# ON THE ISOMORPHISM OF FORMAL LINEARIZATION FOR NONLINEAR SYSTEMS

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**Abstract**

Nonlinear systems are formally transformed into linear systems by introducing a sequence of linearly independent functions. In this paper, we study the relationship between the given nonlinear system and the corresponding formal linear system. The conditions for isomorphic linearization are acquired. The isomorphic linearization between Euclidean spaces is carried out by finitely many independent functions. An analytic nonlinear system is isomorphically linearized on Hilbert space by the Taylor expansion and a periodic nonlinear system, by the Fourier expansion.

## I. INTRODUCTION

There has been considerable interest in linearizing nonlinear systems (see [1]–[4], for example). In [2], a formal linearization approach of nonlinear systems has been proposed and applied to estimation and control problems. This approach transforms a given nonlinear system on Euclidean space into a linear system on a certain function space by introducing a sequence of linearly independent functions. Thus the well-developed linear theory of estimation and control has been successfully applied to the nonlinear system.

The purpose of this paper is to study the relationship between the given nonlinear system and the formal linear system. The isomorphism of the two systems is defined so that the systems are related by diffeomorphism between two manifolds. One of the manifolds is the state space of the given nonlinear system on Euclidean space, and the other is the state space of the formal linear system on the function space. Conditions for the two systems to be isomorphic is investigated. Moreover the following is studied here. A nonlinear system for which finitely many independent functins suffice is isomorphically linearized on Euclidean space. An ananytic nonlinear system is linearized on Hilbert space by introducing certain polinomials as the linearly independent functions, namely by the Taylor expansion. A periodic nonlinear system is also linearized on Hilbert space by the trigonometric functions, namely by the Fourier expansion.

## II. FORMAL LINEARIZATION

We consider a nonlinear system described by the differential equation

$$\Sigma_1 : \dot{x}(t) = f(t, x(t)) \qquad ((t_0, x(t_0)) \in \Gamma) \tag{1}$$

which is defined on an open cylinder $\Gamma \triangleq T \times M_1 \subset R^{n_0+1}$, where $\cdot = d/dt$, $x \in R^{n_0}$ is an $n_0$-dimensional state vector, $R^k$ is a $k$-dimensional real Euclidean space with a natural topology, $R = R^1$, $T \subset R$ is an interval of time $t$, $M_1 \subset R^{n_0}$ is a state space of $x$, $f : \Gamma \to R^{n_0}$ is a vector valued function of class $C^r$. A function of class $C^r$ means an $r$ times continuously differentiabel function if $r = 0, 1, 2, \ldots, \infty$ and an analytic function $r = \omega$.

Introducing a sequence of linearly independent functions of real values $\{1, \phi_1(x), \phi_2(x),$

..., $\phi_N(x)$,...}, the nonlinear system $\sum_1$ is transformed into a formal linear system $\sum_2$ on a function space $Z$ as follows (see [2]).

Let the function space $Z$ include

$$M_2=\phi(M_1)\triangleq\{\phi(x)\colon x\in M_1\}$$

where

$$\phi(x)=[\phi_1(x),\ \phi_2(x),\ldots,\ \phi_N(x),\ldots]^T. \tag{2}$$

The superscript $T$ means transposing and $\triangleq$ denotes defining. The dynamic equation of $\phi_N(x)$ ($N=1, 2, 3,\ldots$) is

$$\dot{\phi}_N(x(t))=\partial\phi_N(x(t)/\partial x^T(t)f(t_1x(t)). \tag{3}$$

The right hand side is associated with

$$\partial\phi_N(x(t))/\partial x^T(t)f(t,\ x(t))\Leftrightarrow\sum_{i=1}^{\infty}\alpha_{Ni}(t)\phi_i(x(t))+\alpha_{N0}(t) \tag{4}$$

where $\alpha_{Ni}\in R$ for $i=0, 1, 2,\ldots$ and $N=1, 2, 3,\ldots$, so it follows that

$$\dot{\phi}(x(t))\Leftrightarrow A(t)\phi(x(t))+b(t) \qquad \phi(x(t_0))\in M_2 \tag{5}$$

where

$$A(t)=\begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t)\cdots\alpha_{1N}(t)\cdots \\ \alpha_{21}(t) & \alpha_{22}(t)\cdots\alpha_{2N}(t)\cdots \\ \vdots & \vdots \quad \vdots \\ \alpha_{N1}(t) & \alpha_{N2}(t)\cdots\alpha_{NN}(t)\cdots \\ \vdots & \vdots \quad \vdots \end{bmatrix}$$

$$b(t)=[\alpha_{10}(t)\quad \alpha_{20}(t)\cdots\alpha_{N0}(t)\cdots]^T$$

$$\phi(x(t_0))=[\phi_1(x(t_0)),\ \phi_2(x(t_0)),\ldots,\ \phi_N(x(t_0))\cdots]^T$$

Let us write an element of $Z$ by $z(t)=[z_1(t),\ z_2(t),\ldots,\ z_N(t),\ldots]^T$. Since $\phi(x)\in M_2\subset Z$, we regard $\phi(x)$ as an element of $Z$ and derive a linear equation from (3) as follows:

$$\textstyle\sum_2\colon\ \dot{z}(t)=A(t)z(t)+b(t)$$

$$z(t_0)=\phi(x(t_0))\in Z \tag{6}$$

which is called "a formal linear system". We here give the definition of isomorphic systems which comes from [5]. For (1) and (6) to make sense, it is necessary to interpret them as Appendix B.

[DEFINITION]

The two systems $\sum_1$ and $\sum_2$ of (1) and (6) are isomorphic if (I) $\phi\colon M_1\to M_2$ is an diffeomorphism and (II) $d\phi(f(t,\ x(t))=A(t)z(t)+b(t)$ for all $(t,\ x(t))\in\Gamma$.

We will consider conditions for the isomorphism in the following sections.

## III. ISOMORPHISM

We now state and prove the isomorphism of a given nonlinear system with a corresponding formal linear system.

[Theorem 1]

Let $\sum_1$ of (1) be a given system defined on $\Gamma$. Let $\sum_2$ of (6) be the corresponding formal linear system. Then $\sum_1$ and $\sum_2$ are isomorphic on the two $n$-dimensional $C^r$-manifolds $M_1(\subset R^{n_0})$ and $M_2(\subset Z)$ if the following assumptions hold:

[A–i] (1) has a unique solution on $\Gamma=\Gamma\times M_1$ where $M_1$ is an $n$-dimensional $C^r$-manifold.
[A–ii] (6) of an initial value problem has a unique solution on $T$.
[A–iii] The equality# of (5) holds for $\Gamma$:

$$\dot{\phi}(x(t))=A(t)\phi(x(t))+b(t). \tag{7}$$

[A–iv] $\phi$: $M_1\rightarrow M_2$ is a homeomorphism.

(Proof)

A differential equation (1) has a unique solution $x(t)(t\in T)$ from [A–i], so its image $\phi(x(t))$ by a mapping $\phi$ is uniquely determined. Let us consider (7) a differential equation defined on a linear space $Z$ and integrate (7) on $[t_0, t]\subset T$:

$$\phi(x(t))-\phi(x(t_0))=\int_{t_0}^{t}(A(\tau)\phi(x(\tau))+b(\tau))d\tau.$$

Similary, regard (6) as a differential equation on $Z$, then we have

$$z(t)=z(t_0)+\int_{t_0}^{t}(A(\tau)z(\tau)+b(\tau))d\tau.$$

From [A–ii] and the initial condition $z(t_0)=\phi(x(t_0))$, the last two equations indicate that, for all $(t, x(t))\in\Gamma$,

$$z(t)=\phi(x(t)). \tag{8}$$

That is, we have had the existence and the value of the solution of (6).

$M_1$ is assumed to be an $n$-dimensional $C^r$-manifold. Let an atlas of $M_1$ be $\{(U_\lambda, \psi_\lambda)$: $\lambda\in\wedge\}$ where $\wedge$ is an index set, $U_\lambda$ is an open set of $M_1$, $O_\lambda$ is an open set of $R^n$, and $\psi_\lambda$ is a homeomorphism from $U_\lambda$ onto $O_\lambda$. From Appendix (A1–c),

$$\psi_\lambda\cdot\psi_\mu^{-1}:\ \psi_\mu(U_\lambda\cap U_\mu)\longrightarrow\psi_\lambda(U_\lambda\cap U_\mu) \tag{9}$$

is bijective and both $\psi_\lambda\circ\psi_\mu^{-1}$ and $(\psi_\lambda\circ\psi_\mu^{-1})^{-1}=\psi_\mu\circ\psi_\lambda^{-1}$ are of class $C^r$.

We turn to $M_2$. From [A–iii], $\phi$: $M_1\rightarrow M_2$ is a homeomorphism, so $V_\lambda=\phi(U_\lambda)$ is an open set of $M_2=\phi(M_1)$ such that $M_2=\bigcup_{\lambda\in\wedge}V_\lambda$ ((A1–a) hols).

Defining $h_\lambda\triangleq\psi_\lambda\circ\phi^{-1}$: $M_2\rightarrow R^n$ for each $\lambda\in\wedge$, $h_\lambda$ is an homeomorphism from $V_\lambda$ onto $O_\lambda$ ((A1–b) holds). It follows that

$$h_\lambda\circ h_\mu^{-1}=(\psi_\lambda\circ\phi^{-1})\circ(\psi_\mu\circ\phi^{-1})^{-1}=\psi_\lambda\circ\psi_\mu^{-1},$$

$$h_\lambda(V_\lambda\cap V_\mu)=\psi_\lambda\circ\phi^{-1}(\phi(U_\lambda)\cap\phi(U_\mu))=\psi_\lambda(U_\lambda\cap U_\mu).$$

$$h_\mu(V_\lambda\cap V_\mu)=\psi_\mu(U_\lambda\cap U_\mu).$$

By these equalities and the property mentioned at (9),

$$h_\lambda\circ h_\mu^{-1}:\ h_\mu(V_\lambda\cap V_\mu)\longrightarrow h_\lambda(V_\lambda\cap V_\mu)$$

is bijective and both $h_\lambda\circ h_\mu^{-1}$ and $(h_\lambda\circ h_\mu^{-1})^{-1}$ are of class $C^r$ ((A1–c) holds). Hence $M_2$ has been given a structure of an $n$-dimensional $C^r$-manifold by $\{(V_\lambda, h_\lambda)$: $\lambda\in\wedge\}$ as shown at Appendix A1. From Appendix A2, both $\phi$ and $\phi^{-1}$ are of class $C^r$ because

$$h_\lambda\circ\phi\circ\psi_\lambda^{-1}=(\psi_\lambda\circ\phi^{-1})\circ\phi\circ\psi_\lambda^{-1}=1$$

$$\psi_\lambda\circ\phi^{-1}\circ h_\lambda^{-1}=\psi_\lambda\circ\phi^{-1}\cdot(\phi\circ\psi_\lambda^{-1})=1$$

# The equalty means the following: The function of the left hand side is expanded into a series of the right hand side. Conversely, the seires of the right hand side converges to the function of the left hand side.

where 1 is an identity map which is of class $C^r(r=\omega)$. Therefore, using [A–iii] and Appendix A3, we have that $\phi$: $M_1 \rightarrow M_2$ is a diffeomorphism (Definition (I) holds).

Pay attention to (1) and (B1) of Appendix B. For fixed $(t_0, x(t_0)) \in \Gamma$ and any function $\xi$ of class $C^r$, it follows that

$$d\phi(f(t, x(t)))(\xi) = d\phi(\dot{x}(t))(\xi) = (dx(t))d/du(\xi \circ \phi)$$
$$= d/du(\xi \circ \phi \circ x)(t) = d/du(\xi \circ \phi(x(t))). \tag{10}$$

Substituting (8) into (10) and using (B2) and (6) yield

$$d\phi(f(t, x(t)))(\xi) = d/du(\xi \circ z(t)) = d/du(\xi \circ z)(t)$$
$$= dz(t)d/du(\xi) = \dot{z}(t)(\xi)$$
$$= (A(t)z(t) + b(t))(\xi),$$

thus we have

$$d\phi(f(t, x(t))) = A(t)z(t) + b(t)$$

(Definition (II) holds).

After all, the two systems $\sum_1$ and $\sum_2$ are isomorphic. (Q. E. D.)

We here remark a replacement for the assumption [A–i] of Theorem 1.

Assume that

[A–v] $M_1$ is an open set of $R^{n_0}$ and $T$ is an open interval of $R$.

$\Gamma = T \times M_1$ is thus an open set of $R^{n_0+1}$. Then there exists a unique solution of the differential equation (1) if the following holds (see [7]):

[A–vi] Both $f(t, x)$ and $\partial f(t, x)/\partial x^T$ are continuous on $\Gamma$.

By choosing an identity map as the coordinate functions $\psi_\lambda$, $M_1$ has a structure of an $n$-dimensional $C^r$-manifold where $n = n_0$ and $r = \omega$. Therefore we have:

[Lemma] The assumption [A–i] of Theorem 1 is replaced by [A–v] and [A–vi].

We will study a case of finitely many independent functions.

[Theorem 2]

Let a nonlinear system of (1) be given. Assume that the number of linearly independent functions required is finite and both $A(t)$ and $b(t)$ of (6) are continuous on the entire interval $T$. If the assumptions of [A–i], [A–iii], and [A–iv] hold, then this nonlinear system is isomorphically transformed into a linear system on a manifold contained in Euclidean space.

(Proof)

Let a vector of the linearly independent functions be $\phi(x) = [\phi_1(x), \ldots, \phi_N(x)]^T$ and the function space be $Z = R^N$. To $R^N$, we assign a natural Euclidean topology. A($t$) and $b(t)$ are continuous, so there exists the unique solution of (6) and [A–ii] holds (see [7]). Consequently the assumptions of Theorem 1 are all satisfied. (Q. E. D.)

The following examples illustrate this situation.

[Example 1]

Consider the nonlinear system:

$$\textstyle\sum_1: \dot{x} = \sqrt{ax^2 + 2bx + c} \qquad (= f(x)) \tag{11}$$

on $\quad M_1 = \{x \in R: ax^2 + 2bx + c > 0\}, \quad (x(t_0) \triangleq x_0 \in M_1)$

where $a$, $b$ and $c$ are all constant. For this system, all assumptions of Theorem 2 are satisfied as follows. Thus this is isomorphically linearizable on $C^\omega$-manifolds.

(Proof)

1) [A–i]: Both $f(x)=\sqrt{ax^2+2bx+c}$ and

$$\partial f(x)/\partial x=(ax+b)/\sqrt{ax^2+2bx+c}$$

are continuous on $M_1$. $T$ is restricted to an open interval of $t$ such that

$$M_2 \ni x_0+\int_{t_0}^{t}\sqrt{ax^2(\tau)+2bx(\tau)+c}\,d\tau.$$

[A–v] and [A–vi] are satisfied, so [A–1] holds by Lemma.

2) [A–iii], A, b: Let us choose $\phi_1(x)=x$ and $\phi_2(x)=\dot{x}=\sqrt{ax^2+2bx+c}$. [Aiii] holds because

$$d\phi_1(x)/dt=\partial\phi_1(x)/\partial x f(x)=\phi_2(x)$$

$$d\phi_2(x)/dt=\partial\phi_2(x)/\partial x f(x)=a\phi_1(x)+b\,.$$

These equations indicate that the linearly independent functions are $\{1, \phi_1(x), \phi_2(x)\}$ and are finite in number. Thus we define

$$\phi(x)=[\phi_1(x),\ \phi_2(x)]^T=[x,\ \dot{x}]^T$$

$$=[x,\ \sqrt{ax^2+2bx+c}]^T\,.$$

In this case, the formal linear system is obtained as follows:

$$\dot{z}(t)=\begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} z(t)+\begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$z(t_0)=\phi(x(t_0))=[x_0,\ \sqrt{ax_0^2+2bx_0+c}]^T \in R^2$$

whose coefficients are constant, i.e., continuous.

3) [A–iv]: The mapping $\phi$: $M_1 \rightarrow M_2=\phi(M_1)$ is bijective, because if $\phi(x)=\phi(x')$ then $\phi_1(x)=\phi_1(x')$ or $x=x' \in M_1$. We assign $M_1$ the induced topology of $R^2$. A $p$-neighborhood of $M_1$ is $\{x \in M_1;\ |x-p|<1/k\}$ and a $\phi(p)$-neighborhood of $M_2$ is

$$\{\phi(x) \in M_2;\ \sum_{i=1}^{2}|\phi_i(x)-\phi_i(p)|^2<1/k^2\}.\quad \text{If } |x-p| \rightarrow 0 \text{ then}$$

$$\sum_{i=1}^{2}|\phi_i(x)-\phi_i(p)|^2=(x-p)^2+(\sqrt{ax^2+2bx+c}-\sqrt{ap^2+2bp+c})^2$$

$$=(x-p)^2(1+(ax+ap+2b)^2)/(\sqrt{ax^2+2bx+c}+\sqrt{ap^2+2bp+c})^2) \longrightarrow 0,$$

so $\phi$ is continuous. Let $\phi^{-1}$ be a projection mapping:

$$\phi^{-1}: M_2 \longrightarrow M_1: \phi(x) \longrightarrow \phi_1(x)\,.$$

If $\sum_{i=1}^{2}|\phi_i(x)-\phi_i(p)|^2 \rightarrow 0$ then $|\phi_1(x)-\phi_1(p)| \rightarrow 0$ or $x \rightarrow p$, so $\phi^{-1}$ is continuous. Therefore $\phi$: $M_1 \rightarrow M_2$ is a homeomorphism.

Appendix $D$ gives the expressions of $M_1$ and $M_2$. (Q.E.D.)

[Example 2]

Consider the scalar nonlinear system

$$\Sigma_1: \dot{x}=(x+a)^{1-1/2k} \qquad (x(t_0) \triangleq x_0 \in M_1) \tag{12}$$

defined on $M_1=\{x \in R: x+a>0\}$ where $a \in R$ and $k$ is a positive integer. This system is also isomorphically linearizable by Theorem 2.

(Proof) (One Linearization)

Paying attention that the $m$-th derivative of $x$ is

$$x^{(m)}=\prod_{j=0}^{m-1}(1-j/2k)(x+a)^{1-m/2k} \qquad (1\leqq m\leqq 2k)$$
$$=0 \qquad (m\geqq 2k+1).$$

Let $N=2k$. Define

$$\begin{aligned}\phi(x)&=[\phi_1(x),\ldots,\phi_N(x)]^T\\ &\triangleq[x,\dot{x},\ldots,x^{(2k-1)}]^T\\ &=[x,(x+a)^{1-1/2k},\ldots,\prod_{j=0}^{m-1}(1-j/2k)(x+a)^{1-m/2k},\\ &\quad\ldots,\prod_{j=0}^{2k-2}(1-j/2k)(x+a)^{1/2k}]^T.\end{aligned} \tag{13}$$

Then we have a $2k$-dimensional formal linear system

$$\Sigma_2:\ \dot{z}(t)=Az(t)+b \qquad z(t_0)\in R^{2k} \tag{14}$$

with

$$A=\begin{bmatrix}0 & 1\cdots 0\\ & \ddots\ \ddots\ \vdots\\ & \ddots\ 1\\ 0 & 0\end{bmatrix}$$

$$b=[0,\ldots,0,\prod_{j=0}^{2k-1}(1-j/2k)]^T$$

$$z(t_0)=[x_0,(x_0+a)^{1-1/2k},\ldots,\prod_{j=0}^{2k-2}(1-j/2k)(x_0+a)^{1/2k}]^T.$$

The isomorphism of $\Sigma_2$ of (14) with $\Sigma_1$ of (12) is easily proven in a way similar to the proof of Example 1. In this case, $Z=R^{2k}$.

$\phi$: $M_1\to M_2$ is continuous because

$$\sum_{i=1}^{2k}|\phi_i(x)-\phi_i(p)|^2$$
$$=(x-p)^2+\sum_{m=1}^{2k-1}\prod_{j=0}^{m-1}(1-j/2k)^2((x+a)^{1-m/2k}-(p+a)^{1-m/2k})^2\longrightarrow 0$$

as $x\to p$. $\phi^{-1}$: $M_2\to M_1$ is continuous because

$$|\phi_1(x)-\phi_1(p)|=|x-p|\longrightarrow 0 \quad \text{as} \quad \sum_{i=1}^{2k}|\phi_i(x)-\phi_i(p)|^2\longrightarrow 0.$$

On the other hand, $\Sigma_1$ is transformed into another form.

(Another Linearization)

Let $N=2$ and $L=2k$. Define

$$\phi(x)=[\phi_1(x),\phi_2(x)]^T=[(x+a)^{1/2k},(x+a)^{1/k}]^T. \tag{15}$$

Then we have a 2-dimensional formal linear system

$$\Sigma_3:\ \dot{z}(t)=Az(t)+b \qquad z(t_0)\in R^2 \tag{16}$$

with

$$A=\begin{bmatrix} 0 & 0 \\ 1/k & 0 \end{bmatrix} \quad b=\begin{bmatrix} 1/2k \\ 0 \end{bmatrix}$$

$$z(t_0)=[(x_0+a)^{1/2k}, (x_0+a)^{1/k}]^T.$$

Eq. (14) is also an isomorphic formal linear system of (12) by Theorem 2. In this case, $Z=R^2$. $\phi$: $M_1 \to M_2$ is continuous because

$$\sum_{i=1}^{2} |\phi_i(x)-\phi_i(p)|^2=((x+a)^{1/2k}-(p+a)^{1/2k})^2$$
$$+((x+a)^{1/k}-(p+a)^{1/k})^2 \longrightarrow 0 \quad \text{as} \quad x \longrightarrow p.$$

$\phi^{-1}$: $M_2 \to M_1$ is continuous because of the following: If $\sum_{i=1}^{2} |\phi_i(x)-\phi_i(p)|^2 \to 0$ then

$$\phi_1(x) \longrightarrow \phi_1(p) \quad \text{or} \quad (\phi_1^{2k}(x)-a) \longrightarrow (\phi_1^{2k}(p)-a), \quad \text{so}$$
$$|(\phi_1^{2k}(x)-a)-(\phi_1^{2k}(p)-a)|=|x-p| \longrightarrow 0.$$

The other conditions of Theorem 2 are shown to be satisfied easily. (Q. E. D.)

Example 2 shows the fact that a nonlinear system can be isomorphically transformed in some different formal linear systems. That is, the linearization is not unique. But the corresponding formal linear systems are isomorphic to one another.

In the next section, we will consider the case of an analytic nonlinear function expanded in a Taylor series.

## IV. TAYLOR EXPANSION

We here study the isomorphism for analytic nonlinear systems by the Taylor exansion.

Let (1) be given where $f$ is analytic. Throughout this section, the Taylor expansion is carried out in a neighborhood of 0 assuming $0 \in M_1$ without loss of generality.

As the linearly independent functions, we choose a set of all polynomials in $x$ which appear in the Taylor series. That is,

$$\phi(x)=[\phi_1(x), \phi_2(x), \ldots, \phi_N(x), \ldots]^T$$
$$=\left[x_1, x_2, \ldots, x_n, \frac{1}{2!}x_1^2, \frac{1}{1!1!}x_1x_2, \ldots, \frac{1}{r_1!\cdots r_n!}\prod_{i=1}^{n} x_i^{r_i}, \ldots\right]^T \tag{17}$$

where the elements are arranged in lexicographic order. From (3),

$$\dot{\phi}(x(t))=\partial\phi(x(t))/\partial x^T(t) f(t, x(t)).$$

Expanding $\partial\phi(x)/\partial x^T f(t, x)$ in a Taylor series about $x=0$ and putting $x=x(t)$, it follows that

$$\dot{\phi}(x(t))=A(t)\phi(x(t))+b(t) \tag{18}$$

with

$$A(t)=\partial(\partial\phi(x)/\partial x^T f(t, x))/\partial\phi^T(x)\,|_{x=0}$$
$$b(t)=(\partial\phi(x)/\partial x^T f(t, x))\,|_{x=0}.$$

In this case, the next theorem follows:

[Theorem 3]

Let a nonlinear system of (1) be given. Assume that $\Gamma=T\times M_1$ is open of $R^{no+1}$, $f$ is analytic on $\Gamma$, and (6) of an initial value problem has a unique solution on $T$. Then by the Taylor expansion, $\sum_1$ and $\sum_2$ are isomorphic on $M_1$ and $M_2$ which is an $n$-dimensional $C^\omega$-manifold contained in Hilbert space $(l^2)$.

(Proof)

$M_1$ is given an induced topology of $R^n$ so that an open neighborhood base at $p\in M_1$ is $\{U_k^{M_1}(p)\colon k\in N\}$ with

$$U_n^{M_1}(p)=\{x\in M_1\colon \sum_{i=1}^{n}(x_i-p_i)^2<1/k^2\}. \tag{19}$$

$M_1$ is then an $n$-dimensional $C^\omega$-manifold by putting

$$\{(U_\lambda, \psi_\lambda)\colon \lambda\in\wedge\}=\{(M_1, 1)\}$$

at Appendix A1. $f$ is analytic on an open set $\Gamma$ so [A–v] and [A–vi] hold, namely [A–1] does. $\partial\phi(x)/\partial x^T f(t, x)$ is also analytic, so [A–iii] holds by using (18) and the property of analytic function.

A Hilbert space is

$$(l^2)=\{\xi=[\xi_1, \xi_2,\ldots]^T\colon \sum_{i=1}^{\infty}|\xi_i|^2<\infty,\ \|\xi\|=\sqrt{\sum_{i=1}^{\infty}|\xi_i|^2}\}, \tag{20}$$

while the elements of $\phi(x)$ of (17) lead to

$$\sum_{i=1}^{\infty}|\phi_i(x)|^2\leqq(\sum_{i=1}^{\infty}|\phi_i(x)|)^2=(\exp\sum_{i=1}^{n}|x_i|-1)^2<\infty.$$

These indicate that $M_2$ is included in $(l^2)$ and

$$Z=(l^2)\supset M_2$$

by choosing $(l^2)$ as $Z$. In such a way, we have fixed a linear space $Z$ which has contained $M_2$ as a subset and assigned a topology $Z$. We now give $M_2$ an induced topology of $Z$.

The mapping $\phi$; $M_1\to M_2$ is onto. For any $\phi(x)=\phi(x')\in M_2$, it follows that

$$x=[\phi_1(x),\ldots, \phi_n(x)]^T=[\phi_1(x'),\ldots, \phi_n(x')]^T=x'\in M_2,$$

so $\phi$ is one to one. For all $z\in(l^2)$, an open $z$-neighborhood base of $(l^2)$ is $\{V_k(z)\colon k\in N\}$ where

$$V_k(z)=\{\xi\in(l^2)\colon \sum_{i=1}^{\infty}|\xi_i-z_i|^2<1/k^2\}.$$

For all $\phi(p)\in M_2$, the induced topology of $M_2$ is generated by an open $\phi(p)$-neighborhood base $\{V_k^{M_2}(\phi(p))\colon k\in N\}$ where

$$V_k^{M_2}(\phi(p))=\{\phi(x)\in M_2\colon \sum_{i=1}^{\infty}|\phi_i(x)-\phi_i(p)|^2<1/k^2\}. \tag{21}$$

In order to prove that $\phi$ is continuous on these topologies on $M_1$ and $M_2$, all we have to do is to show $\sum_{i=1}^{\infty}|\phi_i(x)-\phi_i(p)|^2\to 0$

as $\sum_{i=1}^{n}|x_i-p_i|^2\to 0$ for $x$, $p\in M_1$ from (19) and (21). From (C3) of Appendix C, it follows that

$$\sum_{i=1}^{\infty}|\phi_i(x)-\phi_i(p)|^2=\sum_{i=1}^{\infty}\sum_{r_1+\cdots+r_n=i}\frac{1}{(r_1!\cdots r_n!)^2}(\prod_{j=1}^{n}x_j^{r_j}-\prod_{j=1}^{n}p_j^{r_j})^2\longrightarrow 0$$

as $\sum_{i=1}^{n}(x_i-p_i)^2\to 0$ thus $\phi\colon M_1\to M_2$ is continuous.

Let $g$ be a projection mapping which projects the first $n$ components of $\phi(x)$:

$$g: M_2 \longrightarrow M_1: [\phi_1(x),\dots, \phi_n(x), \phi_{n+1}(x),\dots]^T \longrightarrow [\phi_1(x),\dots, \phi_n(x)]^T.$$

Since $\phi_i(x)=x_i$ for $i=1, 2, 3,\dots, n$, $U_k^{M_1}(p)$ of (19) may be written as

$$U_k^{M_1}(p)=\{[\phi_1(x),\dots, \phi_n(x)]^T \in M_1: \sum_{i=1}^{n} |\phi_i(x)-\phi_i(p)|^2<1/k^2\}.$$

Therefore in order that $g$ is continuous, it must be that $\sum_{i=1}^{n} |\phi_i(x)-\phi_i(p)|^2 \to 0$ as $\sum_{i=1}^{\infty} |\phi_i(x)-\phi_i(p)|^2 \to 0$. This is clearly satisfied, so $g$ is continuous. Now, this $g$ is the inverse mapping of $\phi$ because $\phi \cdot g=1$ and $g \cdot \phi=1$. Thus $\phi^{-1}=g: M_2 \to M_1$ is continuous.

After all, it has been proven that $\phi: M_1 \to M_2$ is a homeomorphism and [A–iv] holds. (Q. E. D.)

In the next section, we will consider the case of an periodic nonlinear function expanded in a Fourier series.

## V. FOURIER EXPANSION

We study the isomorphism by the Fourier expansion. For simplicity, we here only treat with the linearization of a scalar periodic system by the trigonometric Fourier expansion. A multi-dimensional case is straightforward by making a direct product of scalars (see [2]). That is, Eq. (1) of $n=1$ is given on $M_1 \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $f(t, x)$ is a periodic function such as sinx, cosx, etc.

The following trigonometric functions are chosen as the linearly independent functions:

$$\begin{cases} \phi_{2r-1}(x)=r^{-\alpha} \sin rx \\ \phi_{2r}(x)=r^{-\alpha} \cos rx \end{cases} \qquad \begin{pmatrix} \alpha>1/2 \\ r=1, 2, 3,\dots \end{pmatrix}.$$

Thus $\phi(x)$ is

$$\begin{aligned} \phi(x)&=[\phi_1(x), \phi_2(x),\dots, \phi_N(x),\dots,]^T \\ &=[\sin x, \cos x,\dots, N^{-\alpha} \sin Nx, N^{-\alpha} \cos Nx,\dots]^T. \end{aligned} \tag{22}$$

Expanding $\partial\phi(x)/\partial x f(t, x)$ in a trigonometirc Fourier series in $x$ and putting $x=x(t)$, it follows that

$$\Sigma_2: \dot{\phi}(x(t))=A(t)\phi(x(t))+b(t) \tag{23}$$

with

$$A(t)=(2\pi \sum_{r=1}^{\infty} r^{-2\alpha})^{-1} \int_{-\pi}^{\pi} \partial\phi(x)/\partial x f(t, x)\phi(x)dx$$

$$b(t)=(2\pi)^{-1} \int_{-\pi}^{\pi} \partial\phi(x)/\partial x f(t, x)dx.$$

In this case the next theorem follows:

[Theorem 4]

Let a scalar nonlinear system of (1) ($n=1$) be given. Assume that $f(t, x)$ is a periodic function of period $2\pi$ in $x \in M_1$ and [A–ii] [A–v] and [A–vi] hold. Then by the trigonometric Fourier expansion, $\Sigma_1$ of (1) and $\Sigma_2$ of (23) are isomorphic on $M_1$ and $M_2$

(Proof)

Since $f(t, x)$ is a differentiable function of period $2\pi$ with respect to $x$, so is $\partial\phi(x)/\partial x f(t, x)$. Thus [A–iii] holds by (23). The elements of $\phi(x)$ of (22) lead to

$$\sum_{r=1}^{\infty} |\phi_r(x)|^2 = \sum_{r=1}^{\infty} r^{-2\alpha}(\sin^2 rx + \cos^2 rx)$$
$$= \sum_{r=1}^{\infty} r^{-2\alpha} < \infty$$

which indicates $M_2 \subset (l^2) \subset Z$ by choosing $(l^2)$ as $Z$. Therefore we have made a linear space $Z$ which has contained $M_2$ as a subset and assigned $Z$ a topology. The induced topology of $Z$ is now given to $M_2$.

It is assumed that $M_1 \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for simplicity. If $\phi(x) = \phi(x') \in M_2$ then $\phi_1(x) = \phi_1(x')$ or $\sin x = \sin x'$ so $x = x' \in M_1$. Hence the mapping $\phi: M_2 \to M_2 = \phi(M_1)$ is bijective. we recall that a $\phi(p)$-neighborhood of $M_2$ is the same form as (21). If $|x-p| \to 0$ or $x \to p$, then

$$\varepsilon \triangleq \sup_{r \in N} ((\sin rx - \sin rp)^2 + (\cos rx - \cos rp)^2) \longrightarrow 0$$

namely

$$\sum_{r=1}^{\infty} |\phi_r(x) - \phi_r(p)|^2 = \sum_{r=1}^{\infty} r^{-2\alpha}((\sin rx - \sin rp)^2 + (\cos rx - \cos rp)^2) \leqq \eta\varepsilon \longrightarrow 0$$

where $\eta = \sum_{r=1}^{\infty} r^{-2\alpha}$.

Thus $\phi$ is continuous from $M_1$ onto $M_2$.

Let a function $g: M_2 \to M_1$ be defined as $g(\phi(x)) = \sin^{-1} \phi_1(x)$ where $\phi(x) = [\phi_1(x), \phi_2(x), \ldots]^T \in M_2$. The function $g$ is an inverse mapping of $\phi$ because $\phi \circ g = 1$ and $g \circ \phi = 1$. Since $M_2$ is assigned the indeced topology of $(l^2)$, the identity mapping 1 is continuous from $M_2$ into $(l^2)$. Define $\tilde{g}(z) = \sin^{-1} z_1: \mathscr{D} \to R$ where

$$\mathscr{D} \triangleq \{z = [z_1, z_2, \ldots]^T \in (l^2): |z_1| < 1\},$$

and $\tilde{g}$ is continuous from $\mathscr{D} \subset (l^2)$ into $R$. Accordingly $g = \tilde{g} \circ 1: M_2 \to M_1 \subset R$ is continuous.

After all, $\phi: M_1 \to M_2$ is a homeomorphism, and [A–iv] holds. (Q. E. D.)

## VI. CONCLUSIONS

We have studied the isomorphism between a given nonlinear system and a formal linear system. The following cases have been treated: a general form including the others as special cases, a linear combination of finitely many independent functions, a Taylor series, and a trigonometric Fourier series.

One would be interested in the study of the isomorphism when the formal linearization method is applied to nonlinear estimation and/or nonlinear control problem. Further studies including it are left for the future.

## APPENDIX A
## DEFINITIONS OF MANIFOLD

⟨A1⟩ If the following conditions (A1–a)∼(A1–c) hold, then a Hausdorff space $M$ has a structure of differentiable manifold by an atlas $\{(V_\lambda, h_\lambda): \lambda \in \Lambda\}$, the dimension of $M$ is $n$, and $M$ is refered to an $n$-dimensional $C^r$-manifold or a manifold simply.

(A1–a) Given an open covering $\{V_\lambda: \lambda \in \wedge\}$ such that $M = \bigcup_{\lambda \in \wedge} V_\lambda$.

(A1–b) Given a homeomorphism $h_\lambda: V_\lambda \to h_\lambda(V_\lambda)$ for each $\lambda \in \wedge$ where $V_\lambda$ and $h_\lambda(V_\lambda)$ are open sets of $M$ and $R^n$ respectively.

(A1–c) Whenever $V_\lambda \cap V_\mu$ is not empty, a function $h_\lambda \circ h_\mu^{-1}: h_\mu(V_\lambda \cap V_\mu) \to h_\lambda(V_\lambda \cap V_\mu)$ is bijective and both $h_\lambda \circ h_\mu^{-1}$ and $(h_\lambda \circ h_\mu)^{-1}$ are of $C^r$ class.

⟨A2⟩ Let $M$ and $N$ be manifolds. A mapping $\phi: M \to N$ is of $C^r$ class at $p \in M$ if there exist a $(U. \psi)$ at $p \in U \subset M$ and a $(V_\lambda. h)$ at $\phi(p) \in V \subset N$ such that

$$h \circ \phi \circ \psi^{-1}: \psi(U) \longrightarrow h(V)$$

is of $C^r$ class at $\psi(p)$.

⟨A3⟩ Let $M$ and $N$ be manifolds. A mapping $\phi: M \to N$ is a diffeomorphism if $\phi$ is a homeomorphism and both $\phi$ and $\phi^{-1}$ are of $C^r$ class.

## APPENDIX B
## SYSTEM ON MANIFOLD

Let $M_1$ be an $n$-dimensional $C^r$-manifold contained in $R^{n_0}$. Let $M_2$ be an $n$-dimensional $C^r$-manifold contained in a topological linear space $Z$ whose dimension may be infinite. An open interval $T \subset R$ is a 1-dimensional $C^r$-manifold $(r = \omega)$ if $u: t \to t$ is a coordinate function.

We only consider the solutions $\{z(t)\} \subset Z$ of (4) such that $z(t) \in M_2$. Then, $\sum_1$ and $\sum_2$ on the manifolds are interpreted as follows. $x: T \to M_1$ and $z: T \to M_2$ are $C^r$-curves. $\dot{x}(t)$ is a tangent vector of $x$ at $t \in T$ so that

$$\dot{x}(t) = dx(t)d/du \tag{B1}$$

where $dx$ is a differential of $x$. $z(t)$ is a tangent vector of $z$ at $t \in T$ so that

$$\dot{z}(t) = dz(t)d/du \tag{B2}$$

where $dz$ is a differential of $z$. $d\phi$ is also a differential of $\phi$. For fixed $t \in T$, we regard that $f(t, \cdot)$ is a vector field on $M_1$ and $A(t)(\cdot) + b(t)$ is a vector field on $M_2$.

## APPENDIX C
## CONTINUITY OF $\phi$

We here study a continuity of $\phi: M_1 \to M_2$ of (11) in the Taylor expansion. If $\sum_{i=1}^{n} (x_i - p_i)^2 \to 0$, then $x_i \to p_i$ for $i = 1, 2, \ldots, n$. For any $\varepsilon > 0$ and a sufficiently large integer $N_0$, it follows that

$$\sum_{r_1+\cdots+r_n=N_0}^{\infty} \frac{1}{(r_1!\cdots r_n!)^2} \left(\prod_{j=1}^{n} x_j^{r_j} - \prod_{j=1}^{n} p_j^{r_j}\right)^2$$

$$= \sum_{r_1+\cdots+r_n=N_0}^{\infty} \left(\prod_{j=1}^{n} \frac{x_j^{r_j}}{r_j!} - \prod_{j=1}^{n} \frac{p_j^{r_j}}{r_j!}\right)^2$$

$$\leqq \left(\sum_{r_1+\cdots+r_n=N_0}^{\infty} \left|\prod_{j=1}^{n} \frac{x_j^{r_j}}{r_j!} - \prod_{j=1}^{n} \frac{p_j^{r_j}}{r_j!}\right|\right)^2$$

$$\leqq \left(\sum_{r_1+\cdots+r_n=N_0}^{\infty} \left(\prod_{j=1}^{n} \frac{|x_j|^{r_j}}{r_j!} + \prod_{j=1}^{n} \frac{|p_j|^{r_j}}{r_j!}\right)\right)^2$$

$$= \left(\left(\exp(|x_1| + \cdots + |x_n|) - \sum_{r_1+\cdots+r_n=0}^{N_0-1} \prod_{j=1}^{n} \frac{|x_j|^{r_j}}{r_j!}\right)\right.$$

$$+\left(\exp(|p_1|+\cdots+|p_n|)-\sum_{r_1+\cdots+r_n=0}^{N_0-1}\prod_{j=1}^{n}\frac{|x_j|^{r_j}}{r_j!}\right)^2$$

$$\leqq \varepsilon/2.$$

The last inequality is due to a property of the exponential series. Let us fix $\varepsilon>0$ and $N_0$ so that (C1) holds. By the mean value theorem, there existes $K>0$ such that

$$\left|\prod_{j=1}^{n}\frac{x_j^{r_j}}{r_j!}-\prod_{j=1}^{n}\frac{p_j^{r_j}}{r_j!}\right|\leqq Kn\delta$$

for $1\leqq r_1+\cdots+r_n\leqq N_0-1$ and $|x_i-p_i|<\delta$ $(i=1,\dots,n)$. Thus, choosing a sufficiently small $\delta>0$ yields

$$\sum_{r_1+\cdots+r_n=1}^{N_0-1}\frac{1}{(r_1!\cdots r_n!)^2}\left(\prod_{j=1}^{n}x_j^{r_j}-\prod_{j=1}^{n}p_j^{r_j}\right)^2\leqq\sum_{r_1+\cdots+r_n=0}^{N_0-1}k^2n^2\delta^2\leqq\varepsilon/2 \tag{C2}$$

The sum of (C1) and (C2) follows

$$\sum_{i=1}^{\infty}\sum_{r_1+\cdots+r_n=i}\frac{1}{(r_1!\cdots r_n!)^2}\left(\prod_{j=1}^{n}x_j^{r_j}-\prod_{j=1}^{n}p_j^{r_j}\right)^2\leqq\varepsilon. \tag{C3}$$

## APPENDIX D
## EXPRESSION OF MANIFOLDS

The manifolds $M_1$ and $M_2$ of Example 1 are expressed as follows.
Define $N\triangleq\{y:\ ay^2+2by+c>0\}$.

(i) A case of $a\neq 0$:

$$M_1=\Big\{x(t):\ x(t)=\frac{1}{2}\left(x_0+\frac{b}{a}\right)(e^{\sqrt{a}t}+e^{-\sqrt{a}t})$$
$$+\frac{1}{2\sqrt{a}}\sqrt{ax_0^2+2bx_0+c}\,(e^{\sqrt{a}t}-e^{-\sqrt{a}t})-\frac{b}{a},\ x(t)\in N\Big\}$$

$$M_2=\left\{\xi(t):\begin{bmatrix}\xi_1(t)\\ \\ \xi_2(t)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\left(x_0+\frac{b}{a}\right)(e^{\sqrt{a}t}+e^{-\sqrt{a}t})+\frac{1}{2\sqrt{a}}\sqrt{ax_0^2+2bx_0+c}\,(e^{\sqrt{a}t}-e^{-\sqrt{a}t})-\frac{b}{a}\\ \frac{\sqrt{a}}{2}\left(x_0+\frac{b}{a}\right)(e^{\sqrt{a}t}-e^{-\sqrt{a}t})+\frac{1}{2}\sqrt{ax_0^2+2bx_0+c}\,(e^{\sqrt{a}t}+e^{-\sqrt{a}t})\end{bmatrix},\quad \xi_1(t)\in N\right\}.$$

(ii) A case of $a=0$:

$$M_1=\Big\{x(t):\ x(t)=\frac{1}{2}bt^2+\sqrt{2bx_0+c}\,t+x_0,\ x(t)\in N\Big\}$$

$$M_2=\left\{\xi(t):\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}bt^2+\sqrt{2bx_0+c}\,t+x_0\\ bt+\sqrt{2bx_0+c}\end{bmatrix},\quad \xi_1(t)\in N\right\}.$$

Both (i) and (ii) are satisfied with $M_2=\{[x,\sqrt{ax^2+2bx+c}]^T:\ x\in M_1\}$.

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