

(INSERT TABLES 2 TO 5 ABOUT HERE.)

6. Discussion

6.1 Example in real data case

In the introductory section, it is noted that the gumbel distribution model is more realistic in impulse breakdown of electrical insulation which does not have a self-restoring property similarly to the Weibull case. In this section, some real data case is provided and the usefulness of adopting the gumbel distribution is explained.

Example 2

We provide a second example data case when the solid insulation is used for the step-up breakdown voltage test; this case is exactly the same as introduced in Example 2 in Hirose [6]. To make more understandable, we added the breakdown strength information in Table 6 because the thickness of the insulation differs from each other. In the table, the final impulse voltage v_k and the breakdown voltage V_k, and the corresponding strengths for them, u_k and U_k, $(k = 1, \cdots, 15)$, are shown. The very first impulse voltage is $v_{1,0} = 45$ and the step-up voltage is 1.5; the impulses applied three times at each stage ($m = 3$). All the insulation is broken by the first impulse at the final stage.

As explained in Hirose [6], the insulation is broken at wave head or wave tail/peak. We regard the former case as the complete data case using $f(V_{i(k)})$, and the latter as the incomplete data case using $F(v_{i(k)})$. Thus, the likelihood function for this case is used as if likelihood functions (4) and (6) were mixed.

If we assume the gumbel distribution for maxima for the underlying distribution, α and β are estimated to be $\hat{\alpha} = 746.85$ and $\hat{\beta} = 2875.9$ in breakdown strength; the corresponding log-likelihood value is -112.708. If we assume the normal distribution $N(\mu, \sigma^2)$ for the underlying distribution, the maximum log-likelihood function becomes -113.550, which means that the gumbel distribution for maxima is superior to the normal distribution in this real data case. This does not imply that we should use the gumbel distribution (*for maxima*) in all the real data cases; the gumbel distribution (*for minima*) might be better to some other real data cases.

Figure 5 shows the optimally fitted log-likelihood values in the Weibull distribution vs. the shape parameter using the method shown in Hirose [7]. When the location parameter is set to 950, then the log-likelihood is maximized to -112.712 with the shape parameter value 4.437; the lower tail shape of this distribution is very similar to that in the gumbel distribution for maxima, as shown in Figure 6.

(INSERT TABLE 6 ABOUT HERE.)

(INSERT FIGURES 5 AND 6 ABOUT HERE.)

6.2 Comparison to other distributions

In reliability distributions, we often use the normal, Weibull/Fréchet, gumbel, gamma, log-normal, generalized extreme-value distributions, and etc. In this paper, we discussed the case in the gumbel distribution. It would be helpful for the reader to compare the results obtained in these reliability distributions. However, this attempt will provide a bulky report. Thus, we intended to describe the major points in the gumbel distribution here. Some other comparative research will be informative if the comparison is focused on some points. For example, Hirose [8] focused the estimation problem for the lower percentile point; although other cases, e.g., the comparison for mean or median can also be informative, we, however, want to wait for the future work.

7. Concluding Remarks

To estimate the impulse breakdown voltages accurately for non-self-restoring electrical insulation, the new step-up test method is recommended when the underlying probability distribution (of breakdown voltage level) is assumed to be a gumbel distribution models for minima and maxima. This paper first recommends the use of the parameters of the underlying probability distribution, e.g., the scale and location parameters. Second, it is advantageous to use the new step-up method if the observed breakdown voltage itself rather than the two-valued information of breakdown and non-breakdown is available. Using the new step-up method, the number of test specimens can be substantially reduced comparing to that in the conventional step-up method for the estimate of location parameter. The optimal test procedure is obtained with larger d/σ. Comparing the maximized log-likelihood value in the gumbel distribution to that in the normal distribution in some case of the real step-up breakdown voltage test, a fit of the gumbel distribution to the data case is found to be superior to that of the normal distribution, which suggests the usefulness of the gumbel distribution for the underlying distribution in the step-up breakdown voltage test.

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Table 1. Simulated breakdown voltages by the step-up method.

testpiece number	final breakdown stress	final setup stress
1	2184	2650
2	2267	2600
3	2343	2950
4	2363	2600
5	2253	2400
6	2020	2950
7	2371	2450
8	1621	1900
9	2228	2600
10	1428	1450
11	2328	2350
12	2244	2350
13	2614	2700
14	2099	2100
15	2460	2550

The first stress is 500, and the step-up distance is 50.

Table 2. $bias(\hat{\theta})$ and $sue(\hat{\theta})$ of the estimates in the conventional step-up method. in the gumbel distribution for minima

d/σ	n	M	$bias(\hat{\alpha})$	$sue(\hat{\alpha})$	$bias(\hat{\beta})$	$sue(\hat{\beta})$
0.1	∞			0.78108		2.28838
0.1	100	1000	−0.00907	0.76075	−0.03178	2.28912
0.1	50	1000	−0.01881	0.76908	−0.05523	2.24918
0.1	20	1000	−0.03616	0.78273	−0.10755	2.24484
0.1	10	1000	−0.07674	0.82474	−0.24834	2.46076
0.2	∞			0.78515		1.83243
0.2	100	1000	−0.00925	0.78326	−0.02015	1.81696
0.2	50	1000	−0.01004	0.81858	−0.02927	1.86622
0.2	20	1000	−0.03910	0.80954	−0.08591	1.88819
0.2	10	1000	−0.07616	0.83693	−0.18548	1.98077
0.5	∞			0.81080		1.36242
0.5	100	1000	−0.00911	0.84407	−0.01344	1.40660
0.5	50	1000	−0.01022	0.81932	−0.02400	1.34834
0.5	20	1000	−0.04582	0.84200	−0.06837	1.40093
0.5	10	1000	−0.07339	0.83114	−0.11163	1.44712
1.0	∞			0.87491		1.20501
1.0	100	1000	−0.00672	0.91308	0.00234	1.20474
1.0	50	1000	−0.01550	0.86128	−0.02620	1.22938
1.0	20	997	−0.04550	0.94075	−0.05269	1.31424
1.0	10	948	−0.03478	1.13718	−0.04333	1.28479

M: number of estimates successfully computed.

Table 3. $bias(\hat{\theta})$ and $sue(\hat{\theta})$ of the estimates in the new step-up method. in the gumbel distribution for minima

d/σ	n	M	$bias(\hat{\alpha})$	$sue(\hat{\alpha})$	$bias(\hat{\beta})$	$sue(\hat{\beta})$
0.1	∞			0.61857		1.92692
0.1	100	1000	−0.00482	0.60450	−0.02113	1.94439
0.1	50	1000	−0.01109	0.61066	−0.03626	1.90500
0.1	20	1000	−0.02521	0.60552	−0.08206	1.83269
0.1	10	1000	−0.04578	0.65556	−0.17475	2.09476
0.2	∞			0.62217		1.59686
0.2	100	1000	−0.00793	0.61542	−0.01810	1.59925
0.2	50	1000	−0.00493	0.65684	−0.02092	1.63692
0.2	20	1000	−0.02494	0.64257	−0.06361	1.65923
0.2	10	1000	−0.04962	0.64038	−0.14599	1.69866
0.5	∞			0.63247		1.24633
0.5	100	1000	−0.00389	0.64174	−0.00886	1.27366
0.5	50	1000	−0.00531	0.65049	−0.02076	1.24978
0.5	20	1000	−0.02284	0.63684	−0.05088	1.29741
0.5	10	1000	−0.04930	0.63469	−0.09987	1.31836
1.0	∞			0.64769		1.07884
1.0	100	1000	−0.00632	0.68225	−0.00125	1.07434
1.0	50	1000	−0.00913	0.62327	−0.02393	1.10698
1.0	20	1000	−0.03342	0.66388	−0.04975	1.13774
1.0	10	1000	−0.04464	0.65500	−0.06826	1.11844

M: number of estimates successfully computed.

Table 4. $bias(\hat{\theta})$ and $sue(\hat{\theta})$ of the estimates in the conventional step-up method. in the gumbel distribution for maxima

d/σ	n	M	$bias(\hat{\alpha})$	$sue(\hat{\alpha})$	$bias(\hat{\beta})$	$sue(\hat{\beta})$
0.1	∞			0.94277		0.94376
0.1	100	1000	−0.00482	0.96621	−0.00073	1.00044
0.1	50	1000	−0.02445	0.96577	−0.01504	0.93948
0.1	20	1000	−0.04263	0.96512	−0.03470	0.93940
0.1	10	1000	−0.10052	0.94433	−0.07025	0.92815
0.2	∞			0.97543		0.87451
0.2	100	1000	−0.01020	0.98767	−0.00838	0.86429
0.2	50	1000	−0.02645	0.98914	−0.01891	0.87292
0.2	20	1000	−0.04736	0.95961	−0.02570	0.88096
0.2	10	1000	−0.09748	0.98808	−0.03859	0.82818
0.5	∞			0.97543		0.87451
0.5	100	1000	−0.01396	1.03059	−0.00458	0.86587
0.5	50	1000	−0.02779	1.08397	−0.00424	0.84150
0.5	20	1000	−0.05149	1.03684	−0.00150	0.83604
0.5	10	994	−0.10296	1.10395	−0.01017	0.85197
1.0	∞			1.15733		0.98549
1.0	100	1000	−0.01142	1.12439	−0.00036	0.95961
1.0	50	999	−0.02126	1.17396	0.00723	0.98912
1.0	20	985	−0.05156	1.22354	0.00586	1.02935
1.0	10	860	−0.04766	1.58684	−0.00239	0.98867

M: number of estimates successfully computed.

Table 5. $bias(\hat{\theta})$ and $sue(\hat{\theta})$ of the estimates in the new step-up method. in the gumbel distribution for maxima

d/σ	n	M	$bias(\hat{\alpha})$	$sue(\hat{\alpha})$	$bias(\hat{\beta})$	$sue(\hat{\beta})$
0.1	∞			0.66485		0.81898
0.1	100	1000	−0.00073	0.67262	0.00085	0.86412
0.1	50	1000	−0.01481	0.68021	−0.01028	0.80933
0.1	20	1000	−0.02550	0.69375	−0.02661	0.84138
0.1	10	1000	−0.05947	0.68646	−0.05095	0.82778
0.2	∞			0.67901		0.80282
0.2	100	1000	−0.00279	0.67430	−0.00519	0.78821
0.2	50	1000	−0.01643	0.68647	−0.01608	0.79576
0.2	20	1000	−0.03195	0.70047	−0.02449	0.83208
0.2	10	1000	−0.05995	0.69577	−0.03468	0.78625
0.5	∞			0.70134		0.81408
0.5	100	1000	−0.00696	0.68692	−0.00802	0.82349
0.5	50	1000	−0.01620	0.73055	−0.00653	0.78306
0.5	20	1000	−0.02975	0.69985	−0.01208	0.79197
0.5	10	1000	−0.06508	0.73575	−0.02998	0.80803
1.0	∞			0.72634		0.84791
1.0	100	1000	−0.00556	0.70940	−0.00189	0.81756
1.0	50	1000	−0.01045	0.73341	−0.00364	0.86080
1.0	20	1000	−0.03283	0.74440	−0.00924	0.85165
1.0	10	1000	−0.07689	0.75680	−0.02272	0.85401

M: number of estimates successfully computed.

Table 6. Actual step-up test data case in the solid electrical insulation.

testpiece number	final impulse voltage	final impulse strength	final breakdown voltage	final breakdown strength	breakdown point
1	139.5	1647.28	138.8	1639.01	head
2	135.0	1725.98	135.0	1725.98	tail/peak
3	127.5	1596.82	123.8	1550.48	head
4	148.5	2234.24	135.1	2032.63	head
5	151.5	1957.32	142.8	1844.92	head
6	148.5	1878.81	142.6	1804.16	head
7	150.0	1980.18	131.1	1730.68	head
8	121.5	1678.08	121.5	1678.08	tail/peak
9	142.5	1766.56	142.5	1766.56	tail/peak
10	157.5	2174.88	157.5	2174.88	tail/peak
11	148.5	2074.78	148.5	2074.78	tail/peak
12	145.5	1963.90	145.5	1963.90	tail/peak
13	160.5	2383.86	160.5	2383.86	tail/peak
14	123.0	1698.78	119.2	1646.30	head
15	141.0	1841.36	141.0	1841.36	tail/peak

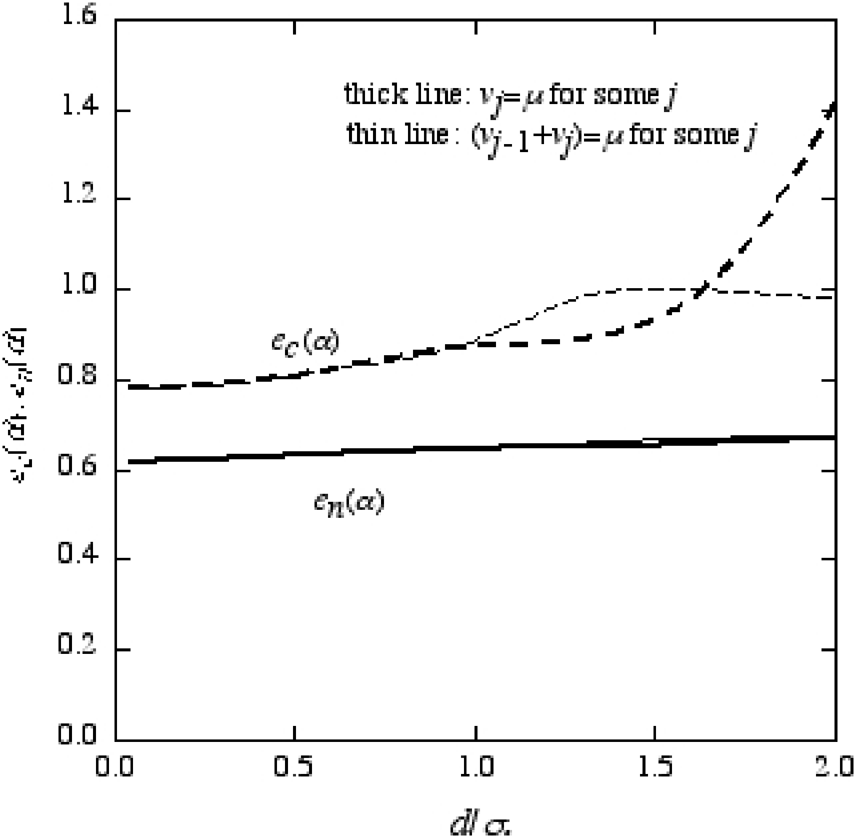

Fig. 1 $e_C(\alpha)$, $e_N(\alpha)$ against d/σ in the gumbel distribution for minima.

$e_C(\alpha)$, $e_N(\alpha)$ denote the standardized error in the conventional and new step-up methods for parameter α.

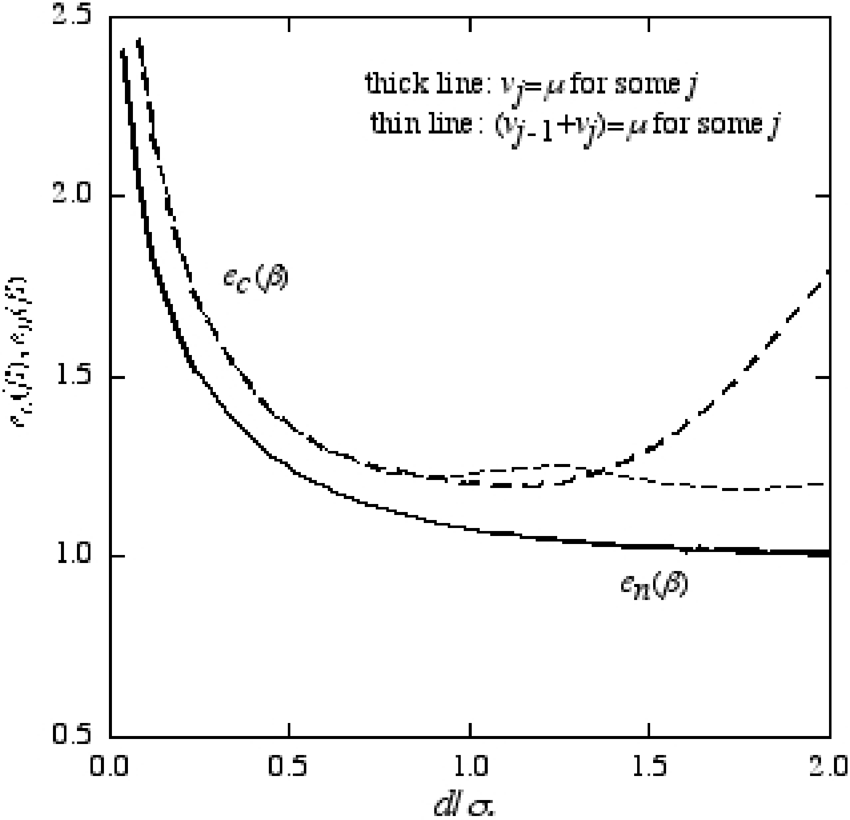

Fig. 2 $e_C(\beta)$, $e_n(\beta)$ against d/σ in the gumbel distribution for minima.

$e_C(\beta)$, $e_n(\beta)$ denote the standardized error in the conventional and new step-up methods for parameter β.

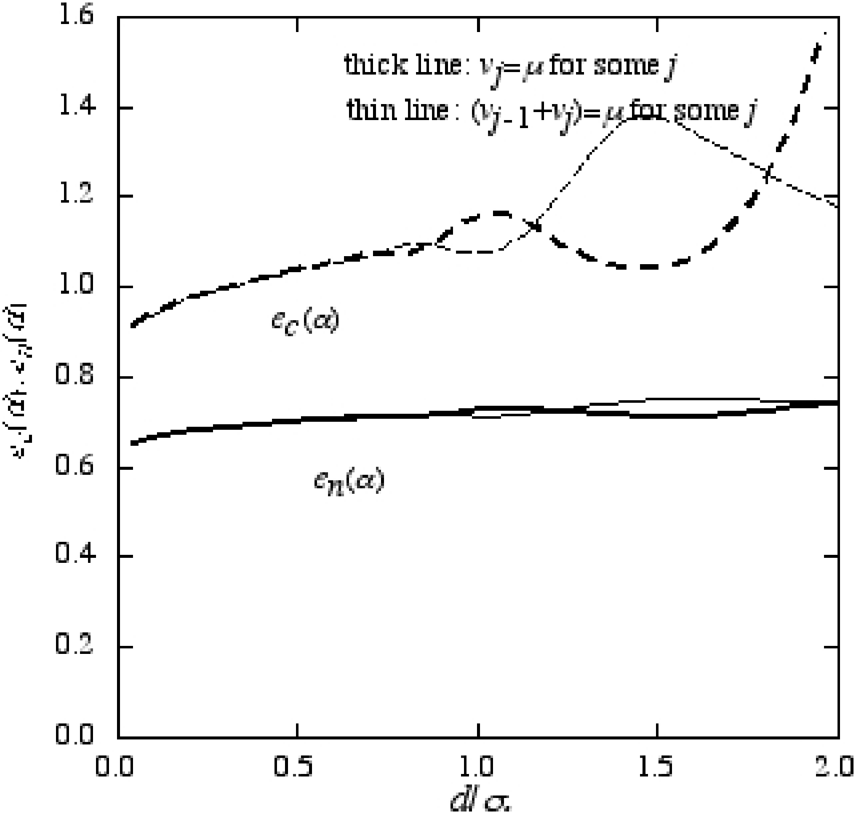

Fig. 3 $e_C(\alpha)$, $e_N(\alpha)$ against d/σ in the gumbel distribution for maxima.

$e_C(\alpha)$, $e_N(\alpha)$ denote the standardized error in the conventional and new step-up methods for parameter α.

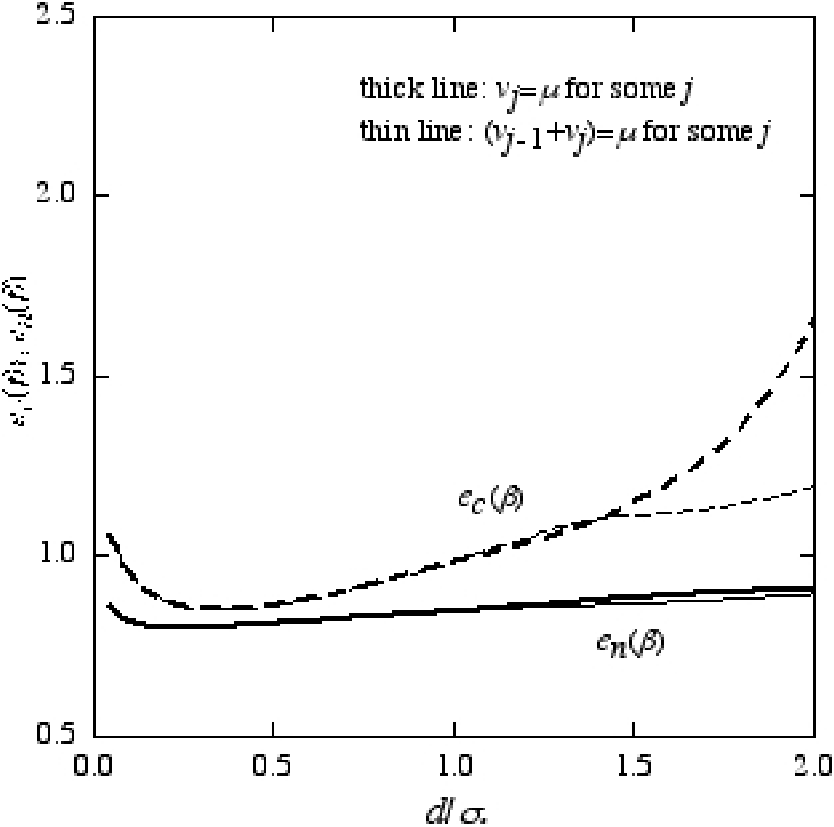

Fig. 4 $e_c(\beta)$, $e_n(\beta)$ against d/σ in the gumbel distribution for maxima.

$e_c(\beta)$, $e_n(\beta)$ denote the standardized error in the conventional and new step-up methods for parameter β.

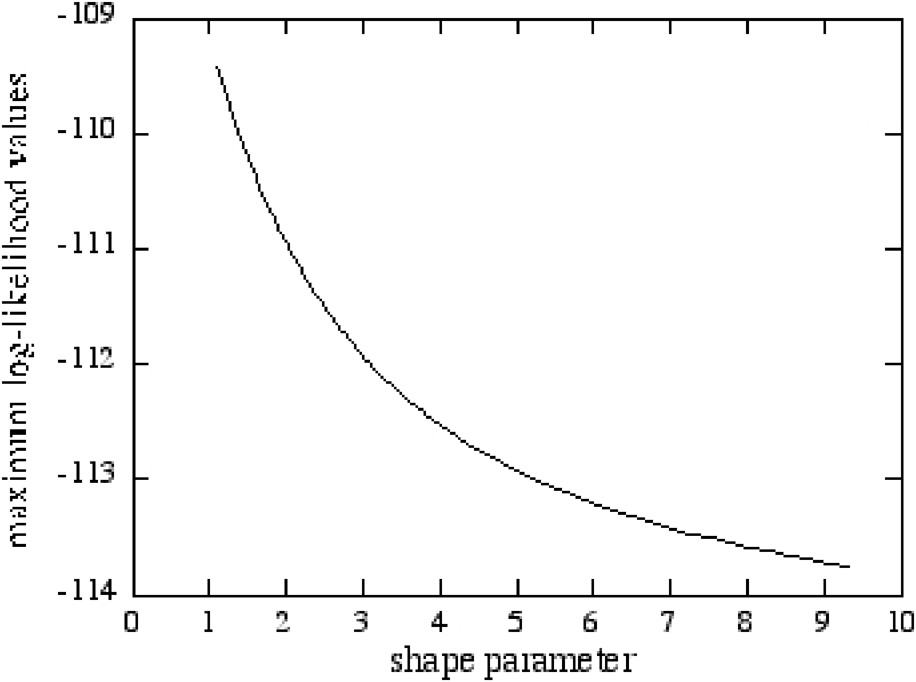

Fig. 5 Weibull shape parameter vs. maximum log-likelihood values.

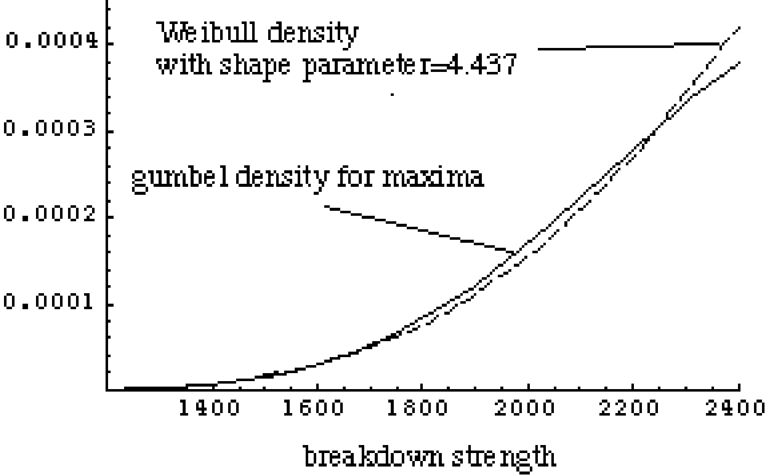

Fig. 6 Densiy functions in the gumbel distribution for maxima and the Weibull distribution with shape parameter=4.437.