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TENSOR PRODUCT OF ANNIHILATING SPACES

By

Kazô TSUJI

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1. This paper is a sequel to [2], [3]. We shall hope that a reader is familiar with [2], [3], and shall use terminologies and results from there without any explanation. In this paper we shall clarify relations between annihilating spaces and corresponding von Neumann algebras in a tensor product Hilbert space. We shall then present a conjecture which is equivalent to a problem raised by Dixmier [1].

Let $\mathfrak{H}$ be a complex Hilbert space. We shall define an operator of rank $\leqq 1$ for every $f,\ g \in \mathfrak{H}$ by the following:

$$(f \circledast \bar{g})h = <h,\ g> f \quad \text{for every } h \in \mathfrak{H}.$$

In the previous papers we denoted this operator by $f \otimes \bar{g}$, but throughout this paper, we devote a notation "$\otimes$" to the tensor product symbol. Let $(\tau c)(\mathfrak{H})$ be a Banach space consisting of the trace class operators on $\mathfrak{H}$, and let $(\tau c)_0(\mathfrak{H})$ be a subspace $\{A : \text{Trace } A = t(A) = 0,\ A \in (\tau c)(\mathfrak{H})\}$. Let $r(\mathfrak{H})$ be a set of all operators of rank$\leqq 1$ on $\mathfrak{H}$, and let $\mathfrak{L}(\mathfrak{H})$ be a space of all bounded linear operators on $\mathfrak{H}$. For an indexed Hilbert space $\mathfrak{H}_i$ we shall denote trace (resp. inner product, identity operator) by $t_i(\cdot)$ (resp. $<,\ >_i$, I_i).

2. Let $\mathfrak{H}_i$ be a complex Hilbert space ($i=1, 2$), and let $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ be their tensor product Hilbert space. Then we have

LEMMA 1. $(f \otimes \varphi) \circledast \overline{(g \otimes \psi)} = (f \circledast \bar{g}) \otimes (\varphi \circledast \bar{\psi})$ *for every* $f \otimes \varphi,\ g \otimes \psi \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$.

PROOF. For every $h \otimes \eta \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$, we have

$$\begin{aligned}((f \otimes \varphi) \circledast \overline{(g \otimes \psi)})(h \otimes \eta) &= <h \otimes \eta,\ g \otimes \psi> (f \otimes \varphi) \\ &= <h,\ g>_1 <\eta,\ \psi>_2 (f \otimes \varphi) \\ &= (<h,\ g>_1 f) \otimes (<\eta,\ \psi>_2 \varphi) \\ &= ((f \circledast \bar{g})h) \otimes ((\varphi \circledast \bar{\psi})\eta) \\ &= ((f \circledast \bar{g}) \otimes (\varphi \circledast \bar{\psi}))(h \otimes \eta).\end{aligned}$$

Hence we obtain our Lemma by continuity of the operator.

Proposition 1. *Let* $\mathscr{T}_1$ *be an annihilating space in* $\mathfrak{H}_1$. *Then a set* $\mathfrak{S}_1=\{(f \circledast \bar{g})\otimes(\varphi\circledast\bar{\phi}) : f\circledast\bar{g} \in \mathscr{T}_1, \varphi\circledast\bar{\phi} \in r(\mathfrak{H}_2)\}$ *is a subset of* $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$. *Let* $\mathscr{T}_1\otimes r(\mathfrak{H}_2)$ *be a closed linear subspace of* $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ *which is generated by the set* $\mathfrak{S}_1$. *Then we have* $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))^{\perp}=\mathscr{T}_1\otimes r(\mathfrak{H}_2)$.

Proof. Since $t((f\circledast\bar{g})\otimes(\varphi\circledast\bar{\phi}))=t_1(f\circledast\bar{g})t_2(\varphi\circledast\bar{\phi})$, it is clear that we have $\mathfrak{S}_1\subset(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$.

If $(f\circledast\bar{g})\otimes(\varphi\circledast\bar{\phi}) \in \mathfrak{S}_1$, then for every $A \in \mathscr{T}_1^{\perp}$ and every $B \in \mathfrak{L}(\mathfrak{H}_2)$,

$$\begin{aligned} t((A\otimes B)((f\circledast\bar{g})\otimes(\varphi\circledast\bar{\phi}))) &= t(A(f\circledast\bar{g})\otimes B(\varphi\circledast\bar{\phi})) \\ &= t((Af\circledast\bar{g})\otimes(B\varphi\circledast\bar{\phi})) \\ &= t_1(A(f\circledast\bar{g}))t_2(B(\varphi\circledast\bar{\phi})) \\ &= 0. \end{aligned}$$

Hence by continuity of the trace we have $A\otimes B \in (\mathscr{T}_1\otimes r(\mathfrak{H}_2))^{\perp}$. But by [2, §1, Lemma], $(\mathscr{T}_1\otimes r(\mathfrak{H}_2))^{\perp}$ is ultra-weakly closed. Therefore we have $\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2)\subset(\mathscr{T}_1\otimes r(\mathfrak{H}_2))^{\perp}$, that is, $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))^{\perp}\supset\mathscr{T}_1\otimes r(\mathfrak{H}_2)$.

On the other hand, we have $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))'=(\mathscr{T}_1^{\perp})'\otimes\{\alpha I_2\}$ and $(\mathscr{T}_1^{\perp})'\otimes\{\alpha I_2\}=\{A\otimes I_2 : A \in (\mathscr{T}_1^{\perp})'\}$. Hence a projection in $(\mathscr{T}_1^{\perp})'\otimes\{\alpha I_2\}$ has a form $E\otimes I_2$ for a projection $E \in (\mathscr{T}_1^{\perp})'$. And we have $I-(E\otimes I_2)=(I_1-E)\otimes I_2$. If $x \in (E\otimes I_2)(\mathfrak{H}_1\otimes\mathfrak{H}_2)$, then there exists a sequence $\{x_n\}$ such that $\lim_n x_n=x$, $x_n=\sum_{i=1}^{p_n} f_i^{(n)}\otimes\varphi_i^{(n)}$, and $f_i^{(n)} \in E(\mathfrak{H}_1)$. Similarly, for $y \in ((I_1-E)\otimes I_2)(\mathfrak{H}_1\otimes\mathfrak{H}_2)$, there is a sequence $\{y_n\}$ such that $\lim_n y_n=y$, $y_n=\sum_{j=1}^{q_n} g_j^{(n)}\otimes\phi_j^{(n)}$, and $g_j^{(n)} \in (I_1-E)(\mathfrak{H}_1)$. And we have by Lemma 1

$$\begin{aligned} x_n\circledast\bar{y}_n &= \left(\sum_{i=1}^{p_n} f_i^{(n)}\otimes\varphi_i^{(n)}\right)\circledast\left(\overline{\sum_{j=1}^{q_n} g_j^{(n)}\otimes\phi_j^{(n)}}\right) \\ &= \sum_{i=1}^{p_n}\sum_{j=1}^{q_n}(f_i^{(n)}\circledast\overline{g_j^{(n)}})\otimes(\varphi_i^{(n)}\circledast\overline{\phi_j^{(n)}}). \end{aligned}$$

By virtue of [3, Theorem], we have that $f_i^{(n)}\circledast\overline{g_j^{(n)}} \in \mathscr{T}_1$. Then $x_n\circledast\bar{y}_n \in \mathscr{T}_1\otimes r(\mathfrak{H}_2)$. And by a routine calculation, we have $\lim_n \tau((x_n\circledast\bar{y}_n)-(x\circledast\bar{y}))=0$, where $\tau(\cdot)$ is a norm in a Banach space $(\tau c)(\mathfrak{H}_1\otimes\mathfrak{H}_2)$. Therefore $x\circledast\bar{y} \in \mathscr{T}_1\otimes r(\mathfrak{H}_2)$. Now again by [3, Theorem], $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))^{\perp}$ is generated by a set

$$\mathfrak{S}_0=\{x\circledast\bar{y} : x \in (E\otimes I_2)(\mathfrak{H}_1\otimes\mathfrak{H}_2),\ y \in ((I_1-E)\otimes I_2)(\mathfrak{H}_1\otimes\mathfrak{H}_2),\ \text{projection } E \in (\mathscr{T}_1^{\perp})'\}.$$

Consequently we have $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))^{\perp}\subset\mathscr{T}_1\otimes r(\mathfrak{H}_2)$.

Thus we obtain that $(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))^{\perp}=\mathscr{T}_1\otimes r(\mathfrak{H}_2)$.

COROLLARY. *Let $\mathscr{T}$ be an annihilating space in a complex Hilbert space $\mathfrak{H}$ such that $\mathscr{T}^{\perp}$ is a discrete factor. Then there are two complex Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and $\mathscr{T}$ is unitary equivalent to the annihilating space $r(\mathfrak{H}_1)\otimes(\tau c)_0(\mathfrak{H}_2)$ in $\mathfrak{H}_1\otimes\mathfrak{H}_2$.*

PROOF. Since we have $(r(\mathfrak{H}_1)\otimes(\tau c)_0(\mathfrak{H}_2))^{\perp}=\mathfrak{L}(\mathfrak{H}_1)\otimes\{\alpha I_2\}$, our Corollary is clear.

PROPOSITION 2. *Let $\mathscr{T}_1$ be an annihilating space in $\mathfrak{H}_1$. Denote symbolically by $\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)$ a closed linear subspace of $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ which is generated by a set*

$$\mathfrak{S}_2=\{(f\circledast\bar{g})\otimes(\varphi\circledast\bar{\psi}): f\circledast\bar{g}\in\mathscr{T}_1,\ \varphi\circledast\bar{\psi}\in r(\mathfrak{H}_2),$$
$$\text{or } f\circledast\bar{g}\in r(\mathfrak{H}_1),\ \varphi\circledast\bar{\psi}\in(\tau c)_0(\mathfrak{H}_2)\}.$$

Then we have $(\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\})^{\perp}=\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)$.

PROOF. By the same argument with one in the second paragraph of proof with respect to Proposition 1, we have

$$(\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\})^{\perp}\supset\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2).$$

On the other hand, by proposition 1, $\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)$ is the smallest closed linear subspace of $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ including two annihilating spaces $\mathscr{T}_1\otimes r(\mathfrak{H}_2)$ and $r(\mathfrak{H}_1)\otimes(\tau c)_0(\mathfrak{H}_2)$. And since $(\mathscr{T}_1\otimes r(\mathfrak{H}_2))^{\perp}=\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2)$ and $(r(\mathfrak{H}_1)\otimes(\tau c)_0(\mathfrak{H}_2))^{\perp}=\mathfrak{L}(\mathfrak{H}_1)\otimes\{\alpha I_2\}$, if $X\in(\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2))^{\perp}$, then we have $X\in(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))\cap(\mathfrak{L}(\mathfrak{H}_1)\otimes\{\alpha I_2\})=\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\}$. Therefore we have $(\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2))^{\perp}\subset\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\}$, that is, $\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)\supset(\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\})^{\perp}$.

Thus we have $(\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\})^{\perp}=\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)$.

PROPOSITION 3. *Let $\mathscr{T}_i$ be an annihilating space in $\mathfrak{H}_i\,(i=1, 2)$. Let $\mathscr{T}_1\hat{\otimes}\mathscr{T}_2$ denote symbolically a closed linear subspace of $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ which is generated by a set*

$$\mathfrak{S}=\{(f\circledast\bar{g})\otimes(\varphi\circledast\bar{\psi}): f\circledast\bar{g}\in\mathscr{T}_1,\ \varphi\circledast\bar{\psi}\in r(\mathfrak{H}_2),$$
$$\text{or } f\circledast\bar{g}\in r(\mathfrak{H}_1),\ \varphi\circledast\bar{\psi}\in\mathscr{T}_2\}.$$

Then we have $((\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})')'^{\perp}=\mathscr{T}_1\hat{\otimes}\mathscr{T}_2$.

PROOF. By definition $\mathscr{T}_1\hat{\otimes}\mathscr{T}_2$ is generated by two annihilating spaces $\mathscr{T}_1\otimes r(\mathfrak{H}_2)$ and $r(\mathfrak{H}_1)\otimes\mathscr{T}_2$. Therefore by Proposition 1, $((\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})')'=(\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2))\cap(\mathfrak{L}(\mathfrak{H}_1)\otimes\mathscr{T}_2^{\perp})\subset(\mathscr{T}_1\hat{\otimes}\mathscr{T}_2)^{\perp}$.

On the other hand if $X\in(\mathscr{T}_1\hat{\otimes}\mathscr{T}_2)^{\perp}$, then $X\in(\mathscr{T}_1\otimes r(\mathfrak{H}_2))^{\perp}=\mathscr{T}_1^{\perp}\otimes\mathfrak{L}(\mathfrak{H}_2)$ and similarly $X\in\mathfrak{L}(\mathfrak{H}_1)\otimes\mathscr{T}_2^{\perp}$. Therefore $X\in((\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})')'$. Consequently $(\mathscr{T}_1\hat{\otimes}\mathscr{T}_2)^{\perp}$

$\subset ((\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})')'$. Thus we have $((\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})')'^{\perp}=\mathscr{T}_1\hat{\otimes}\mathscr{T}_2$.

Proposition 4. *Let* $\mathscr{T}_i$ *be an annihilating space in* $\mathfrak{H}_i$ $(i=1, 2)$. *Let* $\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$ *be the largest annihilating space in* $\mathfrak{H}_1\otimes\mathfrak{H}_2$ *which is contained in* $\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)\cap(\tau c)_0(\mathfrak{H}_1)\hat{\otimes}\mathscr{T}_2$. (cf. [4]). *Then we have* $(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})^{\perp}=\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$.

Proof. Since $\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2\subset\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)\cap(\tau c)_0(\mathfrak{H}_1)\hat{\otimes}\mathscr{T}_2$, we have $\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\}\subset(\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2)^{\perp}$ and $\{\alpha I_1\}\otimes\mathscr{T}_2^{\perp}\subset(\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2)^{\perp}$. And $\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp}$ is a von Neumann algebra generated by two von Neumann algebras $\mathscr{T}_1^{\perp}\otimes\{\alpha I_2\}$ and $\{\alpha I_1\}\otimes\mathscr{T}_2^{\perp}$. So we have $\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp}\subset(\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2)^{\perp}$, that is, $(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})^{\perp}\supset\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$.

If $x\circledast\bar{y}\in(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})^{\perp}$, $(x, y\in\mathfrak{H}_1\otimes\mathfrak{H}_2)$, then for every $A_1\otimes I_2$ $(A_1\in\mathscr{T}_1^{\perp})$ and every $I_1\otimes A_2$ $(A_2\in\mathscr{T}_2^{\perp})$, we have $t((A_1\otimes I_2)(x\circledast\bar{y}))=0$ and $t((I_1\otimes A_2)(x\circledast\bar{y}))=0$. Therefore $(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})^{\perp}\subset\mathscr{T}_1\hat{\otimes}(\tau c)_0(\mathfrak{H}_2)\cap(\tau c)_0(\mathfrak{H}_1)\hat{\otimes}\mathscr{T}_2$. Hence by the largestness of $\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$, we have $(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})^{\perp}=\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$.

Corollary 1. *Let* $\mathscr{T}_i$ *be an annihilating space in* $\mathfrak{H}_i$ $(i=1, 2)$. *Then* $(\mathscr{T}_1^{\perp}\otimes\mathscr{T}_2^{\perp})'=(\mathscr{T}_1^{\perp})'\otimes(\mathscr{T}_2^{\perp})'$ *if and only if* $\mathscr{T}_1\hat{\otimes}\mathscr{T}_2=\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$.

Corollary 2. *Let* $\mathscr{T}_i$ *be a semi-finite annihilating space in* $\mathfrak{H}_i$, *where to say that* $\mathscr{T}_i$ *is semi-finite means to say that* $\mathscr{T}_i^{\perp}$ *is a semi-finite von Neumann algebra* $(i=1, 2)$. *Then we have* $\mathscr{T}_1\hat{\otimes}\mathscr{T}_2=\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$.

Open question. *Does* $\mathscr{T}_1\hat{\otimes}\mathscr{T}_2=\mathscr{T}_1\widehat{\widehat{\otimes}}\mathscr{T}_2$ *hold for every annihilating spaces* $\mathscr{T}_1$ *and* $\mathscr{T}_2$?

The question is equivalent to a Dixmier's Problem [1, p. 30]. Finally we remark that the space $(\tau c)_0(\mathfrak{H}_1\otimes\mathfrak{H}_2)$ in $\mathfrak{H}_1\otimes\mathfrak{H}_2$ is generated by a set

$$\{(f\circledast\bar{g})\otimes(\varphi\circledast\bar{\psi}) : f\circledast\bar{g}\in(\tau c)_0(\mathfrak{H}_1),\ \varphi\circledast\bar{\psi}\in r(\mathfrak{H}_2),$$
$$\text{or } f\circledast\bar{g}\in r(\mathfrak{H}_1),\ \varphi\circledast\bar{\psi}\in(\tau c)_0(\mathfrak{H}_2)\}.$$

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Department of Mathematics,
Kyushu Institute of Technology
Tobata, Kitakyushu, Japan