ESTMATION OF THE PHASE SPECTRUM AND ITS APPLICATION

by

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SYNOPSIS

A technique is introduced for estimating the phase spectrum of the desired signal from an observation and a few a priori informations and the signal restorations are presented as an application. The discussion about the characteristics of this technique is included by comparing the results of this method with that obtained by the Wiener filtering.

1. INTRODUCTION

For restoring the signal from the blurred observation the optimal linear filter (Wiener) is in general used. This is the linear filter whose mean square estimation error is minimum.

But the phase spectrum of the output of this filter (i.e. restored signal) depends much on that of the observation. In fact under the some condition the phase spectrum of the output is equal to that of the observation.

This would be one of the defects of the optimum linear filter since generally the phase spectrum of the desired signal is not equal to that of the observation when it is degraded by the noise.

In this paper we will consider an approach to estimating the phase spectrum of the desired signal from the blurred observation. As its application the image restoration is also considered. Further we discuss the characteristics of the results by comparing it with that restored by optimum linear filtering.

2. PHASE SPECTRUM ESTIMATION

The observation model is given as

$$y(t_1, t_2)=\int_{\tau_1}\int_{\tau_2} \boldsymbol{h}(\tau_1, \tau_2)\boldsymbol{f}(t_1-\tau_1, t_2-\tau_2)d\tau_1 d\tau_2+\boldsymbol{v}(t_1, t_2) \tag{1}$$

where $y(t_1, t_2)$ is an observed signal and $\boldsymbol{h}(t_1, t_2)$ is a known two dimensional impulse response often called a point-spread-function in the image processing. $f(t_1, t_2)$ is a desired signal and $\boldsymbol{v}(t_1, t_2)$ is an observation noise with zero mean and known covariance. $\boldsymbol{v}(t_1, t_2)$ may or may not correlate with $f(t_1, t_2)$.

With

$$\int_{\tau_1}\int_{\tau_2} \boldsymbol{h}(\tau_1, \tau_2)f(t_1-\tau_1, t_2-\tau_2)d\tau_1 d\tau_2=\boldsymbol{h}(t_1, t_2)\otimes f(t_1, t_2)\triangle f_h(t_1, t_2) \tag{2}$$

Eq. (1) is rewritten as

$$y(t_1, t_2)=f_h(t_1, t_2)+\boldsymbol{v}(t_1, t_2) \tag{1$'$}$$

where $\otimes$ sign is a convolution operator.

Now, assuming that $f_h(t_1, t_2)$ is known, we obtain from Eq. (1)′

$$y(t_1, t_2)-f_h(t_1, t_2)=v(t_1, t_2)\,|\,f_h(t_1, t_2).$$

The autocorrelation of the above equation is[1]

$$E\{E[(y(t_1, t_2)-f_h(t_1, t_2))(y(t_1', t_2')-f_h(t_1', t_2'))\,|\,f_h(t_1, t_2)]\}$$
$$=E\{E[v(t_1, t_2)v(t_1', t_2')\,|\,f_h(t_1, t_2)]\}.$$

Thus we can get

$$E[(y(t_1, t_2)-f_h(t_1, t_2))(y(t_1', t_2')-f_h(t_1', t_2'))]$$
$$=E[v(t_1, t_2)v(t_1', t_2')]. \tag{3}$$

Assuming $y(t_1, t_2), f_h(t_1, t_2)$ and $v(t_1, t_2)$ are stationary processes, Eq. (3) can be rewritten as

$$\phi_{yy}(t_1-t_1', t_2-t_2')+\phi_{f_hf_h}(t_1-t_1', t_2-t_2')-\phi_{f_hy}(t_1-t_1', t_2-t_2')-\phi_{yf_h}(t_1-t_1', t_2-t_2')$$
$$=\phi_{vv}(t_1-t_1', t_2-t_2') \tag{4}$$

where

$$\left.\begin{aligned}
&\phi_{yy}(t_1-t_1', t_2-t_2')=E[y(t_1, t_2)\cdot y(t_1', t_2')]\\
&\phi_{f_hf_h}(t_1-t_1', t_2-t_2')=E[f_h(t_1, t_2)\cdot f_h(t_1', t_2')]\\
&\phi_{f_hy}(t_1-t_1', t_2-t_2')=E[f_h(t_1, t_2)\cdot y(t_1', t_2')]\\
&\phi_{yf_h}(t_1-t_1', t_2-t_2')=E[y(t_1, t_2)\cdot f_h(t_1', t_2')]\\
&\phi_{vv}(t_1-t_1', t_2-t_2')=E[v(t_1, t_2)\cdot v(t_1', t_2')].
\end{aligned}\right\} \tag{5}$$

Taking Fourier transform of both sides of Eq. (4),

$$\varphi_{yy}(\omega_1, \omega_2)+\varphi_{f_hf_h}(\omega_1, \omega_2)-|\varphi_{f_hy}(\omega_1, \omega_2)|\,e^{-j\theta(\omega_1,\omega_2)}-|\varphi_{yf_h}(\omega_1, \omega_2)|e^{j\theta(\omega_1,\omega_2)}$$
$$=\varphi_{vv}(\omega_1, \omega_2) \tag{6}$$

is obtained where $\varphi_{yy}(\omega_1, \omega_2)$, $\varphi_{f_hf_h}(\omega_1, \omega_2)$, $\varphi_{f_hy}(\omega_1, \omega_2)$, $\varphi_{yf_h}(\omega_1, \omega_2)$ and $\varphi_{vv}(\omega_1, \omega_2)$ are the Fourier transforms of $\phi_{yy}(\tau_1, \tau_2)$, $\phi_{f_hf_h}(\tau_1, \tau_2)$, $\phi_{f_hy}(\tau_1, \tau_2)$, $\phi_{yf_h}(\tau_1, \tau_2)$ and $\phi_{vv}(\tau_1, \tau_2)$ respectively (i.e. spectral densities or cross spectra).

$\theta(\omega_1, \omega_2)$ is the difference of the phase of the complex spectrums of $f_h(t_1, t_2)$ and $y(t_1, t_2)$;

$$\theta(\omega_1, \omega_2)=\theta_{f_h}(\omega_1, \omega_2)-\theta_y(\omega_1, \omega_2) \tag{7}$$

where $\theta_{f_h}(\omega_1, \omega_2)$ and $\theta_y(\omega_1, \omega_2)$ are the phase spectrums of $f_h(t_1, t_2)$ and $y(t_1, t_2)$ respectively.

Since it is well known that

$$|\varphi_{f_hy}(\omega_1, \omega_2)|=\sqrt{\varphi_{yy}(\omega_1, \omega_2)\cdot\varphi_{f_hf_h}(\omega_1, \omega_2)}=|\varphi_{yf_h}(\omega_1, \omega_2)| \tag{8}$$

Eq. (6) becomes

$$\varphi_{yy}(\omega_1, \omega_2)+\varphi_{f_hf_h}(\omega_1, \omega_2)-2\sqrt{\varphi_{yy}(\omega_1, \omega_2)\cdot\varphi_{f_hf_h}(\omega_1, \omega_2)}\cdot\left[\frac{e^{j\theta(\omega_1,\omega_2)}+e^{-j\theta(\omega_1,\omega_2)}}{2}\right]=\varphi_{vv}(\omega_1, \omega_2). \tag{9}$$

Since

$$\frac{e^{j\theta(\omega_1,\omega_2)}+e^{-j\theta(\omega_1,\omega_2)}}{2}=\cos[\theta(\omega_1, \omega_2)]$$

we obtain from Eq. (9)

$$\cos\left[\theta(\omega_1, \omega_2)\right] = \frac{\varphi_{yy}(\omega_1, \omega_2)+\varphi_{f_h f_h}(\omega_1, \omega_2)-\varphi_{vv}(\omega_1,\omega_2)}{2\cdot\sqrt{\varphi_{yy}(\omega_1, \omega_2)\cdot\varphi_{f_h f_h}(\omega_1, \omega_2)}} \tag{10}$$

From Eq. (2)

$$\theta_{f_h}(\omega_1, \omega_2)=\theta_h(\omega_1, \omega_2)+\theta_f(\omega_1, \omega_2) \tag{11}$$

where $\theta_h(\omega_1, \omega_2)$ and $\theta_f(\omega_1, \omega_2)$ are the phase spectrums of $\boldsymbol{h}(t_1, t_2)$ and $\boldsymbol{f}(t_1, t_2)$.

Finally from Eqs. (7), (10) and (11) we get

$$\theta_f(\omega_1, \omega_2)=\cos^{-1}\frac{\varphi_{yy}+\varphi_{f_h f_h}-\varphi_{vv}}{2\cdot\sqrt{\varphi_{yy}\cdot\varphi_{f_h f_h}}}+\theta_y-\theta_h. \tag{12}$$

For the simplicity of the notation in Eq. (12) the arguments are omitted.

The function $\cos^{-1}(\cdot)$ is a two-valued function (i.e. plus or minus) for the domain $(-\pi, \pi)$.

Thus we must find which sign is true.

Here we take the following method which may be simple and accurate.

1. With the spectral density and the candidates of the phase spectrum of $\boldsymbol{f}_h(t_1, t_2)$, we define ${}_+\hat{F}_h(\omega_1, \omega_2)$ and ${}_-\hat{F}_h(\omega_1, \omega_2)$ as

$${}_+\hat{F}_h(\omega_1, \omega_2)=\sqrt{\varphi_{f_h f_h}(\omega_1, \omega_2)}\cdot e^{j\hat{\theta}^+_{f_h}(\omega_1,\omega_2)} \tag{13}$$

$${}_-\hat{F}_h(\omega_1, \omega_2)=\sqrt{\varphi_{f_h f_h}(\omega_1, \omega_2)}\cdot e^{j\hat{\theta}^-_{f_h}(\omega_1,\omega_2)} \tag{14}$$

where

$$\left.\begin{aligned}\hat{\theta}^+_{f_h}(\omega_1, \omega_2)&=\theta_h(\omega_1, \omega_2)+\hat{\theta}^+_f(\omega_1, \omega_2)\\ \hat{\theta}^-_{f_h}(\omega_1, \omega_2)&=\theta_h(\omega_1, \omega_2)+\hat{\theta}^-_f(\omega_1, \omega_2)\end{aligned}\right\} \tag{15}$$

and $\hat{\theta}^+_f(\omega_1, \omega_2)$ is a positive value of $\cos^{-1}(\cdot)$ and $\hat{\theta}^-_f(\omega_1, \omega_2)$ is a negative value of it.

2. Compute ${}_\pm Z(\omega_1, \omega_2)=|{}_\pm\hat{F}^*_h(\omega_1, \omega_2)\cdot Y(\omega_1, \omega_2)-\varphi_{f_h f_h}(\omega_1, \omega_2)|^2$. (See Appendix)

If $$|{}_+Z(\omega_1, \omega_2)-\varphi_{f_h f_h}\cdot\varphi_{vv}|>|{}_-Z(\omega_1, \omega_2)-\varphi_{f_h f_h}\cdot\varphi_{vv}|$$

then ${}_-Z(\omega_1, \omega_2)$ is taken, that is $\hat{\theta}^-_{f_h}(\omega_1, \omega_2)$ is chosen, and if the above inequality is converse then ${}_+Z(\omega_1, \omega_2)$ is taken, where $\hat{F}^*_h(\omega_1, \omega_2)$ is a complex conjugate of $\hat{F}_h(\omega_1, \omega_2)$ and $Y(\omega_1, \omega_2)$ is the Fourier transform of the observation $\boldsymbol{y}(t_1, t_2)$.

3. RESTORED SIGNAL AND ITS EVALUATION

From the above discussion the restored signal is expressed in the frequency domain as

$$\hat{F}(\omega_1, \omega_2)=\sqrt{\varphi_{ff}(\omega_1, \omega_2)}\cdot e^{j\theta_f(\omega_1,\omega_2)}, \tag{16}$$

where $\hat{F}(\omega_1, \omega_2)$ is the Fourier transform of the estimate $\hat{\boldsymbol{f}}(t_1, t_2)$.

This estimate is no longer the output of the linear optimal filter.

But the estimate obtained by the linear optimal filtering is not necessarily the best one because it is optimum if and only if the density functions of the corresponding random variables are normal. Indeed the estimate given by Eq. (16) may be better than that obtained by the linear optimal filtering when the spectral densities φ_{ff}, φ_{vv} and φ_{yy} is completely known. If these spectral densities are unknown we may estimate these quantities by using MEM [maximum Entropy Method] or CLSE-Wiener method.[3]

In a restoration problem or any other certain problem, we often want the periodgrams (the spectrums along with the observation) rather than a true spectrum. In such a case the CLSE-Wiener method[3] may be applied to obtain the periodgrams[4] of the corresponding signals. Using these periodgrams our method in this paper is useful to estimate the phase spectrum and consequently the complex spectrum of the original signal.

4. CONCLUSIONS

We have shown the method by which the phase spectrum is estimated.

In general the concept of the phase spectrum has rarely been introduced in restoration problems.

We convince that the phase spectrum is one of the important factors in a signal processing as a spectral density is so.

This paper shows that there is an important relationship between the phase spectrum and the spectral densities.

This method can be easily employed when only the knowledges of the corresponding spectral densities are available.

Furthermore this is also applied successfully when the date length is not necessarily sufficient because this method is based on the a posteriori information rather than the a priori information, while, in such a case a linear optimal filter which is based on the a priori information is not applied successfully.

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APPENDIX

Determination of the sign of the value of the two valued function $\cos^{-1}(\cdot)$ in the region $(-\pi, \pi)$.

For the simplicity in notation, one-dimensional signal is considered.

Two-dimensional signal is also able to treated similarly.

Since the observation is written as

$$\boldsymbol{y}(t)=\boldsymbol{f}_h(t)+\boldsymbol{v}(t), \tag{A-1}$$

it is expressed in terms of its Fourier transform;

$$Y(\omega)=F_h(\omega)+V(\omega) \tag{A-2}$$

where $V(\omega)$ is the frequency component of Fourier-Stieltjes integral of the noise $\boldsymbol{v}(t)$.

Now we introduce the following quantities.

$$_+\hat{F}_h^*(\omega)\cdot Y(\omega)={}_+\hat{F}_h^*(\omega)\cdot(F_h(\omega)+V(\omega)) \tag{A-3}$$

$$_-\hat{F}_h^*(\omega)\cdot Y(\omega)={}_-\hat{F}_h^*(\omega)\cdot(F_h(\omega)+V(\omega)) \tag{A-4}$$

where $_+\hat{F}_h(\omega)$ and $_-\hat{F}_h(\omega)$ are defined by Eqs. (13) and (14) respectively and $_+\hat{F}_h^*(\omega)$ and $_-\hat{F}_h^*(\omega)$ are those complex conjugates.

Suppose now that $_+\hat{F}_h(\omega)$ is true and $_-\hat{F}_h(\omega)$ is false;

$$_+\hat{F}_h(\omega)=F_h(\omega), \quad {}_-\hat{F}_h(\omega)\neq F_h(\omega) \tag{A-5}$$

then from Eq. (A-3)

$$_+\hat{F}_h^*(\omega)\cdot Y(\omega)=|F_h(\omega)|^2+F_h^*(\omega)V(\omega) \tag{A-6}$$

and from Eq. (A-4)

$$_{-}\hat{F}_h^*(\omega)Y(\omega)=|F_h(\omega)|^2\cdot e^{j(\theta_{fh}-\theta_{\bar{f}_h})}+{}_{-}\hat{F}_h^*(\omega)V(\omega) \tag{A-7}$$

are obtained where

$$_{-}\hat{F}_h(\omega)=|F_h(\omega)|\cdot e^{j\theta_{\bar{f}_h}}$$

and

$$F_h(\omega)=|F_h(\omega)|\cdot e^{j\theta_{fh}}.$$

Subtracting $|F_h(\omega)|^2$ from the both sides of Eq. (A–6), we get

$$_{+}\hat{F}_h^*(\omega)\cdot Y(\omega)-|F_h(\omega)|^2=F_h^*(\omega)\cdot V(\omega).$$

Taking the squared absolute value of the above equation,

$$|_{+}\hat{F}_h^*(\omega)\cdot Y(\omega)-|F_h(\omega)|^2|^2=|F_h^*(\omega)\cdot V(\omega)|^2$$
$$=|F_h(\omega)|^2\cdot|V(\omega)|^2 \tag{A-8}$$

is gotten. Since

$$|F_h(\omega)|^2=\varphi_{f_hf_h}(\omega),\quad |V(\omega)|^2=\varphi_{vv}(\omega),$$

Eq. (A–8) becomes

$$|_{+}\hat{F}_h^*(\omega)\cdot Y(\omega)-\varphi_{f_hf_h}(\omega)|^2=\varphi_{f_hf_h}(\omega)\cdot\varphi_{vv}(\omega). \tag{A-9}$$

Thus we conclude that if Eq. (A–9) is satisfied then the $\hat{\theta}^+_{f_h}(\omega)$ is true.

On the otherhand as for the false phase $\hat{\theta}^-_{f_h}(\omega)$, subtracting $|F_h(\omega)|^2$ from the both sides of Eq. (A–7), we get

$$_{-}\hat{F}_h^*(\omega)\cdot Y(\omega)-|F_h(\omega)|^2=|F_h(\omega)|^2\cdot(e^{j(\theta_{fh}-\theta_{\bar{f}_h})}-1)+{}_{-}\hat{F}_h^*(\omega)\cdot V(\omega)$$
$$=P(\omega)+{}_{-}\hat{F}_h^*(\omega)V(\omega) \tag{A-10}$$

where

$$P(\omega)=|F_h(\omega)|^2(e^{j(\theta_{fh}-\theta_{\bar{f}_h})}-1).$$

Taking the squared absolute value of the above equation,

$$|_{-}\hat{F}_h^*(\omega)\cdot Y(\omega)-|F_h(\omega)|^2|^2=|P(\omega)|^2+|_{-}\hat{F}_h^*(\omega)\cdot V(\omega)|^2+P(\omega)\cdot(\hat{F}_h^*(\omega)\cdot V(\omega))^*$$
$$+P^*(\omega)(_{-}\hat{F}_h^*(\omega)V(\omega))$$
$$=|P(\omega)|^2+|F_h(\omega)|^2|V(\omega)|^2+2\cdot\text{real}\,[P(\omega)\cdot(_{-}\hat{F}_h^*(\omega)\cdot V(\omega))^*]$$
$$=|P(\omega)|^2+\varphi_{f_hf_h}(\omega)\cdot\varphi_{vv}(\omega)+2\cdot\text{real}\,[P(\omega)\cdot(_{-}\hat{F}_h^*(\omega)\cdot V(\omega))^*] \tag{A-11}$$

where

$$|P(\omega)|^2=||F_h(\omega)|^2\cdot(e^{j(\theta_{fh}-\theta_{\bar{f}_h})}-1)|^2$$
$$=|F_h(\omega)|^4\cdot|e^{j(\theta_{fh}-\theta_{\bar{f}_h})}-1|^2$$
$$=|F_h(\omega)|^4(e^{j(\theta_{fh}-\theta_{\bar{f}_h})}-1)(e^{-j(\theta_{fh}-\theta_{\bar{f}_h})}-1)$$
$$=|F_h(\omega)|^4(2-2\cdot\text{real}\,e^{j(\theta_{fh}-\theta_{\bar{f}_h})})$$
$$=2\cdot|F_h(\omega)|^4(1-\cos(\theta_{f_h}-\hat{\theta}_{\bar{f}_h})). \tag{A-12}$$

Consequently the left hand side of Eq. (A–11) is clearly different from $\varphi_{f_hf_h}(\omega)\cdot\varphi_{vv}(\omega)$ when

$$|P(\omega)|^2 \neq -2\,\mathrm{real}\,[P(\omega)\cdot(-\hat{F}_h^*(\omega)\cdot V(\omega))^*]. \tag{A-13}$$

If the both sides of Eq. (A–13) are identical, the above evaluation is not available. In this case we take the following method:

1. Taking the arithmetic mean of the phase spectrum of the neighborhood of the corresponding frequency.
2. If $\hat{\theta}_f^+(\omega)$ is closer to the mean than $\hat{\theta}_f^-(\omega)$, then $\hat{\theta}_f^-(\omega)$ is taken as $\theta_f(\omega)$.
3. If $\hat{\theta}_f^-(\omega)$ is closer to the mean than $\hat{\theta}_f^+(\omega)$, then $\hat{\theta}_f^+(\omega)$ is taken as $\theta_f(\omega)$.

This means that we choose $\hat{\theta}_f(\omega)$ uncorrelated, since if $\boldsymbol{f}(t)$ is a stationary process with Gaussian distribution its phase spectrum is a process with white uniform distribution.[1)]

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