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ON SPACES OF FINITE PARTS OF HOMOGENEOUS FUNCTIONS

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Let P be a linear partial differential operator on R^n with smooth coefficients and of homogeneous order m. Then $P=F\Lambda^m$ where $\Lambda=\frac{1}{2\pi}\sqrt{-\Delta}$ and F is a singular integral operator [3]. Hence particular studies of such operators F have been made in their connection with the operator Λ [3], [4], [5]. The kernel of the operator F is given by $F(x, x-\xi)$: $F(\varphi)=\int F(x, x-\xi)\varphi(\xi)d\xi$ where $F(x, \xi)$ is a distribution expressed as

$$F(x, \xi)=c(x)\delta(\xi)+\underset{\xi}{\mathrm{v.p.}}\, f(x, \xi).$$

Here $c(x)$ and $f(x, \xi)$ are m times boundedly continuously differentiable functions with respect to x, $\delta(\xi)$ is the Dirac measure relative to ξ and $\underset{\xi}{\mathrm{v.p.}}$ denotes the Cauchy's principal value in ξ. The existence of $\underset{\xi}{\mathrm{v.p.}}\, f(x, \xi)$ is assured by the assumption that $f(x, \xi)$ is in $\xi\in R^n-\{0\}$ a function, infinitely continuously differentiable, homogeneous of degree $-n$ and with spherical mean 0. The space of such distributions is denoted by $\mathscr{B}^m(\Sigma)$ [11]. If $c(x)$ and $f(x, \xi)$ are actually independent of x, then $F(\varphi)$ is easily computed by taking Fourier transforms. Distributions of such a particular type constitute a nuclear space Σ of type (F). Then we may observe that $\mathscr{B}^m(\Sigma)=\mathscr{B}^m \underset{\pi}{\hat{\otimes}} \Sigma$, where $\mathscr{B}^m$ is the space of m times continuously differentiable functions on R^n with bounded derivatives.

Stimulated by these investigations, the author of the present paper has examined a generalization of the space of kernels of such singular integral operators. Let E be any quasi-complete locally convex Hausdorff space, and let $\Sigma_\lambda(E)$ be the set of functions defined on $R^n-\{0\}$ with values in E, infinitely continuously differentiable and homogeneous of degree λ, where λ is any complex number. In case that E is the complex number field, we write Σ_λ in place of $\Sigma_\lambda(E)$. Then Σ_λ is a nuclear space of type (F) and we may infer that $\Sigma_\lambda(E)=\Sigma_\lambda\, \varepsilon E$ for any E. Furthermore if in particular E is a space of type (F) we obtain $\Sigma_\lambda(E)=\Sigma_\lambda \underset{\pi}{\hat{\otimes}} E$. Hence in such a case various properties of elements of $\Sigma_\lambda(E)$ may be derived from those of elements of Σ_λ and E.

Section 1 is devoted to the study of Σ_λ and of Pf. Σ_λ, the space of finite parts of functions in Σ'_λ. We shall prove by means of the Fourier transformation that Pf.Σ'_λ is isomorphic with Pf.$\Sigma'_{-(n+\lambda)}$, if $\lambda\neq h$ and $\lambda\neq-(n+h)$ for any integer $h\geq 0$. As for the case $\lambda=h$, Pf.Σ'_h is transformed into $\Sigma'^1_{-(n+h)}\subset$ Pf.$\Sigma'_{-(n+h)}$. We shall investigate the structure of $\Sigma'^1_{-(n+h)}$ and show that it is isomorphic with Σ'_h. Thus in particular we obtain $\Sigma'^1_{-n}=\Sigma'$. In Section 2 we shall consider the space $\Sigma'_\lambda(E)$. We first prove that $\Sigma'_\lambda(E)=\Sigma'_\lambda\,\varepsilon E$. Then defining the finite part of the vector valued function in $\Sigma'_\lambda(E)$, and denoting the set consisting of them by Pf.$\Sigma'_\lambda(E)$ we may prove that Pf.$\Sigma'_\lambda(E)$ is mapped onto Pf.$\Sigma'_{-(n+\lambda)}(E)$ by the Fourier transformation provided that $\lambda\neq h$ and $\lambda\neq -(n+h)$ for any integer $h\geq 0$. The argument for the singular cases $\lambda=h$ or $\lambda=-(n+h)$ is quite similar to that in Section 1. Applying the preceding results to cases $\lambda=0$ and $\lambda=-n$ we shall study singular integral operators in Section 3. There we shall make use of the fact that $\mathscr{B}^m(\Sigma')=\Sigma'^1_{-n}\hat{\otimes}_\pi\mathscr{B}^m$. Then various problems concerning the continuity of singular integral operators in their connection with the operator Λ may be reduced to the next type: what could be said about the continuity of the operator $T_j(\varphi)=s*\left(b\dfrac{\partial\varphi}{\partial x_j}\right)-b\left(s*\dfrac{\partial\varphi}{\partial x_j}\right)$, where $s\in\Sigma'^1_{-n}$ and $b\in\mathscr{B}^m$ are prescribed elements. We shall give an answer by Proposition 7, so to speak a lemma of Friedrichs' type if we may use such an expression [6]. Thus the rest of this section may be seen as a mere application of Proposition 7.

Notations and terminologies of this paper are essentially those of [1] (locally convex spaces) and those of [12], [15] (distributions).

1. Spaces Σ'_λ and Pf.Σ'_λ. Let R^n be the real Euclidean n-space and let λ be any complex number. A distribution $T\in\mathscr{S}'$ is called *homogeneous of degree* λ if $\langle T,\varphi(r^{-1}x)\rangle=r^{n+\lambda}\langle T,\varphi\rangle$ foy any $\varphi\in\mathscr{S}$ and for any $r>0$. In case that $T=f$ is a locally integrable function, the said condition is reduced to that for any $r>0$ it holds that $f(rx)=r^\lambda f(x)$ almost everywhere on R^n. Thus a continuous function on R^n is homogeneous of degree 0 if and only if it is a constant, and an infinitely continuously differentiable function on R^n might be homogeneous of degree λ only if λ is a non-negative integer. According to [14] we write

$$\varphi^\natural(x)=\frac{1}{\omega_n}\int_{|x'|=1}\varphi(|x|x')dx'$$

for any continuous function φ, where ω_n is the surface area of the unit sphere $|x'|=1$. $\varphi^\natural$ is called the *spherical mean* of φ. Then it is not difficult to see that $\varphi\to\varphi^\natural$ is a continuous linear application of $\mathscr{D}$ into $\mathscr{D}$ [14]. Therefore its adjoint, denoted by the same symbol "$\natural$", may be defined by

$$\langle T^\natural,\varphi\rangle=\langle T,\varphi^\natural\rangle\quad\text{for}\quad T\in\mathscr{D}',\ \varphi\in\mathscr{D}.$$

Since $\varphi^{\natural}$ may be expressed as the mean of φ with respect to the rotation group on R^n, it is not difficult to see that $\widehat{T^{\natural}}=\hat{T}^{\natural}$ for any $T\in\mathcal{S}'$. Here and in the following sections of this paper we often make use of $\hat{T}$ to denote the Fourier transform $\mathcal{F}(T)(y)=\int T(x)e^{-2\pi i\langle x,y\rangle}dx$.

Let $\sum_\lambda$ be the set of functions $f(x)$ defined on $R^n-\{0\}$, infinitely continuously differentiable and homogeneous of degree λ. It is a space of type (F) provided with the subspace topology of $\mathcal{E}(R^n-\{0\})$. For any integer $h\geq 0$, we write

$$\sum\nolimits_h^{\natural 0}=\{f;\, f\in\sum\nolimits_h \text{ and } (D^p f)^{\natural}=0 \text{ for any } p,\ |p|=h\}$$

and

$$\sum\nolimits_{-(n+h)}^{\natural 0}=\{f;\, f\in\sum\nolimits_{-(n+h)} \text{ and } (x^p f)^{\natural}=0 \text{ for any } p,\ |p|=h\},$$

where $p=(p_1,\cdots,p_n)$, $|p|=p_1+\cdots+p_n$, and

$$D^p f=\frac{\partial^{|p|}}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}}f,\qquad x^p=x_1^{p_1}\cdots x_n^{p_n}.$$

For any $f\in\sum_\lambda$ the *finite part* Pf.f of f is defined in the following way [7, pp. 367–389]. Letting $\varphi\in\mathcal{D}$ and putting

$$u(r)=\int_{|x'|=1}f(x')\varphi(rx')dx'$$

we write

$$\langle \mathrm{Pf}.f,\varphi\rangle=\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty r^{n+\lambda-1}u(r)dr+\sum_{l=0}^{k}\frac{u^{(l)}(0)}{l!\,(n+\lambda+l)}\,\varepsilon^{n+\lambda+l}\right\},$$

where k is any non-negative integer such that $-1<\Re\lambda+n+k$. This expression may have no meaning if λ is a negative integer of the form $\lambda=-(n+l)$. In such a case $\dfrac{\varepsilon^{n+\lambda+l}}{n+\lambda+l}$ must be replaced by $\log\varepsilon$. Thus for simplicity's sake let us agree to think of $\dfrac{\varepsilon^0}{0}$ as $\log\varepsilon$ if such is the case. Here we remark that

$$u^{(l)}(0)=\sum_{|p|=l}\varphi^{(p)}(0)\int_{|x|=1}f(x)x^p\,dx.$$

As an immediate consequence of the definition we obtain

PROPOSITION 1. *Let $\psi\in\mathcal{E}(R^n)$ be homogeneous. Then it holds that ψPf.$f=$ Pf.ψf for any $f\in\sum_\lambda$.*

PROOF. We first note that the degree h of the homogeneity of ψ is a non-negative integer. Letting $\varphi\in\mathscr{D}$, we write

$$\langle\psi\,\mathrm{Pf}.f,\ \varphi\rangle=\langle\mathrm{Pf}.f,\ \varphi\psi\rangle=\lim_{\varepsilon\to0}\left\{\int_\varepsilon^\infty r^{n+\lambda-1}\,u(r)\,dr+\sum_{l=0}^{k}\frac{u^{(l)}(0)}{l!\,(n+\lambda+l)}\,\varepsilon^{n+\lambda+l}\right\}$$

and

$$\langle\mathrm{Pf}.\psi f,\ \varphi\rangle=\lim_{\varepsilon\to0}\left\{\int_\varepsilon^\infty r^{n+\lambda+h-1}\,v(r)dr+\sum_{l=0}^{k'}\frac{v^{(l)}(0)}{l!\,(n+\lambda+h+l)}\,\varepsilon^{n+\lambda+h+l}\right\},$$

where

$$u(r)=\int_{|x|=1}f(x)\,\psi(rx)\,\varphi(rx)dx=r^h\int_{|x|=1}f(x)\,\psi(x)\varphi(rx)dx=r^h\,v(r)\,.$$

Then since we have

$$u^{(l)}(0)=\begin{cases}0 & \text{for } 0\le l\le h-1,\\ \dfrac{l!}{(l-h)!}\,v^{(l-h)}(0) & \text{for } h\le l,\end{cases}$$

it is not difficult to see that $\langle\psi\,\mathrm{Pf}.f,\ \varphi\rangle=\langle\mathrm{Pf}.\psi f,\ \varphi\rangle$. This completes the proof.

PROPOSITION 2. *It holds that* $(\mathrm{Pf}.f)^\natural=\mathrm{Pf}.f^\natural$ *for any* $f\in\sum_\lambda$.

PROOF. Letting $\varphi\in\mathscr{D}$ and putting

$$\varphi^\natural(x)=\frac{1}{\omega_n}\int_{|x'|=1}\varphi(|x|x')\,dx'=h(|x|),\qquad \bar f=\int_{|x|=1}f(x)dx,$$

we obtain $f^\natural(x)=\dfrac{1}{\omega_n}\,|x|^\lambda\bar f$, and thus

$$\int_{|x|=1}f(x)\varphi^\natural(rx)\,dx=h(r)\bar f=\int_{|x|=1}f^\natural(x)\varphi(rx)dx.$$

Therefore it yields that

$$\langle(\mathrm{Pf}.f)^\natural,\ \varphi\rangle=\langle\mathrm{Pf}.f,\ \varphi^\natural\rangle=\langle\mathrm{Pf}.f^\natural,\ \varphi\rangle$$

for any $\varphi\in\mathscr{D}$. This completes the proof.

It is not difficult to see that $\mathrm{Pf}.f$ is a distribution $\in\mathscr{S}'$ and in case that $\lambda\neq-(n+h)$, $h=0, 1, 2,\cdots$, $\mathrm{Pf}.f$ is homogeneous of degree λ. Although $\mathrm{Pf}.f=f$ for $f\in\sum_\lambda$ if $\Re\lambda>-n$, yet in general the set of distributions $\mathrm{Pf}.f$, $f\in\sum_\lambda$, should be denoted by $\mathrm{Pf}.\sum_\lambda$. The topology of $\mathrm{Pf}.\sum_\lambda$ is given by that of $\sum_\lambda$. For the case $\lambda=-(n+h)$, $h=0, 1, 2,\cdots$, we obtain

PROPOSITION 3. *Let* $f\in\sum'_{-(n+h)}$. *Then* Pf.f *is homogeneous of degree* $-(n+h)$ *if and only if* $f\in\sum'^{0}_{-(n+h)}$.

PROOF. Letting $\varphi\in\mathscr{D}$ and letting

$$u(r)=\int_{|x|=1} f(x)\varphi(rx)\,dx$$

we obtain

$$\begin{aligned}\langle \mathrm{Pf}.f,\ \varphi(t^{-1}x)\rangle &=\lim_{\varepsilon\to0}\left\{\int_{\varepsilon}^{\infty} r^{-h-1}\,u(t^{-1}r)\,dr+\sum_{l=0}^{h-1}\frac{t^{-l}u^{(l)}(0)}{l!(-h+l)}\varepsilon^{-h+l}+\frac{t^{-h}u^{(h)}(0)}{h!}\log\varepsilon\right\}\\ &=t^{-h}\lim_{\varepsilon\to0}\left\{\int_{t^{-1}\varepsilon}^{\infty} r^{-h-1}u(r)\,dr+\sum_{l=0}^{h-1}\frac{u^{(l)}(0)}{l!(-h+l)}(t^{-1}\varepsilon)^{-h+l}+\frac{u^{(h)}(0)}{h!}\log(t^{-1}\varepsilon)\right\}\\ &\quad+t^{-h}\frac{u^{(h)}(0)}{h!}\log t\\ &=t^{-h}\left\{\langle \mathrm{Pf}.f,\ \varphi\rangle+\frac{u^{(h)}(0)}{h!}\log t\right\}\end{aligned}$$

Thus Pf.f is homogeneous of degree $-(n+h)$ if and only if $u^{(h)}(0)=0$ for any $\varphi\in\mathscr{D}$. This means $\int_{|x|=1} f(x)x^p\,dx=0$ for any p, $|p|=h$, that is $(fx^p)'=0$ for any p, $|p|=h$. This completes the proof.

We now prove the next lemma for later use.

LEMMA 1. *Let* $T\in\mathscr{S}'$ *be a function on* $R^n-\{0\}$, *infinitely continuously differentiable and homogeneous of degree* λ. *Then its Fourier transform* $\hat{T}$ *is also a function on* $R^n-\{0\}$, *infinitely continuously differentiable and homogeneous of degree* $-(n+\lambda)$.

PROOF. For any $\varphi\in\mathscr{D}$ we obtain

$$\langle\hat{T},\ \varphi(r^{-1}y)\rangle=\langle T,\ r^n\hat{\varphi}(rx)\rangle=r^{-\lambda}\langle T,\ \hat{\varphi}\rangle.$$

Therefore $\hat{T}$ is homogeneous of degree $-(n+\lambda)$. To see that $\hat{T}$ is infinitely continuously differentiable on $R^n-\{0\}$ it is enough to show that $\Delta^k\hat{T}$ is a continuous function on $R^n-\{0\}$ for any k [12, II, p. 47]. For this purpose let $h\in\mathscr{D}$ be

$$h(x)=\begin{cases}1 & \text{for } |x|\leq1,\\ 0 & \text{for } |x|\geq2,\end{cases}$$

and let l be any integer $\geq\frac{1}{2}(2k+1+n+\Re\lambda)$. Then by the theorem of Paley-Wiener we get

$$\mathscr{F}(\Delta^l|x|^{2k}hT)\in\mathscr{E}.$$

On the other hand we get $\Delta^l |x|^{2k}(1-h)\, T \in L^1$, since there exists a positive constant c such that

$$|\Delta^l |x|^{2k}(1-h)\, T(x)| \leq c|x|^{-2l+2k+\Re\lambda} \quad \text{for} \quad |x| \geq 2.$$

Thus $\mathscr{F}(\Delta^l |x|^{2k}(1-h)T)$ are continuous function. Therefore $\mathscr{F}(\Delta^l |x|^{2k}\, T)$ and hence $|y|^{2l} \Delta^k \hat{T}$ are continuous functions on R^n. This completes the proof.

We now prove the following theorem.

THEOREM 1. *Let* λ *be a complex number such that* $\lambda \neq h$ *and* $\lambda \neq -(n+h)$ *for any integer* $h \geq 0$. *Then* $\mathscr{F}$ *is a topological isomorphism of* Pf.$\sum_\lambda$ *onto* Pf.$\sum_{-(n+\lambda)}$.

PROOF. It is enough to consider the case $\Re\lambda > -n$, because if $\Re\lambda \leq -n$ then the problem is reduced to the case $\Re(-(n+\lambda)) \geq 0$ concerning the inverse $\mathscr{F}^{-1}$. We first prove that $\hat{f} \in \text{Pf.}\sum_{-(n+\lambda)}$ for any $f \in \sum_\lambda$. It is known by Lemma 1 that $\hat{f}$ is identical with an element $g \in \sum_{-(n+\lambda)}$ on $R^n - \{0\}$. Therefore $\hat{f}$ and Pf.g can differ only at the origin of R^n. Hence $\hat{f} - \text{Pf.}g = \sum c_p D^p \delta$ and thus we have $f - \mathscr{F}^{-1}(\text{Pf.}g) = \sum c_p(-2\pi i y)^p$. The left side of this identity is a homogeneous distribution of degree λ. Therefore $c_p = 0$ except p with $|p| = \lambda$, and the latter is excluded by the assumption $\lambda \neq h$. Hence we get $\hat{f} = \text{Pf.}g$ as desired. Quite similarly we may prove that for any $g \in \sum_{-(n+\lambda)}$ there exists an $f \in \sum_\lambda$ such that $\hat{f} = \text{Pf.}g$. Thus we see that $\mathscr{F}$ is a linear isomorphism of $\sum_\lambda$ onto Pf.$\sum_{-(n+\lambda)}$. To see that this is topological we need only to prove that it is continuous because $\sum_\lambda$ and Pf.$\sum_{-(n+\lambda)}$ are both spaces of type (F). The closed graph theorem [9, Chap. I, p. 16] enables us to reduce the problem to show that if $f_k \to 0$ in $\sum_\lambda$ and $\hat{f}_k \to \text{Pf.}g$ in Pf.$\sum_{-(n+\lambda)}$, then $g=0$. This is shown as follows. Since it holds that $f_k \to 0$ and $\hat{f}_k \to \text{Pf.}g$ in $\mathscr{S}'$, we may obtain for any $\varphi \in \mathscr{S}$

$$\langle \text{Pf.}g, \varphi \rangle = \lim_{k\to\infty} \langle \hat{f}_k, \varphi \rangle = \lim_{k\to\infty} \langle f_k, \hat{\varphi} \rangle = 0,$$

that is, Pf.$g=0$. This completes the proof.

For the case of singular values of λ: $\lambda = h$ or $\lambda = -(n+h)$, we first prove the next theorem.

THEOREM 2. *Let* h *be any non-negative integer. Then* $\mathscr{F}$ *is a topological isomorphism of* $\sum_h^0$ *onto* Pf.$\sum_{-(n+h)}^0$.

PROOF. We begin the proof by showing $\hat{f} \in \text{Pf.}\sum_{-(n+h)}^0$ for any $f \in \sum_h^0$. It is known by Lemma 1 that $\hat{f} = g \in \sum_{-(n+h)}$ on $R^n - \{0\}$. Then since $\hat{f}$ and Pf.g can differ only at the origin of R^n, we may write $\hat{f} - \text{Pf.}g = \sum c_p D^p \delta$, and thus

$$\langle \hat{f}, \varphi(r^{-1}y) \rangle - \langle \text{Pf.}g, \varphi(r^{-1}y) \rangle = \sum c_p \langle D^p \delta, \varphi(r^{-1}y) \rangle$$

for any $\varphi\in\mathscr{D}$. Using an identity obtained in the proof of Proposition 3 we get

$$\langle\hat{f},\varphi\rangle-\langle\mathrm{Pf.}g,\varphi\rangle-\frac{u^{(h)}(0)}{h!}\log r=\sum c_p(-1)^{|p|}r^{h-|p|}\varphi^{(p)}(0)$$

for any $\varphi\in\mathscr{D}$, where $u(r)=\int_{|x|=1}g(x)\varphi(rx)dx$. It now follows first that $c_p=0$ for any p, $|p|\neq h$, and next that $u^{(h)}(0)=0$ for any $\varphi\in\mathscr{D}$. This means $\int_{|x|=1}g(x)x^p\,dx=0$ for any p, $|p|=h$. Consequently we get $g\in\sum^0_{-(n+h)}$ and $\hat{f}=\mathrm{Pf.}g+\sum_{|p|=h}c_p D^p\delta$. Hence we may write

$$f=\mathscr{F}^{-1}(\mathrm{Pf.}g)+\sum_{|p|=h}c_p(-2\pi i)^h x^p,$$

and thus for any p, $|p|=h$, we see

$$D^p f=\mathscr{F}^{-1}((2\pi i)^h y^p\,\mathrm{Pf.}g)+c_p(-2\pi i)^h p!,$$

where $p!=p_1!\cdots p_n!$. Therefore it follows that

$$(D^p f)^\natural=(2\pi i)^h\mathscr{F}^{-1}(\mathrm{Pf.}(y^p g)^\natural)+c_p(-2\pi i)^h p!,$$

since $\natural$ is commutative with $\mathscr{F}^{-1}$ and Pf. is commutative with y^p and $\natural$. Thus we may conclude that $c_p=0$, because $(D^p f)^\natural=0$ and $(y^p g)^\natural=0$. This proves $\hat{f}=\mathrm{Pf.}g$, $g\in\sum^0_{-(n+h)}$. Let conversely $g\in\sum^0_{-(n+h)}$. Then it follows that $f=\mathscr{F}^{-1}(\mathrm{Pf.}g)\in\sum_h$ by Lemma 1. Thus to get $f\in\sum^0_h$ it is enough to see that $(D^p f)^\natural=(2\pi i)^h\mathscr{F}^{-1}(\mathrm{Pf.}(y^p g)^\natural)$ for any p, $|p|=h$. Hence $\mathscr{F}$ is a linear isomorphism of $\sum^0_h$ onto $\mathrm{Pf.}\sum^0_{-(n+h)}$. That $\mathscr{F}$ is topological may be seen just in the same way as in the proof of Theorem 1. This completes the proof.

In order to develop further arguments concerning the spaces $\sum_h$ and $\mathrm{Pf.}\sum_{-(n+h)}$, we shall make use of the following notations. Let Π_h be the set of homogeneous polynomials of degree h: $\sum_{|p|=h}c_p x^p$ and let $\Pi_{-(n+h)}$ be the set of distributions of the form $\sum_{|p|=h}c_p D^p\delta$. Then Π_h, $\Pi_{-(n+h)}$ may be considered as complex Euclidean $\frac{(n+h-1)!}{h!(n-1)!}$-spaces, and $\Pi_{-(n+h)}$ is the Fourier transform of Π_h. Here we should like to show that the set of distributions $\mathrm{Pf.}\sum^0_{-(n+h)}$ and $\Pi_{-(n+h)}$ have no element in common except 0. In fact let $\mathrm{Pf.}f=\sum_{|p|=h}c_p D^p\delta$ for $f\in\sum^0_{-(n+h)}$. Then for any $\varphi\in\mathscr{D}$ such that $\varphi(x)\equiv0$ on a neighbourhood of the origin $|x|<\delta$, we obtain

$$\int_{\delta\le|x|}f(x)\varphi(x)dx=\langle\mathrm{Pf.}f,\varphi\rangle=\langle\sum_{|p|=h}c_p D^p\delta,\varphi\rangle=0.$$

Hence it follows that $f(x)=0$ for $|x|>\delta$, and since $\delta>0$ is arbitrary it yields that $\mathrm{Pf}.f=0$ as desired. Therefore we may consider that $\mathrm{Pf}.\sum^{0}_{-(n+h)}+\Pi_{-(n+h)}$ is a topological direct sum which we shall denote by $\sum^{1}_{-(n+h)}$. We now state the next

THEOREM 3. *$\mathscr{F}$ is a topological isomorphism of $\sum_h$ onto $\sum^{1}_{-(n+h)}$.*

PROOF. For any $f\in\sum_h$ we obtain $\hat{f}=g\in\sum_{-(n+h)}$ on $R^n-\{0\}$. Then just in the same manner as in the proof of Theorem 2 we may conclude that $g\in\sum^{0}_{-(n+h)}$ and $\hat{f}-\mathrm{Pf}.g=\sum_{|p|=h} c_p D^p\delta$, that is $\hat{f}\in\sum^{1}_{-(n+h)}$. Let conversely $T=\mathrm{Pf}.g+\sum_{|p|=h} c_p D^p\delta$ with $g\in\sum^{0}_{-(n+h)}$. Then we get

$$\mathscr{F}^{-1}(T)=\mathscr{F}^{-1}(\mathrm{Pf}.g)+\sum_{|p|=h} c_p(-2\pi ix)^p\in\textstyle\sum^{0}_{h}+\Pi_h\subset\sum_h\,,$$

that is $T\in\mathscr{F}(\sum_h)$. Therefore we have proved that $\mathscr{F}(\sum_h)=\sum^{1}_{-(n+h)}$. By means of Theorem 2 it is not difficult to see that $\mathscr{F}^{-1}$ is a continuous application of $\sum^{1}_{-(n+h)}$ onto $\sum_h$. Hence $\mathscr{F}$ is a topological isomorphism of $\sum_h$ onto $\sum^{1}_{-(n+h)}$. This completes the proof.

COROLLARY. *It holds that $\sum_h=\sum^{0}_{h}+\Pi_h$ in the sense of topological direct sum.*

PROOF. It is sufficient to remark that $\mathrm{Pf}.\sum^{0}_{-(n+h)}+\Pi_{-(n+h)}$ is a topological direct sum. This completes the proof.

2. Spaces $\sum_\lambda(E)$ and $\mathrm{Pf}.\sum_\lambda(E)$. Let E be any quasi-complete locally convex Hausdorff space. A vector valued distribution $\vec{T}$ on R^n with values in E: $\vec{T}\in\mathscr{D}'(E)$, is called homogeneous of degree λ if the scalar valued distribution $\langle\vec{T},\vec{e}'\rangle$ is homogeneous of degree λ for any $\vec{e}'\in E'$. Let $\sum_\lambda(E)$ be the set of vector valued functions $\vec{f}$ defined on $R^n-\{0\}$ with values in E, infinitely continuously differentiable and homogeneous of degree λ. It is a locally convex Hausdorff space provided with the subspace topology of $\mathscr{E}(R^n-\{0\};E)$ [13]. We first show that

PROPOSITION 4. $\sum_\lambda(E)=\sum_\lambda\varepsilon E$.

PROOF. Since $\sum_\lambda$ is a closed subspace of $\mathscr{E}(R^n-\{0\})$, it is a complete Montel space. Therefore its strong dual $\sum'_\lambda$ coincides with $(\sum'_\lambda)_c$ and hence we have $\sum_\lambda\varepsilon E=\mathscr{L}_\varepsilon(\sum'_\lambda;E)$. Thus the statement is proved if we show that $\sum_\lambda(E)=\mathscr{L}_\varepsilon(\sum'_\lambda;E)$. Let i be the injection of $\sum_\lambda$ into $\mathscr{E}(R^n-\{0\}$. Then its adjoint ${}^t i$ is a continuous linear application of $\mathscr{E}'(R^n-\{0\})$ onto $\sum'_\lambda$. We first prove that any $\vec{f}\in\mathscr{L}(\sum'_\lambda;E)$ may be considered as an element of $\sum_\lambda(E)$. For this purpose we put

$$\vec{f}(x)=\vec{f}({}^t i(\delta_{(x)})),\ x\in R^n-\{0\},$$

where $\delta_{(x)}=\tau_x\,\delta$ and $\vec{f}(\cdot)$ on the right side means the image in E by the linear application $\vec{f}$. Since it holds that

$$\langle {}^t i\,\delta_{(rx)}, f\rangle=f(rx)=r^\lambda f(x)=r^\lambda\langle {}^t i\,\delta_{(x)}, f\rangle$$

for any $f\in\sum_\lambda$ and $r>0$, we have ${}^t i\,\delta_{(rx)}=r^{\lambda\,t} i\,\delta_{(x)}$. Thus $\vec{f}(rx)=r^\lambda\vec{f}(x)$, that is, $\vec{f}(x)$ is a homogeneous function of degree λ defined on $R^n-\{0\}$ with values in E. To prove that $\vec{f}(x)$ is infinitely differentiable it is enough to see

$$\vec{f}({}^t i(D^p\,\delta_{(x)}))=(-1)^{|p|}\,\vec{f}({}^t i\,D^p_x\,\delta_{(x)})=(-1)^{|p|}\,D^p_x\,\vec{f}({}^t i\,\delta_{(x)})=(-1)^{|p|}\,D^p_x\,\vec{f}(x)$$

for any p. Since the mapping $R^n-\{0\}\ni x\to D^p\,\delta_{(x)}\in\mathcal{E}'(R^n-\{0\})$ is continuous, the derivatives of $\vec{f}(x)$ are all continuous. Thus we get $\vec{f}\in\sum_\lambda(E)$. Next we prove that any $\vec{f}\in\sum_\lambda(E)$ may be considered as an element of $\mathcal{L}(\sum'_\lambda; E)$. To this end let S be any element of $\mathcal{E}'(R^n-\{0\})$ and define a linear form $\langle\vec{f}, S\rangle$ on E' by the following identity:

$$\langle\langle\vec{f}, S\rangle, \vec{e}'\rangle=\langle i\langle\vec{f}, \vec{e}'\rangle, S\rangle\quad\text{for } \vec{e}'\in E'.$$

Then it is known that $\langle\vec{f}, S\rangle\in E$ because E is quasi-complete [13]. Let now $T\in\sum'_\lambda$ and observe that there exists an $S\in\mathcal{E}'(R^n-\{0\})$ such that $T={}^t i\,S$. Putting

$$\vec{f}(T)=\langle\vec{f}, S\rangle\in E$$

we must show that $\langle\vec{f}, S\rangle=0$ for any $S\in\mathcal{E}'(R^n-\{0\})$ with ${}^t i\,S=0$. In fact for any $\vec{e}'\in E'$ it holds that

$$\langle\langle\vec{f}, S\rangle, \vec{e}'\rangle=\langle i\langle\vec{f}, \vec{e}'\rangle, S\rangle=\langle\langle\vec{f}, \vec{e}'\rangle, {}^t i\,S\rangle=0.$$

Hence we have $\langle\vec{f}, S\rangle=0$ as desired. Therefore $\vec{f}(T)$ is uniquely determined. To prove the continuity of $\vec{f}(T)$ we proceed as follows. We first note that $\mathcal{E}(R^n-\{0\})$ and its closed subspace $\sum_\lambda$ are Schwartz (F) space [8]. Therefore $\mathcal{E}'(R^n-\{0\})$ and $\sum'_\lambda$ are Silva spaces and thus the continuous linear application ${}^t i$ of $\mathcal{E}'(R^n-\{0\})$ onto $\sum'_\lambda$ is a topological homomorphism [16]. Hence it follows easily that $\vec{f}(T)=\langle\vec{f}, S\rangle$ is continuous and thus $\vec{f}\in\mathcal{L}(\sum'_\lambda; E)$. We may now say that $\sum_\lambda(E)$ and $\mathcal{L}(\sum'_\lambda; E)$ are the same vector space. To see that they have also the same topology, we consider the fundamental systems of neighourhoods of the origin in respective spaces. Let

$$V=V(C, m, n)=\{f;\, f\in\textstyle\sum_\lambda,\ |D^p f(C)|\le\dfrac{1}{n}\ \text{for any } p,\ |p|\le m\}$$

where C is any compact subset of $R^n-\{0\}$ and $m\ge 0$, $n>0$ are any integers.

Then by definition $\{V(C, m, n)\}$ is a fundamental system of neighbourhoods of the origin in $\sum_\lambda'$. Taking any convex circled neighbourhood U of the origin in E and putting

$$W = W(U, V) = \{\vec{f};\ \vec{f} \in \mathscr{L}(\textstyle\sum_\lambda'; E),\ \vec{f}(U^\circ) \subset V\},$$

we obtain a fundamental system $\{W(U, V)\}$ of neighbourhoods of the origin in $\mathscr{L}_\varepsilon(\sum_\lambda'; E)$. By writing

$$\begin{aligned} W &= \{\vec{f};\ \vec{f} \in \mathscr{L}_\varepsilon(\textstyle\sum_\lambda'; E),\ |D^p \vec{f}(U^\circ)(C)| \leq \dfrac{1}{n} \text{ for any } p,\ |p| \leq m\} \\ &= \{\vec{f};\ \vec{f} \in \textstyle\sum_\lambda(E),\ D^p \vec{f}(C) \subset \dfrac{1}{n} U \text{ for any } p,\ |p| \leq m\}, \end{aligned}$$

we infer that W is a neighbourhood of the origin in $\sum_\lambda(E)$. Furthermore this identity shows us that any neighbourhood of the origin in $\sum_\lambda(E)$ contains a member of $\{W(U, V)\}$ because the last side of the identity written above forms a fundamental system of neighbourhoods of the origin in $\sum_\lambda(E)$. This completes the proof.

We now define for any $\vec{f} \in \sum_\lambda(E)$ a vector valued distribution Pf.$\vec{f} \in \mathscr{D}'(E)$ in the following way:

$$\langle \text{Pf.}\vec{f}, \varphi \rangle = \lim_{\varepsilon \to 0} \left\{ \int_\varepsilon^\infty r^{n+\lambda-1} \vec{u}(r) dr + \sum_{l=0}^{k} \frac{\vec{u}^{(l)}(0)}{l!(n+\lambda+l)} \varepsilon^{n+\lambda+l} \right\}$$

for any $\varphi \in \mathscr{D}$, where k is any non-negative integer such that $\Re\lambda + n + k > -1$ and $\vec{u}(x) = \int_{|x|=1} \vec{f}(x)\varphi(rx)dx$. As in Section 1, this expression may have no meaning when λ is a negative integer of the form $\lambda = -(n+l)$, and in such a case $\log \varepsilon$ must be written in place of $\dfrac{\varepsilon^{n+\lambda+l}}{n+\lambda+l}$. The set of distributions Pf.$\vec{f}$, $\vec{f} \in \sum_\lambda(E)$, is denoted by Pf.$\sum_\lambda(E)$ provided with the topology of $\sum_\lambda(E)$. Then by means of the following sequence of isomorphisms:

$$\text{Pf.}\textstyle\sum_\lambda(E) \to \sum_\lambda(E) \to \sum_\lambda \varepsilon E \to (\text{Pf.}\sum_\lambda)\varepsilon E = \mathscr{L}_\varepsilon(E_c'; \text{Pf.}\sum_\lambda),$$

it holds that $\langle \text{Pf.}\vec{f}, \vec{e}' \rangle = \text{Pf.}\langle \vec{f}, \vec{e}' \rangle \in \text{Pf.}\sum_\lambda$ for any $\vec{e}' \in E'$. Keeping this in mind we may write Pf.$\sum_\lambda(E) = (\text{Pf.}\sum_\lambda)\varepsilon E$. In case of $\lambda = h$ or $\lambda = -(n+h)$ where h is any integer ≥ 0 we have defined $\sum_\lambda^0$ in Section 1. In just the same way we put

$$\textstyle\sum_h^0(E) = \{\vec{f};\ \vec{f} \in \sum_h(E) \text{ and } (D^p \vec{f})^\natural = 0 \text{ for any } p,\ |p| = h\}$$

and

$$\textstyle\sum_{-(n+h)}^0(E) = \{\vec{f};\ \vec{f} \in \sum_{-(n+h)}(E) \text{ and } (x^p \vec{f})^\natural = 0 \text{ for any } p,\ |p| = h\},$$

where $\vec{\varphi}^{\natural}$ denotes as before the spherical mean of $\vec{\varphi}$: $\vec{\varphi}^{\natural}(x)=\frac{1}{\omega_n}\int_{|x'|=1}\vec{\varphi}(|x|x')dx'$.

We now prove the next

Proposition 5. $\sum_h^{\cdot 0}(E)=\sum_h^{\cdot 0}\varepsilon E$, $\quad \sum_{-(n+h)}^{\cdot 0}(E)=\sum_{-(n+h)}^{\cdot 0}\varepsilon E$.

Proof. It is enough to prove $\sum_h^{\cdot 0}(E)=\mathscr{L}_\varepsilon(\sum_h^{\cdot 0\prime}; E)$ and $\sum_{-(n+h)}^{\cdot 0}(E)=\mathscr{L}_\varepsilon(\sum_{-(n+h)}^{\cdot 0\prime}; E)$.

Ad $\sum_h^{\cdot 0}(E)=\mathscr{L}_\varepsilon(\sum_h^{\cdot 0\prime}; E)$: Let j be the injection $\sum_h^{\cdot 0}\to\mathscr{E}(R^n-\{0\})$ and make use of the notations in the proof of Proposition 4. We first prove that any $\vec{f}\in\mathscr{L}(\sum_h^{\cdot 0\prime}; E)$ may be considered as an element of $\sum_h^{\cdot 0}(E)$. To this end an E-valued function $\vec{f}(x)$ is defined by the next identity:

$$\vec{f}(x)=\vec{f}({}^t j(\delta_{(x)})),\quad x\in R_n-\{0\}.$$

Then quite similarly as before we infer that $\vec{f}\in\sum_h(E)$. To prove $(D^p\vec{f})^{\natural}=0$, we first observe that $D^p\vec{f}(x)=(-1)^{|p|}\vec{f}({}^t j\,D^p\,\delta_{(x)})$ and next that for any $f\in\sum_h^{\cdot 0}$ we obtain

$$\int_{|x|=1}\langle {}^t j\,D^p\,\delta_{(x)}, f\rangle\,dx=(-1)^{|p|}\int_{|x|=1}\langle\delta_{(x)}, D^p f\rangle\,dx=(-1)^{|p|}\int_{|x|=1}D^p f(x)dx=0.$$

Then it follows that $\int_{|x|=1}{}^t j\,D^p\,\delta_{(x)}\,dx=0$ and hence $(D^p\vec{f})^{\natural}=0$ as desired. We next show that any $\vec{f}\in\sum_h^{\cdot 0}(E)$ may be considered as an element of $\mathscr{L}(\sum_h^{\cdot 0\prime}; E)$. The proof is sketched as follows. For any $S\in\mathscr{E}'(R^n-\{0\})$ we define a linear form on E' by the identity:

$$\langle\langle\vec{f}, S\rangle, \vec{e}'\rangle=\langle j\langle\vec{f}, \vec{e}'\rangle, S\rangle\quad\text{for } \vec{e}'\in E'.$$

Then it may be seen that $\langle\vec{f}, S\rangle\in E$. Hence putting for any $T\in\sum_h^{\cdot 0\prime}$

$$\vec{f}(T)=\langle\vec{f}, S\rangle\in E$$

where $T={}^t j\,S$ with $S\in\mathscr{E}'(R^n-\{0\})$, we obtain a linear application of $\sum_h^{\cdot 0\prime}$ into E. By observing that ${}^t j$ is a topological homomorphism of $\mathscr{E}'(R^n-\{0\})$ onto $\sum_h^{\cdot 0\prime}$ it is not difficult to see the continuity of $\vec{f}(T)$. This proves $\vec{f}\in\mathscr{L}(\sum_h^{\cdot 0\prime}; E)$ and we say that $\sum_h^{\cdot 0}(E)$ and $\mathscr{L}_\varepsilon(\sum_h^{\cdot 0\prime}; E)$ are the same vector space. Finally we may prove that they have also the same topology just in the same way as in the proof of Proposition 4 and details are omitted.

Ad $\sum_{-(n+h)}^{\cdot 0}(E)=\mathscr{L}_\varepsilon(\sum_{-(n+h)}^{\cdot 0\prime}; E)$: Let j' be the injection $\sum_{-(n+h)}^{\cdot 0}\to\mathscr{E}(R^n-\{0\})$. Then by putting

$$\vec{f}(x)=\vec{f}({}^t j'(\delta_{(x)}),\quad x\in R^n-\{0\}$$

for any $\vec{f}\in\mathscr{L}(\sum_{-(n+h)}^{\cdot 0\prime}; E)$, we must prove that the function $\vec{f}$ belongs to

$\sum'^{0}_{-(n+h)}(E)$. Except that $(x^p \vec{f})^t=0$, the proof goes along the same line as in the previous case. That $(x^p \vec{f})^t=0$ is seen as follows. For any $f\in\sum'^{0}_{-(n+h)}$ it holds that

$$\int_{|x|=1} \langle {}^t j' x^p \delta_{(x)}, f\rangle\, dx=\int_{|x|=1} \langle x^p \delta_{(x)}, f\rangle\, dx=\int_{|x|=1} x^p f(x)\, dx=0$$

Therefore $\int_{|x|=1} {}^t j' x^p \delta_{(x)}\, dx=0$ and hence $(x^p \vec{f})^t=0$. This completes the proof.

If the set of distributions Pf.$\vec{f}$, $\vec{f}\in\sum'^{0}_{-(n+h)}(E)$, is denoted by Pf.$\sum'^{0}_{-(n+h)}(E)$, then we may write Pf.$\sum'^{0}_{-(n+h)}(E)=(\text{Pf.}\sum'^{0}_{-(n+h)})\,\varepsilon E$. Let $\Pi_h(E)$ be the set of homogeneous polynomials of degree h with coefficients in E: $\sum_{|p|=h} x^p \vec{e}_p$, $\vec{e}_p\in E$, and let $\Pi_{-(n+h)}(E)$ be the set of linear combinations of the form $\sum_{|p|=h} D^p\delta\otimes\vec{e}_p$. Then $\Pi_h(E)=\Pi_h\,\varepsilon E$ and $\Pi_{-(n+h)}(E)=\Pi_{-(n+h)}\,\varepsilon E$. Hence in the sense of the topological direct sum we obtain $\sum'_h(E)=\sum'^{0}_h(E)+\Pi_h(E)$. Let us denote the topological direct sum Pf.$\sum'^{0}_{-(n+h)}(E)+\Pi_{-(n+h)}(E)$ by $\sum'^{1}_{-(n+h)}(E)$. Then we obtain $\sum'^{1}_{-(n+h)}(E)=\sum'^{1}_{-(n+h)}\,\varepsilon E$. Since $\sum'_\lambda$, $\sum'^{0}_h$, $\sum'^{0}_{-(n+h)}$ are all defined to be closed subspaces of the complete nuclear space $\mathscr{E}(R^n-\{0\})$, we may infer that they are nuclear complete spaces of type (F) [9, Chap. 2, p. 47], and thus they have also the property of approximation [9, Chap. 1, p. 165]. Therefore if E is a space of type (F) we may write that $\sum'_\lambda(E)=\sum'_\lambda\hat{\otimes}_\pi E$, $\sum'_h(E)=\sum'_h\hat{\otimes}_\pi E,\cdots$ [15]. Thus, for example, any element $\vec{f}\in\sum'_\lambda(E)$ is expressed as

$$\vec{f}=\sum_{k=1}^{\infty}\alpha_k f_k \vec{e}_k,$$

where $\sum_{k=1}^{\infty}|\alpha_k|<\infty$, $f_k\to0$ in $\sum'_\lambda$, $\vec{e}_k\to0$ in E [9, Chap. 1, p. 51]. The same holds also for Pf.$\sum'_\lambda(E)$, $\sum'^{1}_{-(n+h)}(E),\cdots$.

It is now a simple matter to define the Fourier transformation on Pf.$\sum'_\lambda(E)$. Let I be the identical application on E. Then the application $\mathscr{F}\varepsilon I$ is called the *Fourier transformation.* It is defined on Pf.$\sum'_\lambda(E)$, $\sum'^{1}_{-(n+h)}(E),\cdots$, because Pf.$\sum'_\lambda(E)=(\text{Pf.}\sum'_\lambda)\,\varepsilon E$, $\sum'^{1}_{-(n+h)}(E)=\sum'^{1}_{-(n+h)}\,\varepsilon E$, $\cdots$. By virtue of the preceding results we obtain

PROPOSITION 6. *The Fourier transformation defines the topological isomorphisms between*

1) Pf.$\sum'_\lambda(E)$ *and* Pf.$\sum'_{-(n+\lambda)}(E)$ (λ *is any complex number other than* $0, 1, 2,\cdots, -n, -(n+1), -(n+2),\cdots$),

2) $\sum'^{0}_h(E)$ *and* Pf.$\sum'^{0}_{-(n+h)}(E)$ ($h=0, 1, 2, \cdots$),

3) $\sum'_h(E)$ *and* $\sum'^{1}_{-(n+h)}(E)$ ($h=0, 1, 2, \cdots$).

PROOF. Evident.

3. Singular integral operators. In the theory of singular integral operators studied by A.P. Calderón and A. Zygmund particular róles are played by the spaces $\sum'_0$ and $\sum'^1_{-n}$ [3], [4], [5], [11]. Therefore in the rest of this paper we shall restrict our attention to $\sum'_0$, $\sum'^1_{-n}$ and investigate the same subject matter as the one discussed in [3], [5] in view of an application of our preceding results.

Let H^m, $m=0, 1, 2, \cdots$, be the space of distributions f on R^n such that $D^p f \in L^2$ for any p, $|p| \leq m$. It is known that H^m is a Hilbert space provided with the scalar product and the norm

$$(f, g)_m = \sum_{|p| \leq m} (D^p f, D^p g)_{L^2}, \ \|f\|_m = (f, f)_m^{\frac{1}{2}}$$

respectively. In the first place let us consider the convolution of $s \in \sum'^1_{-n}$ by $h \in L^2$. Since $s \in \mathscr{D}'_{L^2}$ and $h \in \mathscr{D}'_{L^2}$, we may conclude that $\mathscr{S}'$-convolution $s \circledast h = s * h$ is defined [10]. Then by virtue of the exchange formula we get $(s*h)^{\wedge} = \hat{s} \cdot \hat{h}$ [10]. Therefore $s*h \in L^2$ and $\|s*h\|_{L^2} \leq \|\hat{s}\|_\infty \|h\|_{L^2}$. In case that $h \in H^m$, we may infer that $s*h \in H^m$ and $\|s*h\|_m \leq \|\hat{s}\|_\infty \|h\|_m$, an immediate consequence of the fact that $D^p(s*h) = s*D^p h$ for any p.

We now remark that for any $s \in \mathrm{Pf}.\sum^{0}_{-n}$ it holds that

$$s*h(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} h(x-y) s(y) dy$$

in the sense of convergence in L^2. It is already known that the right hand side of this identity really exists in the sense of convergence in L^2 as well as of convergence almost everywhere [2]. Therefore the statement is nearly evident. But here we shall prove it by using the Fourier transformation. Let for any a, b, $0 < a < b \leq \infty$

$$s_{a,b}(x) = \begin{cases} s(x) & \text{for } a \leq |x| < b, \\ 0 & \text{otherwise.} \end{cases}$$

We first consider the case $b < \infty$. Then the Fourier transform of $s_{a,b}$ may be written as

$$\hat{s}_{a,b}(\xi) = \int_{|x'|=1} s(x') \, I(\langle x', \xi' \rangle; a, b, \rho) \, dx'$$

where $\xi = \rho \xi'$, $\rho \geq 0$, $|\xi'| = 1$ and we put

$$I(\alpha) = I(\alpha; a, b, \rho) = \int_{2\pi\rho a}^{2\pi\rho b} \frac{e^{-it\alpha} - e^{-it}}{t} \, dt.$$

Since $I'(\alpha) = \frac{1}{\alpha} (e^{-2\pi i\rho b\alpha} - e^{-2\pi i\rho a\alpha})$, we obtain

$$I(\alpha)=\begin{cases}\displaystyle\int_1^\alpha \frac{e^{-2\pi i\rho b\beta}-e^{-2\pi i\rho a\beta}}{\beta}\,d\beta & \text{for } \alpha>0,\\ \displaystyle I(-1)+\int_{-1}^\alpha \frac{e^{-2\pi i\rho b\beta}-e^{-2\pi i\rho a\beta}}{\beta}\,d\beta & \text{for } \alpha<0.\end{cases}$$

Therefore we get

$$|I(\alpha)|\leq\begin{cases}-2\log\alpha & \text{for } \alpha>0,\\ c-2\log|\alpha| & \text{for } \alpha<0\end{cases}$$

where

$$c=2\sup_{0<a<b,\ 0<\rho}\left|\int_{2\pi\rho a}^{2\pi\rho b}\frac{\sin t}{t}\,dt\right|.$$

We now pass to the case $b=\infty$. By Lebesgue's convergence theorem we may infer that

$$\lim_{b\to\infty}\hat{s}_{a,b}(\xi)=\int_{|x'|=1}s(x')\,I(\langle x',\xi'\rangle;a,\infty,\rho)\,dx'$$

for any ξ, and hence it holds that

$$\hat{s}_{a,\infty}(\xi)=\int_{|x'|=1}s(x')\,I(\langle x',\xi'\rangle;a,\infty,\rho)\,dx',$$

because $\hat{s}_{a,b}\to\hat{s}_{a,\infty}(b\to\infty)$ in L^2. Consequently it holds that for any $0<a<b\leq\infty$

$$|\hat{s}_{a,b}(\xi)|\leq\int_{|x'|=1}|s(x')|(c-2\log|\langle x',\xi'\rangle|)\,dx'\leq c_1$$

where c_1 is a constant independent of a, b, ξ. Thus for any $h\in L^2$ we get $\hat{s}_{a,b}\hat{h}\in L^2$ with $\|\hat{s}_{a,b}\hat{h}\|_{L^2}\leq c_1\|\hat{h}\|_{L^2}$. Again by Lebesgue's convergence theorem it holds that $\hat{s}_{a,b}\hat{h}\to\hat{s}_{a,\infty}\hat{h}$ in L^2, because $\hat{s}_{a,b}(\xi)\hat{h}(\xi)\to\hat{s}_{a,\infty}(\xi)\hat{h}(\xi)$ almost everywhere. By the inverse Fourier transformation we get $s_{a,b}*h\to s_{a,\infty}*h$ in L^2. On the other hand it is clear that for almost every x we get

$$s_{a,b}*h(x)=\int_{a\leq|y|<b}h(x-y)s(y)dy\to\int_{a\leq|y|}h(x-y)s(y)dy\qquad(b\to\infty).$$

Hence we obtain

$$s_{a,\infty}*h(x)=\int_{a\leq|y|}h(x-y)s(y)dy\in L^2.$$

We now consider the case $a\to 0$. For any fixed $\rho\geq 0$ we observe that

$$|I(\alpha)|\leq\begin{cases}2\int_0^1 \frac{|\sin \pi\rho(b-a)\beta|}{\beta}d\beta\leq 2\pi\rho(b-a) & \text{for } \alpha>0,\\ 2\int_{2\pi\rho a}^{2\pi\rho b}\frac{|\sin t|}{t}dt+2\int_{-1}^{0}\left|\frac{\sin \pi\rho(b-a)\beta}{\beta}\right|d\beta\leq 6\pi\rho(b-a) & \text{for } \alpha<0.\end{cases}$$

Therefore it holds that $I(\alpha; a, b, \rho)\to 0$ $(b-a\to 0)$ uniformly in α, $|\alpha|\leq 1$, and in ρ confined in a bounded set of positive numbers. Hence for any $h\in L^2$ we see $\hat{s}_{a,b}(\xi)\hat{h}(\xi)\to 0$ $(b-a\to 0)$. Thus by the inverse Fourier transformation we may assert that there exists a function $g\in L^2$ such that $h*s_{\varepsilon,\infty}\to g$ $(\varepsilon\to 0)$ in L^2. Hence for any $\varphi\in\mathscr{D}$ it holds that

$$\begin{aligned}\langle g, \varphi\rangle &= \lim_{\varepsilon\to 0}\int\varphi(x)dx\int_{|y|\geq\varepsilon}h(x-y)s(y)\,dy\\ &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}\varphi*\check{h}(y)s(y)dy=\langle s, \varphi*\check{h}\rangle=\langle s*h, \varphi\rangle,\end{aligned}$$

that is $s*h=g=\lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon}h(x-y)s(y)dy$, as desired. Here we remark that for any $s\in\sum_{-n}^{0}$ it holds that Pf.s=v.p. s, *Cauchy's principal value* of s. By the way we note that the $\mathscr{S}'$-convolution is defined for any $s_1, s_2\in\sum_{-n}^{1}$ and we have $s_1*s_2\in\sum_{-n}^{1}$, because $s_1, s_2\in\mathscr{D}'_{L^2}$ and $(s_1*s_2)^\wedge=\hat{s}_1\cdot\hat{s}_2\in\sum_0$.

Before stating the next proposition we prove

LEMMA 2. *Let* $f\in\sum_{-n}$ *and* v. p. f *be defined. Then* $f\in\sum_{-n}^{0}$.

PROOF. Letting $h(t)\in\mathscr{D}(R^1)$, $h(t)\geq 0$ and

$$h(t)=\begin{cases}1 & \text{for } |t|\leq 1,\\ 0 & \text{for } |t|\geq 2,\end{cases}$$

we put $\varphi(x)=h(|x|)$ for $x\in R^n$. Then

$$\int_{|x|\geq\varepsilon}f(x)\,\varphi(x)dx=\int_{r\geq\varepsilon}\frac{h(r)}{r}\,dr\int_{|x|=1}f(x)dx.$$

This shows us that $\int_{|x|=1}f(x)dx=0$, because if it is not the case we get

$$\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}f(x)\varphi(x)dx\to\pm\infty$$

contradicting the hypothesis. This completes the proof.

Let $\mathscr{B}^m$, $m=0, 1, 2, \cdots$, be the space of m times continuously differentiable functions on R^n with bounded derivatives $(m<\infty)$, or $\mathscr{B}^m=\mathscr{B}$ $(m=\infty)$. It is a Banach space $(m<\infty)$ or a space of type (F) $(m=\infty)$, with the usual topology:

uniform convergence on R^n of derivatives of all orders up to m. Next proposition is a lemma of Friedrichs' type, if we may use such an expression [6].

PROPOSITION 7. *Let* $b\in\mathscr{B}^m$, $m\geq 2$, $s\in\sum_{-n}^{1}$ *and let for* $j=1,\cdots,n$,

$$T_j(f)=s*\left(b\frac{\partial f}{\partial x_j}\right)-b\left(s*\frac{\partial f}{\partial x_j}\right).$$

Then T_j *is a continuous linear application of* H^l *into* H^l, $0\leq l\leq m-2$.

PROOF. We first remark that $T_j(f)$ is meaningful because s, $\dfrac{\partial f}{\partial x_j}\in\mathscr{D}'_{L^2}$, and hence the convolutions are defined. Furthermore it is not difficult to see that $T_j(f)$ is a continuous linear application of H^l into $\mathscr{D}'$. Therefore to prove the statement it is enough to see that there exists a constant c such that $\|T_j(\varphi)\|_l\leq c\|\varphi\|_l$ for any $\varphi\in\mathscr{D}$. Since we may write

$$D^p T_j(\varphi)=\sum_{r\leq p}\binom{p}{r}\left\{s*\left(D^{p-r}b\cdot\frac{\partial D^r\varphi}{\partial x_j}\right)-(D^{p-r}b)\left(s*\frac{\partial D^r\varphi}{\partial x_j}\right)\right\},$$

the problem is reduced to show that there exists a constant c such that $\|T_j(\varphi)\|_{L^2}\leq c\|\varphi\|_{L^2}$. This is proved as follows. We assume $j=1$ and $s\in\sum_{-n}^{0}$. Then by an easy computation we first infer that $T_1(\varphi)(x)=\lim_{\varepsilon\to 0}I_\varepsilon(x)$ everywhere on R^n, where

$$\begin{aligned}(1)\qquad I_\varepsilon(x)&=\int_{|y|=\varepsilon}(b(x-y)-b(x))\varphi(x-y)s(y)\eta_1(y)dy\\&\quad-\int_{|y|\geq\varepsilon}b_{x_1}(x-y)\varphi(x-y)s(y)dy+\int_{|y|\geq\varepsilon}(b(x-y)-b(x))\varphi(x-y)s_{y_1}(y)dy\\&=I_\varepsilon^{(1)}(x)+I_\varepsilon^{(2)}(x)+I_\varepsilon^{(3)}(x).\end{aligned}$$

Here $\eta_1(y)$ is the first component of the outer normal unit vector on the sphere $|y|=\varepsilon$. Since $b\in\mathscr{B}^2$ it is not difficult to prove $|\lim_{\varepsilon\to 0}I_\varepsilon^{(1)}(x)|\leq c_1|\varphi(x)|$ and thus

$$(2)\qquad \|\lim_{\varepsilon\to 0}I_\varepsilon^{(1)}\|_{L^2}\leq c_1\|\varphi\|_{L^2},$$

where c_1 is a constant depending on b_{x_j}, $j=1,\cdots,n$, and s. We next dominate $I_\varepsilon^{(2)}$ as follows:

$$(3)\qquad \|\lim_{\varepsilon\to 0}I_\varepsilon^{(2)}\|_{L^2}\leq\|(b_{x_1}\varphi)*s\|_{L^2}\leq\|\hat{s}\|_\infty\|b_{x_1}\varphi\|_{L^2}\leq c_2\|\varphi\|_{L^2},$$

where the onstant c_2 depends on b_{x_1} and s. To estimate $I_\varepsilon^{(3)}(x)$ we write

$$(4)\qquad I_\varepsilon^{(3)}(x)=\int_{|y|\geq 1}+\int_{1>|y|\geq\varepsilon}(b(x-y)-b(x))\varphi(x-y)s_{y_1}(y)dy=J(x)+J_1(x).$$

Since $|(b(x-y)-b(x))\varphi(x-y)|\leq 2\ \|b\|_\infty|\varphi(x-y)|$ and $|s_{y_1}(y)|\leq c_3|y|^{-(n+1)}$ for $|y|\geq 1$ with $c_3=\sup_{|y|=1}|s_{y_1}(y)|$, we infer that

$$(5)\qquad \|J\|_{L^2}\leq c_4\|\varphi\|_{L^2},$$

c_4 being a constant depending merely on b and s_{y_1}. Finally, $J_1(x)$ is dominated in the following way. Taylor's expansion shows us that

$$b(x-y)-b(x)=-\sum_{k=1}^{n} y_k\, b_{x_k}(x)+b(x, y),$$

where $|b(x, y)|\leq c_5|y|^2$ and c_5 is the maximum of $\|b_{x_i x_j}\|_\infty$, $i, j=1,\cdots, n$. Therefore

$$(6)\quad J_1(x)=-\sum_{k=1}^{n} b_{xk}(x)\int_{1\geq|y|\geq\varepsilon}\varphi(x-y)y_k\, s_{y_1}(y)dy+\int_{1\geq|y|\geq\varepsilon}\varphi(x-y)b(x, y)s_{y_1}(y)dy.$$

The absolute value of the second term is dominated by $c_6\int_{1>|y|}|\varphi(x-y)|\,|y|^{-n+1}dy$ where c_6 is a constant depending on c_5 as well as on $\sup_{|y|=1}|s_{y_1}(y)|$ and not on ε. Therefore its L^2-norm does not exceed $c_7\|\varphi\|_{L^2}$, c_7 being a constant depending only on c_6. As for the first term we remark that $y_k s_{y_1}(y)\in\sum_{-n}^{0}$ because v.p. $y_k\, s_{y_1}(y)$ does exist. Consequently to dominate the L^2-norm of the first term by a constant multiple of $\|\varphi\|_{L^2}$, it is sufficient to consider the integral of the type: $\int_{1\geq|y|\geq\varepsilon}\varphi(x-y)s(y)\,dy$, where $\varphi\in\mathscr{S}$ and $s\in\sum_{-n}^{0}$. Then, using the notation introduced above we see that it is nothing but the convolution $\varphi * s_{\varepsilon,1}$. Hence we get $\|\varphi * s_{\varepsilon,1}\|_{L^2}\leq c_8\|\varphi\|_{L^2}$, where $c_8=\sup_{\varepsilon>0}\|\hat{s}_{\varepsilon,1}\|_\infty$. Thus it follows from (6) that

$$(7)\qquad \|J_1\|_{L^2}\leq c_9\|\varphi\|_{L^2},$$

c_9 being a constant independent of ε. By means of (1), (2), (3), (4), (5) and (7) we may conclude that $\|T_1(\varphi)\|_{L^2}\leq c\|\varphi\|_{L^2}$ where the constant c depends only on s and b. This completes the proof.

REMARK 1. To estimate $|\hat{s}_{\varepsilon,1}(\xi)|$ we may put $J(\alpha)=\int_{2\pi\rho\varepsilon}^{2\pi\rho}\frac{e^{-it\alpha}-e^{-t}}{t}\,dt$ instead of $I(\alpha)$. Then using $|J(\alpha)|\leq c-\log|\alpha|$ we obtain the same result [3].

REMARK 2. Further observations show us that when s and b vary within bounded subsets of $\sum_{-n}^{0}$ and $\mathscr{B}^m$ respectively, the corresponding c will also be confined in a bounded set of positive numbers. So that in such a case various T_j may be extended equicontinuously to linear applications of H^l into H^l.

As an application of the preceding proposition we state

PROPOSITION 8. *Let* $b \in \mathscr{B}^m$, $m \geq 2$, $s \in \sum_{-n}^{0}$ *and let*

$$T(f) = s * (bf) - b(s * f).$$

Then T is a continuous linear application of H^l *into* H^{l+1}, $0 \leq l \leq m-3$.

PROOF. By means of the Fourier transformation it is not difficult to see that the application

$$(f_0, f_1, \cdots, f_n) \to f = f_0 + \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}$$

is a continuous homomorphism of $H^{l+1} \times \cdots \times H^{l+1}((n+1)$ factors) onto H^l. Hence $T(f) = T(f_0) + T_1(f_1) + \cdots + T_n(f_n)$ is a continuous application of H^l into H^{l+1}. This completes the proof.

REMARK 3. By virtue of Remark 2, it is noted that if $\{T_\alpha\}$ is a family of T's defined by s_α and b_α, confined in bounded subsets of $\sum_{-n}^{0}$ and $\mathscr{B}^m$ respectively, then $\{T_\alpha\}$ is an equicontinuous subset of $\mathscr{L}(H^l; H^{l+1})$.

We now define the operator Λ by the identity:

$$\Lambda f = c_n\Big(f * \text{Pf.} \frac{1}{r^{n+1}}\Big) = d_n\Big(\Delta f * \frac{1}{r^{n-1}}\Big)$$

where $c_n = \dfrac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+2}{2}}\Gamma(-\frac{1}{2})} = (n-1)d_n$. It is seen that Λ is a continuous linear operator of H^l into H^{l-1}, $l = 1, 2, \cdots$, because v.p. $\dfrac{x_j}{r^{n+1}} \in \sum_{-n}^{0}$. Since it is known that $\mathscr{F}\Big(\text{Pf.} \dfrac{1}{r^{n+1}}\Big) = \dfrac{1}{c_n} r$ [12, II, p. 113], we obtain for any $\varphi \in \mathscr{S}$

$$\Lambda\varphi = \mathscr{F}^{-1}(\hat{\varphi} \cdot r).$$

Therefore it is not difficult to see that $\Lambda^2 = -\dfrac{1}{4\pi^2}\Delta$ and hence we may write $\Lambda = \dfrac{1}{2\pi}\sqrt{-\Delta}$.

To define the operator F corresponding to $F \in \sum_{-n}^{1}(\mathscr{B}^m)$ we first remark that $\sum_{-n}^{1}(\mathscr{B}^m)$ may be considered as the space of m times continuously differentiable functions $F(x)$ defined on R^n with values in $\sum_{-n}^{1}$ and with bounded derivatives of all orders up to m. Thus for any $x \in R^n$ and $h \in H^m$ we may define $F(x) * h$. We now prove

PROPOSITION 9. *For any* $F \in \sum_{-n}^{1}(\mathscr{B}^m)$, *there exists a uniquely determined con-*

tinuous linear operator $F: h\to F(h)$ on H^m such that for any $\varphi\in\mathcal{S}$

$$F(\varphi)\,(x)=(F(x)*\varphi)\,(x).$$

Proof. It is known that $\sum_{-n}^{1}(\mathcal{B}^m)=\mathcal{B}^m\hat{\otimes}\sum_{-n}^{1}$, and that both $\mathcal{B}^m$ and $\sum_{-n}^{1}$ are spaces of type (F). Therefore F may be expressed as

$$F=\sum_{k=1}^{\infty}\alpha_k\,b_k\,s_k+b\delta$$

where $\sum_{k=1}^{\infty}|\alpha_k|<\infty$, $b\in\mathcal{B}^m$, $b_k\to 0$ in $\mathcal{B}^m$ and $s_k\to 0$ in $\sum_{-n}^{0}$ [9, Chap. I, p. 51]. Here the right hand sum is taken in $\sum_{-n}^{1}(\mathcal{B}^m)$, and thus it is not difficult to see that for any $\varphi\in\mathcal{S}$ we obtain

$$(F(x)*\varphi)\,(x)=\sum_{k=1}^{\infty}\alpha_k\,b_k(x)\,s_k*\varphi(x)+b(x)\,\varphi(x)$$

everywhere $x\in R^n$. On the other hand if we put

$$F(h)=\sum_{k=1}^{\infty}\alpha_k\,b_k(s_k*h)+bh$$

for any $h\in H^m$, we get $F(h)\in H^m$. To see that F is continuous it is enough to remark that

$$\|b_k(s_k*h)\|_m\leq\sum_{|p|\leq m}\|D^p(b_k(s_k*h))\|_{L^2}$$
$$\leq\sum_{|p|\leq m}\sum_{r\leq p}\binom{p}{r}\|D^{p-r}b_k\|_\infty\|\hat{s}_k\|_\infty\|D^r h\|_{L^2}\leq c_k\|h\|_m$$

where c_k is a constant such that $c_k\to 0$ $(k\to\infty)$. Since $F(\varphi)\,(x)=(F(x)*\varphi)\,(x)$ for any $\varphi\in\mathcal{S}$, it is clear that such an F is uniquely determined. This completes the proof.

We now introduce some operators related to F. To simplify the wording we call $\hat{F}\in\sum_0(\mathcal{B}^m)$ the *symbol* of the operator F. Then $\bar{\hat{F}}$ is also the symbol of an operator $F^\sharp$. The adjoint of the operator F relative to the space L^2 is denoted by F^*: $(Ff,g)_{L^2}=(f,F^*g)_{L^2}$. Letting F_1, F_2 be operators with the symbols $\hat{F}_1$, $\hat{F}_2\in\sum_0(\mathcal{B}^m)$, we obtain an operator $F_1\circ F_2$ with the symbol $\hat{F}_1\,\hat{F}_2\in\sum_0(\mathcal{B}^m)$. Applying now the preceding results we shall prove the next proposition [3], [5].

Proposition 10. *$FA-AF$, F^*A-AF^*, $(F^*-F^\sharp)\,A$, $A(F^*-F^\sharp)$, $(F_1\circ F_2-F_1F_2)A$, $A(F_1\circ F_2-F_1\,F_2)$ and $F_1\,F_2\,A-AF_1\,F_2$ are all continuous linear operators on H^l, $0\leq l\leq m-2$, $m\geq 2$.*

Proof. It is clear that each of the operators in question belongs to

$\mathscr{L}(H^l; \mathscr{D}')$. Thus to prove the statements it is enough to show that their restriction on $\mathscr{D}$ are continuous from H^l into H^l. To this end let the symbols of the operators F, F_1 and F_2 be expanded in the following forms:

$$\hat{F}(x, \xi)=\sum_{j=1}^{\infty} \alpha_j\, b_j(x)\, \hat{s}_j(\xi)+c(x),$$
$$\hat{F}_k(x, \xi)=\sum_{j=1}^{\infty} \alpha_{kj}\, b_{kj}(x)\, \hat{s}_{kj}(\xi)+c_k(x), \qquad k=1, 2,$$

where $\sum_{j=1}^{\infty} |\alpha_j|<\infty$, $\sum_{j=1}^{\infty} |\alpha_{kj}|<\infty$, $c, c_k\in\mathscr{B}^m$, $b_j, b_{kj}\to 0$ $(j\to\infty)$ in $\mathscr{B}^m$ and $s_j, s_{kj}\to 0$ $(j\to\infty)$ in $\sum_{-n}^{0}$. Then the statements are proved as follows.

Ad $FA-AF$: Letting $\varphi\in\mathscr{D}$, we write

$$(FA-AF)(\varphi)=\sum_{j=1}^{\infty} \alpha_j\{b_j(s_j * A\varphi)-A(b_j(s_j*\varphi))\}+cA\varphi-Ac\varphi$$
$$=d_n\left\{\sum_{j=1}^{\infty}\alpha_j\left(b_j\left(s_j*\frac{1}{r^{n-1}}*\Delta\varphi\right)-\frac{1}{r^{n-1}}*\Delta(b_j(s_j*\varphi))\right)+c\left(\frac{1}{r^{n-1}}*\Delta\varphi\right)-\frac{1}{r^{n-1}}*\Delta(c\varphi)\right\}.$$

Thus the problem is reduced to prove the equicontinuity of

$$P(\varphi)=b\left(\frac{1}{r^{n-1}}*\Delta g\right)-\frac{1}{r^{n-1}}*\Delta(bg)$$

with $g=s*\varphi$ or $g=\varphi$, when b and s are confined in a bounded subset of $\mathscr{B}^m$ and a bounded subset of $\sum_{-n}^{0}$ respectively, Writing

$$P(\varphi)=\sum_{j=1}^{n}\left\{b\left(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial g}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial}{\partial x_j}(bg)\right\}$$
$$=\sum_{j=1}^{n}\left\{b\left(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial g}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*b\frac{\partial g}{\partial x_j}-\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial b}{\partial x_j}g\right\}$$

and observing $\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}=-(n-1)\,\text{v.p.}\frac{x_j}{r^{n+1}}\in\sum_{-n}^{0}$, we may dominate the last term as follows:

$$\left\|\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial b}{\partial x_j}g\right\|_l\leq c_1'\left\{\begin{matrix}\left\|\frac{\partial b}{\partial k_j}(s*\varphi)\right\|_l\\ \left\|\frac{\partial b}{\partial x_j}\varphi\right\|_l\end{matrix}\right\}\leq c_1\|\varphi\|_l,$$

where the constant c_1 depends only on s and b. On the other hand, applying Proposition 7 to the first two terms we get

$$\left\|b\left(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*\frac{\partial g}{\partial x_j}\right)-\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}*b\frac{\partial g}{\partial x_j}\right\|_l\leq c_2'\|g\|_l\leq c_2\|\varphi\|_l,$$

where the constant c_2 depends only on s and b. Thus we see $\|P(\varphi)\|_l \leq c_3 \|\varphi\|_l$ and the constant c_3 may be chosen independently of s and b confined in a bounded subset of $\sum'^{0}_{-n}$ and a bounded subset of $\mathscr{B}^m$ respectively. This completes the proof.

Ad $F^*\Lambda - \Lambda F^*$: Letting $\varphi \in \mathscr{D}$, we get

$$(F^* \Lambda - \Lambda F^*)(\varphi) = \sum_{j=1}^{\infty} \bar{\alpha}_j \{\check{s}_j * (\bar{b}_j \Lambda\varphi) - \Lambda(\check{s}_j * (\bar{b}_j\varphi))\} + \bar{c}\Lambda\varphi - \Lambda\bar{c}\varphi.$$

Hence it is enough to prove the equicontinuity of

$$Q(\varphi) = s * (b\Lambda\varphi) - \Lambda(s * b\varphi),$$

when b and s are restricted in a bounded subset of $\mathscr{B}^m$ and a bounded subset of $\sum'^{0}_{-n}$ respectively. To this end, we write

$$Q(\varphi) = d_n \sum_{j=1}^{n} \left\{ s * b\left(\frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial\varphi}{\partial x_j}\right) - \frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial}{\partial x_j}(s * b\varphi)\right\}$$

$$= d_n\, s * \sum_{j=1}^{n} \left\{ b\left(\frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial\varphi}{\partial x_j}\right) - \frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \left(b \frac{\partial\varphi}{\partial x_j}\right) - \frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial b}{\partial x_j} \varphi\right\}.$$

Then paying attention to $\frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} \in \sum'^{0}_{-n}$, and utilizing Proposition 7, it is not difficult to see $\|Q(\varphi)\|_l \leq c_4 \|\varphi\|_l$, where c_4 is a constant independent of b and s restricted in a bounded subset of $\mathscr{B}^m$ and a bounded subset of $\sum'^{0}_{-n}$ respectively. This completes the proof.

Ad $(F^* - F^\sharp)\Lambda$: For any $\varphi \in \mathscr{D}$, we get

$$(F^* - F^\sharp)\, \Lambda(\varphi) = \sum_{j=1}^{\infty} \bar{\alpha}_j \{\check{s}_j * (\bar{b}_j\, \Lambda\varphi) - \bar{b}_j(\check{s}_j * \Lambda\varphi)\}.$$

Therefore, as repeatedly done, the problem is reduced to show that the application

$$\varphi \to s * (b\Lambda\varphi) - b(s * \Lambda\varphi)$$

is equicontinuous when b and s are confined in fixed bounded subsets of $\mathscr{B}^m$ and $\sum'^{0}_{-n}$ respectively. This again turns into the same problem concerning the application

$$R(\varphi) = s * b\left(\frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial\varphi}{\partial x_j}\right) - b\left(s * \frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial\varphi}{\partial x_j}\right)$$

$$= s * \left\{ b\left(\frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * \frac{\partial\varphi}{\partial x_j}\right) - \frac{\partial}{\partial x_j} \frac{1}{r^{n-1}} * b \frac{\partial\varphi}{\partial x_j}\right\}$$

$$+\left(s*\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}\right)*b\frac{\partial\varphi}{\partial x_j}-b\left(\left(s*\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}}\right)*\frac{\partial\varphi}{\partial x_j}\right).$$

Then observing $s*\dfrac{\partial}{\partial x_j}\dfrac{1}{r^{n-1}}\in\sum_{-n}^{\prime 0}$ and using Proposition 7, it is not difficult to conclude that $\|R(\varphi)\|_l\leq c_5\|\varphi\|_l$ where the constant c_5 is independent of b and s confined in prescribed bounded subsets of $\mathscr{B}^m$ and $\sum_{-n}^{\prime 0}$ respectively. This completes the proof.

Remark 4. Decomposing the application $\varphi\to s*(b\Lambda\varphi)-b(s*\Lambda\varphi)$ into $\varphi\to\Lambda\varphi\to s*(b\Lambda\varphi)-b(s*\Lambda\varphi)$ and applying Proposition 8, we may conclude the same result when $l\geq 1$. But to overcome the case $l=0$ along the same line it seems desirable to consider the space H^m for negative m.

Ad $\Lambda(F^*-F^\sharp)$: This is a consequence of the following identity:

$$\Lambda(F^*-F^\sharp)=(\Lambda F^*-F^*\Lambda)+(F^*-F^\sharp)\Lambda+(F^\sharp\Lambda-\Lambda F^\sharp).$$

Remark 5. It is not difficult to see by Proposition 8 that $F^*-F^\sharp$ is a continuous operator of H^l into H^{l+1} for $0\leq l\leq m-3$. Hence by composing Λ on the left of $F^*-F^\sharp$, we obtain the desired result for $0\leq l\leq m-3$. But it seems to the author that the case $l=m-2$ requires a particular argument.

Ad $(F_1\circ F_2-F_1F_2)\Lambda$: An expansion of $(F_1\circ F_2-F_1F_2)\Lambda(\varphi)$ shows us that the problem becomes to prove that the application

$$\varphi\to b_1\{b_2(s_1*(s_2*\Lambda\varphi))-s_1*b_2(s_2*\Lambda\varphi)\}$$

is equicontinuous when b_1, b_2 describe a bounded subset of $\mathscr{B}^m$ and s_1, s_2 describe a bounded subset of $\sum_{-n}^{\prime 0}$. This turns out to the same problem under the similar circumstances relative to the application of the form

$$\varphi\to b\left(s_1*s_2*\frac{\partial\varphi}{\partial x_j}\right)-s_1*b\left(s_2*\frac{\partial\varphi}{\partial x_j}\right)$$

$$=b\left(s_1*s_2*\frac{\partial\varphi}{\partial x_j}\right)-s_1*s_2*\left(b\frac{\partial\varphi}{\partial x_j}\right)+s_1*\left\{s_2*\left(b\frac{\partial\varphi}{\partial x_j}\right)-b\left(s_2*\frac{\partial\varphi}{\partial x_j}\right)\right\}.$$

Thus the same argument as in the case $(F^*-F^\sharp)\Lambda$ shows us the desired conclusion. This completes the proof.

Ad $F_1F_2\Lambda-\Lambda F_1F_2$: Quite similarly as before, an expansion of $(F_1F_2\Lambda-\Lambda F_1F_2)(\varphi)$ reduces the problem to show the equicontinuity of the application

$$\varphi\to b_1(s_1*b_2(s_2*\Lambda\varphi))-\Lambda\{b_1(s_1*b_2(s_2*\varphi))\}$$

when b_1, b_2 and s_1, s_2 run over prescribed bounded subsets of $\mathscr{B}^m$ and $\sum_{-n}^{\prime 0}$ respectively. Then it is not difficult to see that this is a direct consequence of the same statement concerning the application

$$\varphi \to b_1\Big(s_1 * b_2\Big(s_2 * \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial \varphi}{\partial x_j}\Big)\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial}{\partial x_j}\{b_1(s_1 * b_2(s_2 * \varphi))\}$$

$$= b_1\Big(s_1 * b_2\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * s_2 * \frac{\partial \varphi}{\partial x_j}\Big)\Big)$$

$$- \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_1\Big(s_1 * \frac{\partial}{\partial x_j} b_2(s_2 * \varphi)\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial b_1}{\partial x_j}(s_1 * b_2(s_2 * \varphi)).$$

As to the last term everything is obvious. Letting $g = s_2 * \varphi$, the first two terms may be written as

$$b_1\Big(s_1 * b_2\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial g}{\partial x_j}\Big)\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_1\Big(s_1 * b_2\frac{\partial g}{\partial x_j}\Big)$$

$$- \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_1\Big(s_1 * \Big(\frac{\partial b_2}{\partial x_j} g\Big)\Big).$$

Again there needs no mention of the last term, while the first two become

$$b_1\Big(s_1 * b_2\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial g}{\partial x_j}\Big)\Big) - b_1\Big(s_1 * \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_2\frac{\partial g}{\partial x_j}\Big)$$

$$+ b_1\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * s_1 * b_2\frac{\partial g}{\partial x_j}\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_1\Big(s_1 * b_2\frac{\partial g}{\partial x_j}\Big)$$

$$= b_1\Big\{s_1 * \Big(b_2\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \frac{\partial g}{\partial x_j}\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_2\frac{\partial g}{\partial x_j}\Big)\Big\}$$

$$+ b_1\Big(\frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * \Big(s_1 * b_2\frac{\partial g}{\partial x_j}\Big)\Big) - \frac{\partial}{\partial x_j}\frac{1}{r^{n-1}} * b_1\Big(s_1 * b_2\frac{\partial g}{\partial x_j}\Big).$$

Then applying Proposition 7 it is not difficult to get the assertion as desired. This completes the proof.

Ad $\Lambda(F_1 \circ F_2 - F_1 F_2)$: An immediate consequence of the expression

$$\Lambda(F_1 \circ F_2 - F_1 F_2) = \{\Lambda(F_1 \circ F_2) - (F_1 \circ F_2)\Lambda\} + (F_1 \circ F_2 - F_1 F_2)\Lambda + (F_1 F_2 \Lambda - \Lambda F_1 F_2).$$

This completes the proof.

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