Bull. Kyushu Inst. Tech.
(M. & N. S.) No. 18. 1971

CONTRACTION OF LIE ALGEBRA OF THE MOTION GROUP OF THREE DIMENSIONAL EUCLIDEAN SPACE

By

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(Receivid Oct. 26, 1970)

1. Introduction. The concept of contraction of Lie groups and Lie algebras was first introduced by segal [3] and was later discussed by İnönü and Wigner [1] and Saletan [2]. The interesting examples of contraction in physics are that the Poincarè algebra is contracted to the algebra of the inhogeneous Galilei group and the rotation algebra in three dimesions to the algebra of the Euclidean motions in the plane. The contracting procedure is the following.

Consider a linear transformation U_t in a Lie algebra which depends on a palameter t ($0 \leqq t \leqq 1$) and define a new product called contracted Lie product, if exist, such as

$$(X, Y)_{\circ} = \lim_{t \to 0} U_t^{-1}(U_t X, U_t Y), \tag{1.1}$$

where U_t is assumed to be nonsingular as $t \neq 0$. So long as U_0 is nonsingular the contracted Lie algebra is isomorphic to the original one. But if U_0 is singular we may obtain a new Lie algebra which is not isomorphic to the original one. In [2], Saletan considered the contraction of Lie algebra G with respect to the linear transformation U_t such as

$$U_t = tE + (1-t)u, \tag{1.2}$$

where E is a unit matrix and u is an arbitrary singular matrix, and he obtained the result that the necessary and sufficient condition for (1.1) to exist is

$$u^2(X, Y)_N - u(uX, Y)_N - u(X, uY)_N + (uX, uY)_N = 0, \tag{1.3}$$

and that the contracted Lie product is

$$(X, Y)_{\circ} = u^{-1}(uX, uY)_R - u(X, Y)_N + (uX, Y)_N + (X, uY)_N. \tag{1.4}$$

Here the subscription R and N of Lie product mean respectively the projections into G_R and G_N defined by $u^n G = G_R$ and $u^n G_N = 0$ ($n = \mathbf{dim}\ G$). Let $f(t)$ and $g(t)$ be arbitrary functions such as $f(0)=0$ and $g(0)=1$, we can easily see by the same way as Saletan that even if we take $U_t = f(t)E + g(t)u$ in place of (1.2), the necessary and sufficient condition for (1.1) to exist is condition (1.3), and then the contracted Lie product is given by (1.4). And if A is a nonsingular matrix the contracted algebras with respsct to U_t and $U_t A$ are isomorphic, because

(1.5) $$\lim_{t\to 0}(U_tA)^{-1}((U_tA)X,\ (U_tA)Y)=A^{-1}\lim_{t\to 0}U_t^{-1}(U_t(AX),\ U_t(AY)).$$

Hence the contracted algebra with respect to $U_t=f(t)A+g(t)uA$ is isomorphic to the contracted algebra with respect to U_t of the form (1.2).

The purpose of this paper is to find all types of singular matrices u of the form (1.2) which can contract the Lie algebra of the motion group of 3-dimensional Euclidean space, and to determine the contracted algebras whith respect to u.

2. Lie subalgebras of Lie algebra of motion group. In this section we shall consider the Lie algebra G of motion group of 3-dimensional Euclidean space E_3 and its Lie subalgebras. In [4], Stoka has been obtained the canonical forms of all Lie subalgebras of the Lie algebra G, i. e., he has been proved that with a suitable coordinate transformation $y^j=a^j_ix^i+b^j$ $(i,\ j=1, 2, 3)$ of E_3 the Lie subalgebra of G is transformed into one of the forms, i. e., the canonical forms:

$$G_1^1=[X_4],\ G_1^2=[X_1],\ G_1^3=[X_3+X_4],$$

(2.1) $$G_2^1=[X_3,\ X_4],\ G_2^2=[X_1,\ X_2],$$

$$G_3^1=[X_4,\ X_5,\ X_6],\ G_3^2=[X_1,\ X_3,\ X_6],\ G_3^3=[X_1,\ X_2,\ X_3],$$

$$G_4=[X_1,\ X_2,\ X_3,\ X_4],$$

where $X_i=\frac{\partial}{\partial x^i}$ $(i=1, 2, 3)$, $x_{hk}=x^h\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^h}$ $(h,\ k=1, 2, 3)$; $(X_{12}=X_4,\ X_{23}=X_5,\ X_{31}=X_6)$. $X_1, X_2, \dots, X_6$ is a basis of G satisfying the following relations of Lie products:

$$(X_1,\ X_2)=(X_1,\ X_3)=(X_2,\ X_3)=(X_1,\ X_5)=(X_2,\ X_6)=(X_3,\ X_4)=0,$$

(2.2) $$(X_1,\ X_4)=-(X_3,\ X_5)=X_2,\ (X_2,\ X_5)=-(X_1,\ X_6)=X_3,\ (X_3,\ X_6)=-(X_2,\ X_4)=X_1,$$

$$(X_4,\ X_5)=-X_6,\ (X_5,\ X_6)=-X_4,\ (X_6,\ X_4)=-X_5.$$

LEMMA 2.1. *The subalgebra containing* G_2^1 *is only* G_4, *the subalgebras containing* G_2^2 *are* $[X_1,\ X_2,\ pX_3+qX_4]$ or $[X_1,\ X_2,\ X_3,\ pX_4+qX_5+rX_6]$, *the subalgebra containig* G_3^1 *does not exist, the subalgebra containing* G_3^2 *is only* $[X_1,\ X_2,\ X_3,\ X_6]$ *and the subalgebras containing* G_3^3 *are* $[X_1, X_2,\ X_3,\ pX_4+qX_5+rX_6]$.

PROOF. From (2.1) we may consider the subalgebra whose dimension is at most 4. Let H be any subalgebra of dimension 3 which contains G_2^1. We may take X_3, X_4, and $X=aX_1+bX_2+cX_5+dX_6$ as a basis of H. Since H contains $(X_3,\ X)=dX_1-cX_2$ and $(X_4,\ X)=-cX_1-dX_2$, $(c^2+d^2)X_1\in H$ and $(c^2+d^2)X_2\in H$. If $c^2+d^2\neq 0$, then $X_1,\ X_2\in H$ and hence **dim** $H\geqq 4$; we have $c=d=0$ i. e., $aX_1+bX_2\in H$. And since $(X_4,\ aX_1+b\ X_2)=bX_1-aX_2\in H$, $(a^2+b^2)X_1\in H$ and $(a^2+b^2)X_2\in H$, we have $a=b=0$. Thus we see that there is no subalgebra of dimension 3 which contains G_2^1.

Let H be any subalgebra of dimenson 4 which contains G_2^1, and let $a_1X_1+b_1X_2+c_1X_5+d_1X_6$ and $a_2X_1+b_2X_2+c_2X_5+d_2X_6$ be two independent vectors of H. If $c_1^2+d_1^2\neq 0$ or $c_2^2+d_2^2\neq 0$, by the same way as the above, we have $X_1,\ X_2\in H$. If $c_1=d_1=c_2=d_2=0$, H has two independent vectors $a_1X_1+b_1X_2$ and $a_2X_1+b_2X_2$, and hence

X_1, $X_2 \in H$. Thus the subalgebra of dimension 4 which contains G_2^1 is only G_4.

Let H be any subalgebra of dimension 3 which contains G_2^2. Let $X=aX_3+bX_4+cX_5+dX_6$ be any vector of H, since $(X_1, X)=bX_2-dX_3$ and $(X_2, X)=-bX_1+cX_3$, we have $bX_2-dX_3 \in H$ and $bX_1-cX_3 \in H$. If $c^2+d^2 \neq 0$, $X_3 \in H$; if $c^2+d^2=0$, $aX_3+bX_4 \in H$. Thus the subalgebras of dimension 3 which contain G_2^2 are $[X_1, X_2, pX_3+qX_4]$.

Let H be any subalgebra of dimension 4 which contains G_2^2, and let $a_1X_3+b_1X_4+c_1X_5+d_1X_5$ and $a_2X_3+b_2X_4+d_2X_6$ be two independent vectors of H. If $c_1^2+d_1^2 \neq 0$ or $c_2^2+d_2^2 \neq 0$, by the same way as the above, we have $X_3 \in H$. If $c_1=d_1=c_2=d_2=0$, H has two independent vectors $a_1X_3+b_1X_4$ and $a_2X_3+b_2X_4$, and hence X_3, $X_4 \in H$. Thus the subalgebras of dimension 4 which contain G_2^2 are $[X_1, X_2, X_3, pX_4+qX_5+rX_6]$.

Let H be any subalgebra of dimension 4 which contains G_3^1. Let $X=aX_1+bX_2+cX_3$ be any vector of H. Since $(X_4, X)=bX_1-aX_2$, $(X_5, X)=cX_2-bX_3$, $(X_6, X)=-cX_1+aX_3$, $(X_4, (X_4, X))=-aX_1-bX_2$, $(X_5, (X_5, X))=-bX_2-cX_3$ and $(X_6, (X_6, X))=-aX_1-cX_3$, we have $(a^2+b^2)X_1 \in H$, $(a^2+c^2)X_1 \in H$, $(a^2+b^2)X_2 \in H$, $(b^2+c^2)X_2 \in H$, $(b^2+c)X_3 \in H$ and $(a^2+c^2)X_3 \in H$. If $a^2+b^2+c^2 \neq 0$, then $X_1 \in H$, $H_2 \in H$ and $X_3 \in H$ and hence **dim** $H=6$; we have $a=b=c=0$. Thus the subalgebra which contains G_3^1 does not exist.

Let H be any subalgebra of dimension 4 which contains G_3^2. Let $X=aX_2+bX_4+cX_6$ be any vector of H, since $(X_1, X)=bX_2$ and $(X_3, X)=-cX_2$, we have bX_2, $cX_2 \in H$. If $b^2+c^2 \neq 0$, $X_2 \in H$; if $b=c=0$, $aX_2 \in H$ i. e., $X_2 \in H$. Thus the the subalgebra which contains G_3^2 is only $[X_1, X_2, X_3, X_6]$.

The last assertion of Lemma 2.1 is obvious, because X_1, X_2, X_3 and $pX_4+qX_5+rX_6$ form a basis of a subspace of dimension 4 which contains G_3^3 and then $[X_1, X_2, X_3, pX_4+qX_5+rX_6]$ becomes a subgebra.

3. Some general properties of contraction.

PROPOSITION 3.1. *Let u be a singulalar matrix such as $uG \supset u^2G \supset \dots \supset u^pG=u^{p+1}G \neq [0]$. Then for arbitrary choice of basis of G_R there exists a basis of G_N, in terms of which u can be written as $u=\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ where* **rank** $A=$ **dim** G_R, **rank** $C=$ **dim** G_N, **det** $A \neq 0$ *and* $C^p=0$.

PROOF. Let $X_1, X_2 \dots, X_m$ be a basis of $G_R=u^pG$ and $X_{m+1}, X_{m+2}, \dots, X_n$ be vectors of G such that $X_1, X_2, \dots, X_n$ form a basis of G. Since u is nonsinglar on G_R, we have

$$u^pX_j=c_j^iX_i=u^p(c_j^i(u^{-1})^pX_i) \; ; \; i=1, \dots, m, \; j=1, \dots, n.$$

If we put $Y_j=X_j-c_j^i(u^{-1})^pX_i$, $X_1, \dots X_m, Y_{m+1}, \dots, Y_n$ form a basis of G and then also u can be written as $u=\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$, where **rank** $A=$ **dim** G_R, **rank** $C=$ **dim** G_N and **det** $A \neq 0$, with respcet to this basis. Moreover $u^pY_j=0$, from which it follows that

$$C^p=0 \text{ and } D=BA^{p-1}+CBA^{p-2}+\dots+C^{p-2}BA+C^{p-1}B=0.$$

Then we have

$$CD=CBA^{p-1}+C^2BA^{p-2}+\ldots+C^{p-2}BA^2+C^{p-1}BA=0,$$

and hence

$$CDA^{-1}=CBA^{p-2}+C^2BA^{p-3}+\ldots+C^{p-2}BA+C^{p-1}B=0.$$

Therefore we have $D=BA^{p-1}=0$ i. e., $B=0$. This completes the proof.

PROPOSITION 3.2. *Let u be a matrix as in Proposition* 3.1 *and assume $p=1$. Then u satisfies* (1.3).

PROOF. From the Proposition 3.1 in this case we have $u=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$. Hence $uG_N=0$. We know that uG becomes a subalgebra of G ([2], p. 5), and since $uG=G_R$ we have $(uG, uG)_N=0$. Hence we see that (1.3) is satisfied.

We shall find the form of singular matrices u satisfying (1.3) for each of following cases:

I. $uG=u^2G\neq[0]$, II. $uG\supset u^2G\supset\ldots\supset u^pG=u^{p+1}G\neq[0]\,(p\neq1)$, III. $u^pG=[0]$.

Throughout the following sections, let X_i be the basis (2.2) of the Lie algebra G of the motion group and $Y_{j_1}, Y_{j_2}, \ldots Y_{j_k}$ be the basis of G_N chosen by Proposition 3.1 with respect to the basis $X_{i_1}, X_{i_2}, \ldots, X_{i_s}$ of G_R. And A and C denote the matrices in Proposition 3.1. It can be seen from the proof of Proposition 3.1 that Y_{j_q} is con- gruent to X_{j_q} modulo G_R $(q=1, \ldots, k)$.

4. Gase I. $uG=u^2G\neq[0]$. In this case from the Proposition 3.2 u satisfies (1.3), and from (2.1) with a suitable linear transformation of variables in E_3 u can be transformed into the following forms:

$$u_1=\begin{bmatrix} 0&0&0&a_1&0&0\\ 0&0&0&a_2&0&0\\ 0&0&0&a_3&0&0\\ 0&0&0&a_4&0&0\\ 0&0&0&a_5&0&0\\ 0&0&0&a_6&0&0 \end{bmatrix},\ \text{where } a_4\neq0, \qquad u_2=\begin{bmatrix} 0&0&a_1&a_1&0&0\\ 0&0&a_2&a_2&0&0\\ 0&0&a_3&a_3&0&0\\ 0&0&a_4&a_4&0&0\\ 0&0&a_5&a_5&0&0\\ 0&0&a_6&a_6&0&0 \end{bmatrix},\ \text{where } a_3+a_4\neq0,$$

$$u_3=\begin{bmatrix} a_1&0&0&0&0&0\\ a_2&0&0&0&0&0\\ a_3&0&0&0&0&0\\ a_4&0&0&0&0&0\\ a_5&0&0&0&0&0\\ a_6&0&0&0&0&0 \end{bmatrix},\ \text{where } a_1\neq0, \qquad u_4=\begin{bmatrix} 0&0&a_1&b_1&0&0\\ 0&0&a_2&b_2&0&0\\ 0&0&a_3&b_3&0&0\\ 0&0&a_4&b_4&0&0\\ 0&0&a_5&b_5&0&0\\ 0&0&a_6&b_6&0&0 \end{bmatrix},\ \text{where } \begin{vmatrix} a_3&b_3\\ a_4&b_4 \end{vmatrix}\neq0,$$

$$u_5=\begin{bmatrix} a_1&b_1&0&0&0&0\\ a_2&b_2&0&0&0&0\\ a_3&b_3&0&0&0&0\\ a_4&b_4&0&0&0&0\\ a_5&b_5&0&0&0&0\\ a_6&b_6&0&0&0&0 \end{bmatrix},\ \text{where } \begin{vmatrix} a_1&b_1\\ a_2&b_2 \end{vmatrix}\neq0, \quad u_6=\begin{bmatrix} 0&0&0&a_1&b_1&c_1\\ 0&0&0&a_2&b_2&c_2\\ 0&0&0&a_3&b_3&c_3\\ 0&0&0&a_4&b_4&c_4\\ 0&0&0&a_5&b_5&c_5\\ 0&0&0&a_6&b_6&c_6 \end{bmatrix},\ \text{where } \begin{vmatrix} a_4&b_4&c_4\\ a_5&b_5&c_5\\ a_6&b_6&c_6 \end{vmatrix}\neq0,$$

$$u_7=\begin{bmatrix} a_1 & 0 & b_1 & 0 & 0 & c_1 \\ a_2 & 0 & b_2 & 0 & 0 & c_2 \\ a_3 & 0 & b_3 & 0 & 0 & c_3 \\ a_4 & 0 & b_4 & 0 & 0 & c_4 \\ a_5 & 0 & b_5 & 0 & 0 & c_5 \\ a_6 & 0 & b_6 & 0 & 0 & c_6 \end{bmatrix}, \text{ where } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_6 & b_6 & c_6 \end{vmatrix} \neq 0, \quad u_8=\begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & 0 & 0 & 0 \\ a_5 & b_5 & c_5 & 0 & 0 & 0 \\ a_6 & b_6 & c_6 & 0 & 0 & 0 \end{bmatrix}, \text{ where } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

$$u_8=\begin{bmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & 0 & 0 & 0 \\ a_5 & b_5 & c_5 & d_5 & 0 & 0 & 0 \\ a_6 & b_6 & c_6 & d_6 & 0 & 0 & 0 \end{bmatrix}, \text{ where } \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \neq 0.$$

5. Case II. $\boldsymbol{uG \supset u^3G \supset \ldots \supset u^pG = u^{p+1}G \neq [0]}$ $(\boldsymbol{p \neq 1})$. Since $u^pG = G_R$ is a subalgebra of G ([2], p. 5), G_R is transformed into one of the canonical forms (2.1) with a suitable linear transformation of variables in E_3. Hence throughout in this section we shall assume that G_R is one of the forms (2.1).

(a). Let **dim** $G_R=1$, then $2 \leq p \leq 4$ because the dimension of a subalgebra of G is at most 4. If X is a nonzero vector of G_R, $uX=aX$ $(a \neq 0)$. From (1.3) we have

$$u(u(X,\ Y)_N-(X,\ uY)_N)=a(u(X,\ Y)_N-(X,\ uY)_N).$$

If we put $Z=u(X, Y)_N-(X, uY)_N$, $uZ=aZ$. Since $u^4Z \in G_R$ and $u^4Z \in G_N$, $u^4Z=a^4Z=0$ i. e., $Z=0$. Hence we have

$$u(X,\ Y)_N-(X,\ uY)_N=0, \text{ for } X \in G_R \text{ and any } Y \in G. \tag{5.1}$$

Let $G_R=G_1^1$. From (5.1) we hane $u(Y_4,\ Y_j)_N-(X_4,\ uY_j)_N=0$, from which it follows that

$$\begin{aligned}
&j=1\,;\ (a_1^2+a_2^1)Y_1-(a_1^1-a_2^2)Y_2+a_2^3Y_3+(a_1^6+a_2^5)Y_5-(a_1^5-a_2^6)Y_6=0,\\
&j=2\,;\ (a_1^1-a_2^2)Y_1+(a_1^2+a_2^1)Y_2+a_1^3Y_3+(a_1^5-a_2^6)Y_5+(a_1^6+a_2^5)Y_6=0,\\
&j=3\,;\ a_3^2Y_1-a_3^1Y_2+a_3^6Y_5-a_3^5Y_6=0,\\
&j=5\,;\ (a_5^2+a_6^1)Y_1-(a_5^1-a_6^2)Y_2+a_6^3Y_3+(a_5^6+a_6^5)Y_5-(a_5^5-a_6^6)Y_6=0,\\
&j=6\,;\ (a_5^1-a_6^2)Y_1+(a_5^2+a_6^1)Y_2+a_5^3Y_3+(a_5^5-a_6^6)Y_5+(a_5^6+a_6^5)Y_6=0,
\end{aligned}$$

where $uY_j=a_j^1Y_1+a_j^2Y_2+a_j^3Y_3+a_j^5Y_5+a_j^6Y_6$. Hence we see that the matrix C is of the form

$$C=\begin{bmatrix} A_{11} & 0 & A_{12} \\ 0 & a_3^3 & 0 \\ A_{21} & 0 & A_{22} \end{bmatrix}, \tag{5.2}$$

Here the matrices A_{ij} are of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. They form a field isomorphic to the complex mumber field. Since **rank** $C^3 \leq 1$ and $C^4=0$, by computing the powers of matrix C we have

$$A_{11}{}^3+A_{12}A_{21}(2A_{11}+A_{22})=A_{22}{}^3+A_{12}A_{21}(A_{11}+2A_{22})=0$$
$$A_{12}(A_{11}{}^2+A_{12}A_{21}+A_{11}A_{22}+A_{22}{}^2)=A_{21}(A_{11}{}^2+A_{12}A_{21}+A_{11}A_{22}+A_{22}{}^2)=0,$$
$$(A_{11}{}^2+A_{12}A_{21})^2+A_{12}A_{21}(A_{11}+A_{22})^2=(A_{22}+A_{12}A_{21})^2+A_{12}A_{21}(A_{11}+A_{22})^2=0,$$
$$A_{12}(A_{11}+A_{22})(A_{11}{}^2+2A_{12}A_{21}+A_{22}{}^2)=A_{21}(A_{11}+A_{22})(A_{11}{}^2+2A_{12}A_{21}+A_{22}{}^2)=0.$$

Hence we have $A_{11}+A_{22}=A_{11}{}^2+A_{12}A_{21}=0$. And since $uY_3=a_3^3Y_3$, we have $a_3^3=0$.

First assume that $A_{11}=0$, then $A_{12}A_{21}=0$ i. e., $A_{12}=0$ or $A_{21}=0$. If $A_{21}=0$; since $uY_3=0$ and $C^2=0$, from (1.3) we have $u(uY_1, Y_3)_N=0$ i. e., $2mnY_5+(m^2-n^2)Y_6=0$, where $A_{12}=\begin{bmatrix} n & m \\ -m & n\end{bmatrix}$. Hence we we have $mn=m^2-n^2=0$, from which it follows that $C=0$. This contradicts to $p\neq 1$. If $A_{12}=0$; since $uY_1=uY_2=uY_3=0$ and $C_2=0$, we have

$$u(Y_5,\ uY_6)_N+u(uY_5,\ Y_6)_N=(p^2+q^2)(a_1Y_1+a_2Y_2)=(uY_5,\ uY_6)_N,$$

where $A_{21}=\begin{bmatrix} p & q \\ -q & p\end{bmatrix}$ and $Y_j=X_j+a_jX_4$. ($j=1, 2, 3, 5, 6$). We can easily see that (1.3) is satisfied by the other pairs of the basis X_4, Y_j ($j=1, 2, 3, 5, 6$). Hence in this case u satisfies (1.3), and it is written as the following form with respect to the basis $X_1, \dots, X_6$ of G:

$$u_{10}=\begin{bmatrix} 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ b & c & 0 & a_5 & 0 & 0 \\ -c & b & 0 & a_6 & 0 & 0 \end{bmatrix} \quad \text{where } a_4\neq 0 \text{ and } b^2+c^2\neq 0.$$

Next assume that $A_{11}\neq 0$. Since $uG_N\in[Y_1,\ Y_2,\ Y_5,\ Y_6]$, from (1.3) the coefficient Y_3 in $(uY_j,\ uY_k)_N$ is zero. Hence we have

$$an+bm=nq-mp=ap+bq=a^2+b^2-np-mq=0,$$

where $A_{11}=\begin{bmatrix} a & b \\ -b & a\end{bmatrix}$, $A_{12}=\begin{bmatrix} n & m \\ -m & n\end{bmatrix}$ and $A_{21}=\begin{bmatrix} p & q \\ -q & p\end{bmatrix}$. And since $A_{11}{}^2+A_{12}A_{21}=0$, we have

$$a^2-b^2-np-mq=2ab+np+mq=0.$$

From these equations we have $a:b=-m:n=-q:p$ and $a^2-mq=b^2-np=0$. Since $A_{11}\neq 0$, we have $m=-ka$, $n=kb$, $p=sb$, $q=-sa$ and $ks=1$. Since from (1.3) the coefficients of Y_5 and Y_6 in $u(Y_3,\ uY_1)_N$ are zero, putting $Y_j=X_j+a_jX_4$ we have $k^2ab=k^2(a^2-b^2)=0$. Hence we have $k=0$ or $a=b=0$, but this contradicts to $A_{11}\neq 0$ or $ks=1$. Thus in this case u does not exist.

Let $G_R=G_1^3$. From (5.1) we have

$$j=1;\ (a_1^2+a_1^6+a_2^1)Y_1+(a_1^1-a_1^5-a_2^2)Y_2+a_2^3Y_3+(a_1^6+a_2^5)Y_5-(a_1^5-a_2^6)Y_6=0,$$
$$j=2;\ (a_1^1-a_2^2-a_2^6)Y_1+(a_1^2+a_2^1+a_2^5)Y_2+a_1^3Y_3+(a_1^5-a_2^6)Y_5+(a_1^6+a_2^5)Y_6=0,$$

$$j=3\,;\ (a_3^2+a_3^6)\,Y_1-(a_3^1+a_3^5)\,Y_3+a_3^6Y_5-a_3^5Y_6=0,$$

$$j=5\,;\ (a_2^1+a_5^2+a_5^6+a_6^1)\,Y_1+(a_2^2-a_5^1-a_5^5+a_6^2)\,Y_2$$
$$+(a_2^3+a_6^3)\,Y_3+(a_2^3+a_5^6+a_6^5)\,Y_5+(a_2^6-a_5^5+a_6^6)\,Y_6=0,$$

$$j=6\,;\ (a_1^1+a_1^5-a_6^2-a_6^6)\,Y_1+(a_1^5+a_5^2+a_6^1+a_6^5)\,Y_2$$
$$+(a_1^3+a_5^3)\,Y_3+(a_1^5+a_5^5-a_6^6)\,Y_5+(a_1^6+a_5^6+a_6^5)\,Y_6=0,$$

where $uY_j=a_j^1Y_1+a_j^2Y_2+a_j^3Y_3+a_j^5Y_5+a_j^6Y_6$. From the first three equations of the above we have $a_1^1-a_2^2=a_1^2+a_2^1=a_1^3=a_2^3=a_3^1=a_3^2=a_3^5=a_3^6=0$. Hence from the other three equations of the above we have $a_5^5-a_6^6=a_5^6+a_6^5=a_5^3=a_6^3=0$. Therefore the matrix C is also of the form (5.2), and hence similarly we have $A_{11}+A_{22}=A_{11}{}^2+A_{12}A_{21}=0$. By the same way as the above; if $A_{11}=A_{21}=0$ or $A_{11}\neq 0$ we have a contradiction, and if $A_{11}=A_{12}=0$, u satisfies (1.3). It is written as the following form with respect to the basis $X_1, \ldots, X_6$ of G:

$$u_{11}=\begin{bmatrix} 0 & 0 & a_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & a_4 & 0 & 0 \\ b & c & a_5 & a_5 & 0 & 0 \\ -c & b & a_6 & a_6 & 0 & 0 \end{bmatrix},\ \text{where } a_3+a_4\neq 0 \text{ and } b^2+c^2\neq 0.$$

Let $G_R=G_1^2$. From (5.1) i. e., $u(X_1,\ Y_j)_N-(X_1,\ uY_j)_N=0$, it follows that

$$j=2,\ 3,\ 5\,;\quad a_j^4Y_2-a_j^6Y_3=0,$$
$$j=4\,;\quad (a_2^2-a_4^4)\,Y_2+(a_2^3+a_4^6)\ Y_3+a_2^4Y_4+a_2^5Y_5+a_2^6Y_6=0,$$
$$j=6\,;\quad (a_3^2+a_6^4)\,Y_2+(a_3^3+a_6^6)\ Y_3+a_3^4Y_4+a_3^5Y_5+a_3^6Y_6=0,$$

where $uY_j=a_j^2Y_2+\ldots+a_j^6Y_6$. Hence we have $a_2^4=a_2^5=a_2^6=a_3^4=a_3^5=a_3^6=a_5^4=a_5^6=0$ and $a_2^2-a_4^4=a_2^3+a_4^6=a_3^2+a_6^4=a_3^3-a_6^6=0$. Then since $uY_j=a_j^2Y_2+a_j^3Y_3$ $(j=2,\ 3)$ and $uY_5=a_5^2Y_2+a_5^3Y_3+a_5^5Y_5$, we have $u^pY_5=kY_2+sY_3+(a_5^5)^pY_5$; and since $u^pG_N=0$, we have $a_5^5=0$. Moreover since $u^pY_j=0$ and $uY_j=a_j^2Y_2+a_j^3Y_3$ $(j=2, 3)$, we have $u^2Y_2=u^2Y_3=0$. Then since $u^2(Y_j,\ Y_5)_N=u(Y_j,\ uY_5)=(uY_j,\ uY_5)=0$ $(j=2,\ 3)$, from (1. 3) we have $u(uY_j,\ Y_5)_N=0$ $(j=2,\ 3)$ i. e.,

$$a_2^2(a_2^3-a_3^2)\,Y_2+((a_2^3)^2-a_2^2a_3^3)\,Y_3=0 \text{ and } (a_2^2a_3^3-(a_3^2)^2)\,Y_2+a_3^3(a_2^3-a_3^2)\,Y_5=0.$$

Hence we have $a_2^3=a_3^2$. From this and $u^2Y_2=u^2Y_3=0$ we have $uY_2=uY_3=0$, and then $a_4^4=a_4^6=a_6^4=a_6^6=0$ i. e., $uY_4=a_4^2Y_2+a_4^3Y_3+a_4^5Y_5$ and $uY_6=a_6^2Y_2+a_6^3Y_3+a_6^6Y_5$. Since $u^pY_4=u^pY_6=0$, by the same way as the above we have $a_4^5=a_6^5=0$. Thus we have $uY_2=uY_3=0$ and $uY_j\in[Y_2,\ Y_3]$ $(j=4,\ 5,\ 6)$. In this case u satisfies (1.3), and u is written as the following form with respect to the basis $X_1, \ldots, X_6$ of G:

$$u_{12}=\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 \\ a_4 & b & e & 0 & 0 & 0 \\ a_5 & c & f & 0 & 0 & 0 \\ a_6 & d & g & 0 & 0 & 0 \end{bmatrix},\ \text{where } a\neq 0 \text{ and } b^2+c^2+\ldots+g^2\neq 0.$$

(b). Let **dim** $G_R=2$. And let $G_R=G_2^1$, then from the Lemma 2.1 we have $uG=G_4=[X_1,\ X_2,\ X_3,\ X_4]$ and $u^2G=G_R$. Hence we have $u^2G_N=0$ and $uG_N\subset[Y_1,\ Y_2]$. Therefore from (1.3) we have $u(uX_3,\ Y_j)_N-(uX_3,\ uY_j)_N=0$ i. e.,

$$j=1\ ;\ a_3^4(a_1^2+a_2^1)\,Y_1-a_3^4(a_1^1-a_2^2)\,Y_2=0,$$
$$j=2\ ;\ a_3^4(a_1^1-a_2^2)\,Y_1+a_3^4(a^2+a_2^1)\,Y_2=0,$$
$$j=5\ ;\ (a_2^1a_3^3+a_3^4a_5^2+a_3^4a_6^1)\,Y_1+(a_2^2a_3^3-a_3^4a_5^1+a_3^4a_6^2)\,Y_2=0,$$
$$j=6\ ;\ (a_1^1a_3^3+a_3^4a_5^1-a_3^4a_6^2)\,Y_1+(a_1^2a_3^3+a_3^4a_5^2+a_3^4a_6^1)\,Y_2=0,$$

where $uX_i=a_i^3X_3+a_i^4X_4$ $(i=3,\ 4)$ and $uY_j=a_j^1Y_1+a_j^2Y_2$ $(j=1,\ 2,\ 5,\ 6)$. If $a_3^4\neq0$, from the first three equations of the above we have $a_1^1-a_2^2=a_1^2+a_2^1=0$; then $u^2Y_1=((a_1^1)^2-(a_1^2)^2)\,Y_1+2a_1^1a_1^2Y_2=0$ i. e., $a_1^1=a_1^2=a_2^1=a_2^2=0$. Hence from the other equations of the above we have $a_5^1-a_6^2=a_5^2+a_6^1=0$. If $a_3^4=0$, since u is nonsingular on G_R we have $a_3^3\neq0$. Hence from the last three equations of the above we have $a_1^1=a_1^2=a_2^1=a_2^2=0$. Then since $u(X_4,\ uY_5)_N=0$, from (1.3) we have $u(uX_4,\ Y_5)_N-u\,(uX_4,\ uX_5)_N=0$ i. e., $a_4^4(a_5^2+a_6^1)\,Y_1-a_4^4(a_5^1-a_6^2)\,Y_2=0$. Since u is nonsingular on G_R, we have $a_4^4\neq0$; and then $a_5^1-a_6^2=a_5^2+a_6^1=0$. Therefore the matrix u must be $uY_1=uY_2=0$, $uY_5=pY_1+qY_2$ and $uY_6=-qY_1+pY_2$. This matrix u satisfies (1.3) ; because we have

$$u(uY_5,\ Y_6)_N+u(Y_5,\ uY_6)_N=(p^2+q^2)\,(b_1Y_1+b_2Y_2)=(uY_5\ ,uY_6)_N,$$
$$u(uX_i,\ Y_5)_N+u(X_i,\ uY_5)_N=a_i^4(qY_1-pY_2)=(uX_i,\ uY_5)_N\ ;\ i=3,\ 4,$$
$$u(uX_i,\ Y_6)_N+u(X_i,\ uY_6)_N=a_i^4(pY_1+qY_2)=(uX_i,\ uY_6)_N\ ;\ i=3,\ 4,$$

where $Y_j=X_j+a_jX_3+b_jX_4$ $(j=1,\ 2,\ 5,\ 6)$, and can easily see that (1.3) is satisfied by the other pairs of the basis $X_i,\ Y_j$. The matrix u is written as the following form with respect to the basis $X_1,\dots,\ X_6$ of G:

$$u_{13}=\begin{bmatrix} 0 & 0 & a_1 & b_1 & 0 & 0\\ 0 & 0 & a_2 & b_2 & 0 & 0\\ 0 & 0 & a_3 & b_3 & 0 & 0\\ 0 & 0 & a_4 & b_4 & 0 & 0\\ c & d & a_5 & b_5 & 0 & 0\\ -d & c & a_6 & b_6 & 0 & 0 \end{bmatrix},\ \text{where}\ \begin{vmatrix} a_3 & b_3\\ a_4 & b_4 \end{vmatrix}\neq0\ \text{and}\ c^2+d^2\neq0.$$

Let $G_R=G_2^2$, then from the Lemma 2.1 we have $uG=[X_1,\ X_2,\ pX_3+qX_4]$ or $uG=[X_1,\ X_2,\ X_3,\ pX_4+pX_5+rX_6]$. First assume that $uG=[X_1,\ X_2,\ pX_3+qX_4]$, then we have $uY_j=k_j(pY_3+qY_4)$ $(j=3,\ 4,\ 5,\ 6)$. Since $u^2G=G_R$, we have $u^2G_N=0$. And since $(uG,\ uG)_N=u(X_i,\ uG)_N=0$ $(i=1,\ 2)$, from (1.3) we have $u(uX_i,\ Y_5)_N=u(uX_i,\ Y_6)_N=0$ i. e., $a_i^1uY_3=a_i^2uY_3=0$ where $uX_i=a_i^1X_1+a_i^2X_2$ $(i=1,\ 2)$. Since u is nonsingular on G_R, there exist $a_i^j\neq0$; hence we have $uY_3=0$. Hence $u^2Y_j=(k_j)^2q(pY_3+qY_4)=0$ i. e., $k_jq=0$. Therefore we may put $uY_j=k_jY_3$. In this case we can easily see that u satisfies (1.3), and it is written as the following form with respect to the basis $X_1,\dots,\ X_6$ of G:

$$u_{14}=\begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 & 0 \\ a_3 & b_3 & 0 & 0 & 0 & 0 \\ a_4 & b_4 & c & 0 & 0 & 0 \\ a_5 & b_5 & d & 0 & 0 & 0 \\ a_6 & b_6 & e & 0 & 0 & 0 \end{bmatrix}, \text{ where } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \text{ and } c^2+d^2+e^2\neq 0.$$

Next assume $uG=[X_1, X_2, X_3, pX_4+qX_5+rX_6]$, then we have $uY_j=k_jY_3+s_j(pY_4+qY_5+rY_6)$ ($j=3, 4, 5, 6$). Since $(X_i, Y_3)=(uX_i, Y_3)=0$ ($i=1, 2$), from (1.3) we have $u(X_i, uY_3)_N-(uX_i, uY_3)_N=0$ i. e.,

$$i=1;\ s_3(k_3r+a_1^2q-a_1^1r)Y_3+s_3^2prY_4+s_3^2qrY_5+s_3^2r^2Y_6=0,$$
$$i=2;\ s_3(k_3q-a_2^2q+a_2^1r)Y_3+s_3^2pqY_4+s_3^2q^2Y_5+s_3^2qrY_6=0,$$

where $uX_i=a_i^1X_1+a_i^2X_2$ ($i=1, 2$). If $s_3\neq 0$ we have $q=r=0$. Hence we may put $uY_j=k_jY_3+s_jY_4$. Then since $u(X_i, uG)_N=u(uX_i, uG)_N=0$ ($i=1, 2$), from (1.3) we have

$$u(uX_1, Y_5)_N=u(uX_2, Y_6)_N=0,$$
$$u(uX_1, Y_6)_N-u^2(X_1, Y_6)_N=0,$$
$$u(uX_2, Y_5)_N-u^2(X_2, Y_5)_N=0,$$

i. e., $a_1^2uY_3=a_2^1uY_3=0$ and $u^2Y_3=a_1^1uY_3=a_2^2uY_3$. Since $s_3\neq 0$ i. e., $uY_3\neq 0$; we have $a_1^2=a_1^2=0$, and from $u^pY_3=(a_1^1)^puY_3=(a_2^2)^puY_3=0$ (p is some integer) we have $a_1^1=a_2^2=0$. This is a contradiction because u is nonsingular on G_R. If $s_3=0$ we have $uY_3=k_3Y_3$, and hence $u^pY_3=(k_3)^pY_3=0$ i. e., $k_3=0$. Then since $u^pY_j=(s_j)^pY_4+(s_j)^{p-1}k_jY_3=0$, we have $s_j=0$ i. e., $uY_j=k_jY_3$ ($j=4, 5, 6$). Hence we have $uG\subset[X_1, X_2, X_3]$, but this contradicts to $\mathbf{dim}\ uG=4$.

(c). Let $\mathbf{dim}\ G_R=3$. Int his case we see that $\mathbf{dim}\ uG=4$, and that $G_R=G_2^3$ or $G_R=G_3^3$ because from the Lemma 2.1 there is no subalgebra containing G_3^1. Let $G_R=G_3^2$, then from the Lemma 2.1 we have $uG=[X_1, X_2, X_3, X_6]$. Since $u^2G=G_R$, we have $u^2G_N=0$. And since $(uG, uG)_N=(X_i, uG)_N=0$ ($i=1, 3, 6$), from (1.3) we have $u(uX_i, Y_4)_N=u(X_i, Y_5)_N=0$ i. e.,

$$a_i^1uY_2-a_i^6uY_5=a_i^3uY_2-a_i^4uY_4=0,\ i=1, 3, 6;$$

where $uX_i=a_i^1X_1+a_i^3X_3+a_i^6X_6$ ($i=1, 3, 6$). Let A_i^j be the cofactor of the matrix

$$A=\begin{bmatrix} a_1^1 & a_1^3 & a_1^6 \\ a_3^1 & a_3^3 & a_3^6 \\ a_6^1 & a_6^3 & a_6^6 \end{bmatrix}.$$

Then since u is nonsingular on G_R i. e., $\mathbf{det}\ A\neq 0$, there exist $A_i^1\neq 0$ and $a_j^1\neq 0$. Hence from $A_i^1\neq 0$ we have $uY_2=uY_4=0$ and $a_i^6uY_5=0$, and from $a_j^1\neq 0$ we have $uY_5=0$. Therefore we have $uG=G_R$, but this contradicts to $\mathbf{dim}\ uG=4$.

Let $G_R=G_3^3$, then from the Lemma 2.1 we have $uG=[X_1, X_2, X_3, pX_4+qX_5+rX_6]$.

Hence we have $uY_j=k_j(pY_4+qY_5+rY_6)$ i. e., the matrix C is the following form

$$C=\begin{bmatrix} k_4p & k_4q & k_4r \\ k_5p & k_5q & k_5r \\ k_6p & k_6q & k_6r \end{bmatrix}.$$

Since $u^2G_N=0$ i. e., $C^2=0$, by computing the powers of matrix C we have $k_4p+k_5q+k_6r=0$. Therefore since $u^2G_N=(uY_j,\ uY_k)=0$ $(j,\ k=4,\ 5,\ 6)$, from (1.3) we have $u(Y_j,\ uY_k)_N+u(Y_j,\ uY_k)_N=0$ i. e.,

$$j=4,\ k=5;\ r(k_4^2+k_5^2+k_6^2)(pY_4+qY_5+rY_6)=0,$$
$$j=4,\ k=6;\ q(k_4^2+k_5^2+k_6^2)(pY_4+qY_5+rY_6)=0,$$
$$j=5,\ k=6;\ p(k_4^2+k_5^2+k_6^2)(pY_4+qY_5+rY_6)=0.$$

Since $pY_4+qY_5+rY_6\neq 0$, we have $k_4=k_5=k_6=0$; but this contradicts to **dim** $uG=4$. Thus in this case the matrix which satisfies (1.3) does not exist.

6. Case III. $u^p\mathbf{G}=[\mathbf{0}]$. Since uG is a subalgebra of G, we shall also assume that uG is the canonical form (2.1) as in section 5. Put $uX_i=a_i^jX_j$; then since $G_R=0$ and $G_N=G$, by straightforward computations of (1.3) we have the followings:

(1) $a_1^4uX_1+a_2^4uX_2-(a_1^5+a_2^6)uX_3-(uX_1,\ uX_2)=0,$

(2) $a_1^6uX_1-(a_1^5+a_3^4)uX_2+a_3^6uX_3-(uX_3,\ uX_1)=0,$

(3) $-(a_2^6+a_3^4)uX_1+a_2^5uX_2+a_3^5uX_3-(uX_2,\ uX_3)=0,$

(4) $-(a_1^3-a_5^4)uX_2+(a_1^2-a_5^6)uX_3+a_1^6uX_4-a_1^4uX_6-(uX_1,\ uX_5)=0,$

(5) $(a_2^3-a_6^4)uX_1-(a_2^1-a_5^5)uX_3-a_2^5uX_4+a_2^4uX_5-(uX_2,\ uX_6)=0,$

(6) $-(a_3^2-a_4^6)uX_1+(a_3^1-a_5^4)uX_2-a_3^6uX_5+a_3^5uX_6-(uX_3,\ uX_4)=0,$

(7) $-u^2X_2-a_1^2uX_1+(a_1^1+a_4^4)uX_2-a_4^6uX_3-a_1^6uX_5+a_1^5uX_6-(uX_1,\ uX_4)=0,$

(8) $u^2X_2+a_5^6uX_1-(a_3^3+a_5^5)uX_2+a_3^2uX_3+a_3^6uX_4-a_3^4uX_6-(uX_3,\ uX_5)=0,$

(9) $-u^2X_3-a_5^4uX_1-a_2^3uX_2+(a_2^2+a_5^5)uX_3+a_2^6uX_4-a_2^4uX_6-(uX_2,\ uX_5)=0,$

(10) $u^2X_3+a_1^3uX_1+a_6^4uX_2-(a_1^1+a_6^6)uX_3-a_1^5uX_4+a_1^4uX_5-(uX_1,\ uX_6)=0,$

(11) $-u^2X_1+(a_3^3+a_6^6)uX_1-a_6^5uX_2-a_3^1uX_3-a_3^5uX_4+a_3^4uX_5-(uX_3,\ uX_6)=0,$

(12) $u^2X_1-(a_2^2+a_4^4)uX_1+a_2^1uX_2+a_4^5uX_3-a_2^6uX_5+a_2^5uX_6-(uX_2,\ uX_4)=0,$

(13) $u^2X_6+a_5^2uX_1-(a_4^3+a_5^1)uX_2+a_4^2uX_3+a_4^6uX_4+a_5^6uX_5-(a_4^4+a_5^5)uX_6-(uX_4,\ uX_5)=0,$

(14) $u^2X_5-(a_4^3+a_6^2)uX_1+a_6^1uX_2+a_4^1uX_3+a_4^5uX_4-(a_4^4+a_6^6)uX_5+a_6^5uX_6-(uX_6,\ uX_4)=0,$

(15) $u^2X_4+a_5^3uX_1+a_6^3uX_2-(a_5^1+a_6^2)uX_3-(a_5^5+a_6^6)uX_4+a_5^4uX_5+a_6^4uX_6-(uX_5,\ uX_6)=0.$

(a). Let **dim** $uG=1$. In this case we have $u^2G=(uG,\ uG)=0$. Let $uG=G_1^1$; then putting $uX_i=a_iX_4$, from (1) and (15) we have $a_1uX_1+a_2uX_2=0$ and $a_5uX_5+a_6uX_6=0$ i. e., $(a_1)^2X_4+(a_2)^2X_4=0$ and $(a_5)^2X_4+(a_6)^2X_4=0$. Hence we have $a_1=a_2=a_5=a_6=0$. Since $u^2X_4=(a_4)^2X_4=0$, we have $a_4=0$. Therefore we have $uX_i=0$ $(i=1,\ 2,\ 4,\ 5,\ 6)$. In this case we can see that u satisfies (1), ..., (15), and it is written as the following form;

$$u_{15}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&a&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \text{ where } a\neq 0.$$

Let $uG=G_1^2$, then putting $uX_i=a_iX_1$, from (11) and (12) we have $a_2uX_2=a_3uX_3=0$ i. e., $(a_2)^2X_1=(a_3)^2X_1=0$. Hence we have $a_2=a_3=0$. Since $u^2X_1=(a_1)^2X_1=0$, we have $a_1=0$. Therefore we have $uX_1=uX_2=uX_3=0$. In this case we can see that u satisfies (1), ..., (15), and it is written as the following form:

$$u_{16}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ a&0&0&0&0&0\\ b&0&0&0&0&0\\ c&0&0&0&0&0 \end{bmatrix}, \text{ where } a^2+b^2+c^2\neq 0.$$

Let $uG=G_1^3$, then putting $uX_i=a_i(X_3+X_4)$, from (1) and (15) we have $a_1uX_1+a_2uX_2=0$ and $a_5uX_1+a_6uX_2+a_5uX_5+a_6uX_6=0$ i. e., $((a_1)^2+(a_2)^2)(X_3+X_4)=0$ and $(a_1a_5+a_2a_6+(a_5)^2+(a_6)^2)(X_3+X_4)=0$. Hence we have $a_1=a_2=a_5=a_6=0$. Since $u^2(X_3+X_4)=(a_3+a_4)^2(X_3+X_4)=0$, we have $a_3+a_4=0$. Therefore we have $uX_1=uX_2=uX_5=uX_6=0$, $uX_3=a_3(X_3+X_4)$ and $uX_4=-a_3(X_3+X_4)$. In this case we can see that u satisfies (1), ..., (15), and it is written as the following form:

$$u_{17}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&a&a&0&0\\ 0&0&-a&-a&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \text{ where } a\neq 0.$$

(b). Let **dim** $uG=2$. Let $uG=G_2^1$. Then putting $uX_i=a_i^3X_3+a_i^4X_4$; since $(uG, uG)=0$, from (1) we have $a_1^4uX_1+a_2^4uX_2=0$ i. e., $(a_1^3a_1^4+a_2^3a_2^4)X_3+((a_1^4)^2+(a_2^4)^2)X_4=0$. Hence we have $a_1^4=a_2^4=0$. Then from (2) and (3) we have $a_3^4uX_1=a_3^4uX_2=0$ i. e., $a_1^3a_3^4X_3=a_2^3a_3^4X_3=0$. If $a_3^4\neq 0$; $a_1^3=a_2^3=0$ i. e., $uX_1=uX_2=0$, and hence from (8) and (11) we have $a_3^4uX_6=a_3^4uX_5=0$ i. e., $uX_5=uX_6=0$. Hence from (9) and (15) we have $u^2X_3=u^2X_4=0$ i. e.,

$$\begin{bmatrix} a_3^3 & a_3^4\\ a_4^3 & a_4^4 \end{bmatrix}^2=0, \text{ i. e., } \begin{vmatrix} a_3^3 & a_3^4\\ a_4^3 & a_4^4 \end{vmatrix}=0.$$

Therefore uX_3 and uX_4 are linearly dependent. Then we have $uG=[uX_3]$, but this contradicts to **dim** $uG=2$. If $a_3^4=0$; since $u^pX_3=(a_3^3)^pX_3=0$, we have $a_3^3=0$ i. e., $uX_3=$

0. Hence (7) and (12) we have $a_1^2a_4^4X_3=a_2^3a_4^4X_3=0$. Assume $a_4^4\neq 0$, then we have $a_1^3=a_2^3=0$ i. e., $uX_1=uX_2=0$. Hence from (14) and (13) we have $u^2X_5-a_4^4uX_5=0$ and $u^2X_6-a_4^4uX_6=0$. Then since $u^pX_5=(a_4^4)^{p-1}uX_5=0$ and $u^pX_6=(a_4^4)^{p-1}uX_6=0$, we have $uX_5=uX_6=0$; and hence we have $uG=[uX_4]$. But this contradicts to **dim** $uG=2$. Therefore $a_4^4=0$; then since $u^2X_4=0$, from (15) we have $a_5^3uX_1+a_6^3uX_2+a_5^4uX_5+a_6^4uX_6=0$. Then since $uX_i=a_i^3X_3$ $(i=1, 2)$, we have $((a_5^4)^2+(a_6^4)^2)X_4=0$ i. e., $a_5^4=a_6^4=0$. Hence from (9) and (10) we have $a_1^3uX_1=a_2^3uX_2=0$ i. e., $(a_1^3)^2X_3=(a_2^3)^2X_3=0$. Hence we have $a_1^3=a_2^3=0$ i. e., $uX_1=uX_2=0$. Threfore we have $uG=[X_3]$, but this contradicts to **dim** $uG=2$. Thus in this case u does not exist.

Let $uG=G_2^2$. Then putting $uX_i=a_i^1X_1+a_i^2X_2$; since $(uG, uG)=0$, from (4) and (5) we have $a_1^2uX_3=a_2^1uX_3=0$. If $uX_3\neq 0$ we have $a_1^2=a_2^1=0$. Then since $u^pX_1=(a_1^1)^pX_1=0$ and $u^pX_2=(a_2^2)^pX_2=0$, we have $a_1^1=a_2^2=0$ i. e., $uX_1=uX_2=0$. Hence from (8) and (11) we have $a_3^1uX_3=a_3^2uX_3=0$, but this contradicts to $uX_3\neq 0$. Therefore we have $uX_3=0$; then from (8) and (11) we have $u^2X_1=u^2X_2=0$, and hence from (7) and (12) we have $a_1^2uX_1-a_1^1uX_2=0$ and $a_2^2uX_1-a_2^1uX_2=0$ i. e., $a_1^1(a_1^2-a_2^1)X_1+((a_1^2)^2-a_1^1a_2^2)X_2=0$ and $(a_1^1a_2^2-(a_2^1)^2)X_1+a_2^2(a_1^2-a_2^1)X_2=0$. Hence we have $a_1^2=a_2^1$. Then since $u^pX_i=0$ $(i=1, 2)$, we have $a_1^1=a_1^2=a_2^1=a_2^2=0$ i. e., $uX_1=uX_2=0$. In this case we can see that u satisfies (1), ..., (15), and it is written as the following form:

$$u_{18}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ a&d&0&0&0&0\\ b&e&0&0&0&0\\ c&f&0&0&0&0 \end{bmatrix}, \text{ where } \textbf{rank} \begin{vmatrix} a & d\\ b & e\\ c & f \end{vmatrix}=2.$$

(c) Let **dim** $uG=3$. Let $uG=G_3^1$. Then since uG is isomorphic to the 3-dimensional rotation algebra, uG has no 2-dimensional subalgebras. Hence the dimension of the subalgebra u^2G of the algebra uG is at most 1. Therefore since $uX_i \in u^2G$ $(i=4, 5, 6)$, we shall put $uX_5=kuX_4$ and $uX_6=suX_4$. Then putting $uX_i=a_i^4X_4+a_i^5X_5+a_i^6X_6$ $(i=1, 2, 3, 4)$; since $(uX_j, uX_k)=0$ $(j, k=4, 5, 6)$, from (13), (14) and (15) we have

$$u^2X_6+a_4^6uX_4+ka_4^6uX_5-(a_4^4+ka_4^5)uX_6=0,$$
$$u^2X_5+a_4^5uX_4-(a_4^4+sa_4^6)uX_5+sa_4^5uX_6=0,$$
$$u^2X_4-(ka_4^5+sa_4^6)uX_4+ka_4^4uX_5+sa_4^4uX_6=0,$$

i. e., $a_4^6(1+k^2+s^2)uX_4=0$, $a_4^5(1+k^2+s^2)uX_4=0$ and $a_4^4(1+k^2+s^2)uX_4=0$. Hence we have $uX_4=0$ and then $uX_5=uX_6=0$. Therefore from (1), ..., (9) we have the following:

(1)′ $$(a_1^4)^2+(a_2^4)^2-(a_1^5+a_2^6)a_3^4+(a_1^5a_2^6-a_1^6a_2^5)=0,$$

(2)′ $$a_1^4a_1^5+a_2^4a_2^5-(a_1^5+a_2^6)a_3^5+(a_1^6a_2^4-a_1^4a_2^6)=0,$$

(3)′ $$a_1^4a_1^6+a_2^4a_2^6-(a_1^5+a_2^6)a_3^6+(a_1^4a_2^5-a_1^5a_2^4)=0,$$

(4)′ $$a_1^6a_1^4-(a_1^5+a_3^4)a_2^4+a_3^6a_3^4+(a_1^6a_3^5-a_1^5a_3^6)=0,$$

(5)′ $$a_1^6a_1^5-(a_1^5+a_3^4)a_2^5+a_3^6a_5^3+(a_1^4a_3^6-a_1^6a_3^4)=0,$$

$$(6)' \qquad (a_1^6)^2-(a_1^5+a_3^4)a_2^6+(a_3^6)^2+(a_1^5a_3^4-a_1^4a_3^5)=0,$$
$$(7)' \qquad -(a_2^6+a_3^4)a_1^4+a_2^4a_2^5+a_3^5a_3^4+(a_2^5a_3^6-a_2^6a_3^5)=0,$$
$$(8)' \qquad -(a_2^6+a_3^4)a_1^5+(a_2^5)^2+(a_3^5)^2+(a_2^6a_3^4-a_2^4a_3^6)=0,$$
$$(9)' \qquad -(a_2^6+a_3^4)a_1^6+a_2^5a_2^6+a_3^5a_3^6+(a_2^4a_3^5-a_2^5a_3^4)=0.$$

From the above we have

$$\begin{bmatrix} a_1^4 & a_2^4 & -(a_1^5+a_2^6) \\ a_1^6 & -(a_1^5+a_3^4) & a_1^6 \\ -(a_2^6+a_3^4) & a_2^5 & a_3^5 \end{bmatrix} \begin{bmatrix} a_1^4 & a_1^5 & a_4^1 \\ a_2^4 & a_2^5 & a_2^6 \\ a_3^4 & a_3^5 & a_3^6 \end{bmatrix} + \begin{bmatrix} A_3^4 & A_3^5 & A_3^6 \\ A_2^4 & A_2^5 & A_2^6 \\ A_1^4 & A_1^5 & A_1^6 \end{bmatrix} = 0,$$

where A_i^j is the cofactor of the coefficient a_i^j of the matrix $A=[a_i^j]$ $(i=1, 2, 3;\ j=4, 5, 6)$. Then we have

$$\mathbf{det}\ A \begin{bmatrix} a_1^4 & a_2^4 & -(a_1^5+a_2^6) \\ a_1^6 & -(a_1^5+a_3^4) & a_1^6 \\ -(a_2^6+a_3^4) & a_2^5 & a_3^5 \end{bmatrix} + \begin{bmatrix} A_3^4 & A_3^5 & A_3^6 \\ A_2^4 & A_2^5 & A_2^6 \\ A_1^4 & A_1^5 & A_1^6 \end{bmatrix} \begin{bmatrix} A_1^4 & A_2^4 & A_3^4 \\ A_1^5 & A_2^5 & A_3^5 \\ A_1^6 & A_2^6 & A_3^6 \end{bmatrix} = 0.$$

Since **dim** $uG=3$ we see **det** $A\neq 0$. Hence from the above we have $a_1^4=a_3^5$, $a_2^4=a_3^6$ and $a_1^6=a_2^5$. Then from (2)′ we have $a_1^4a_2^6-a_2^4a_1^6=0$ i. e., $a_1^4:\ a_1^6=a_2^4:\ a_2^6$; and hence putting $a_2^4=ka_1^4$ and $a_2^6=ka_1^6$, from (2)′ we have $a_1^4(a_1^6-ka_1^5)=0$. If $a_1^4\neq 0$ we see that $a_1^6=ka_1^5$, and that $a_2^4:a_2^5:\ a_2^6=a_2^4:a_1^6:a_2^6=ka_1^4:\ ka_1^5:\ ka_1^6$. If $k\neq 0$, uX_1 and uX_2 are linearly dependent; and if $k=0$, $a_2^4=a_2^5=a_2^6=0$ i. e., $uX_2=0$. Hence we have $uG=[uX_1, uX_3]$, but this contradicts to **dim** $uG=3$. Therefore we have $a_1^4=a_3^5=0$, and then from (2)′, (3)′ and (5)′ we have $a_1^6a_2^4=a_1^5a_2^4=a_1^6a_3^4=0$. If $a_2^4\neq 0$ we have $a_1^5=a_1^6=0$, i. e., $uX_1=0$, but this contradicts to **dim** $uG=3$. Hence we have $a_2^4=a_3^6=0$. Since $uX_3\neq 0$ and $a_3^5=a_3^6=0$, we have $a_3^4\neq 0$; and then we have $a_1^6=a_2^5=0$. Then from (1)′ and (6)′ we have $(a_1^5+a_2^6)a_3^4-a_1^5a_2^6=0$ and $(a_1^5+a_3^4)a_2^6-a_1^5a_3^4=0$, from which we have $a_2^6a_3^4=0$. Since $a_3^4\neq 0$, we have $a_2^6=0$ i. e., $uX_3=0$; but this contradicts to **dim** u$G=3$. Thus in this case u does not exist.

Let $uG=G_3^2$. Then putting $uX_i=a_i^1X_1+a_i^3X_3+a_i^6X_6$; from (2) we have $a_1uX_1+a_3uX_3+(uX_1, uX_3)=0$. Then since $(uX_1, uX_3)\in[X_1, X_3]$, from which we have $((a_1^6)^2+(a_3^6)^2)X_6=0$ i. e., $a_1^6=a_3^6=0$. From (1) we have $a_2^6uX_3+(uX_1, uX_2)=0$ i. e., $a_2^6(a_1^3+a_3^1)X_1-a_2^6(a_1^1-a_3^3)X_3=0$. If $a_2^6=0$, we have $uX_i\in[X_1, X_3]$ $(i=1, 2, 3)$, and then from (13) we have $((a_4^6)^2+(a_5^6)^2+(a_6^6)^2)X_6=0$ i. e., $a_4^6=a_5^6=a_6^6$. Hence we have $uG\subset[X_1, X_3]$, but this contradicts to **dim** $uG=3$. Therefore we have $a_2^6\neq 0$, and then $a_1^1-a_3^3=a_1^3+a_3^1=0$. Since $u^pX_i=0$ $(i=1, 2)$, from which we have $a_1^1=a_1^3=a_3^1=a_3^3=0$ i. e., $uX_1=uX_3=0$. Then from (7) we have $u^2X_2=0$ i. e., $a_2^6uX_6=0$; and since $a_2^6\neq 0$ we have $uX_6=0$. Therefore u must be $uX_1=uX_3=uX_6=0$, and in this case we can see that u satisfies (1), (2),…, (8), (10), (11), (14) and (15). From (9), (12) and (13) we have

$$a_2^3uX_2-a_2^6uX_4+(uX_2,\ uX_5)=0,\ a_2^1uX_2-a_2^6uX_5-(uX_2,\ uX_4)=0,$$
$$(a_4^3+a_5^1)uX_2-a_4^6uX_4-a_5^6uX_5+(uX_4,\ uX_5)=0,$$

i. e.,

$$(a_2^1a_2^3-a_2^6a_4^1+a_2^3a_5^6-a_2^6a_5^3)X_1+((a_2^3)^2-a_2^6a_4^3-a_2^1a_5^6+a_2^6a_5^1)X_3+a_2^6(a_2^3-a_4^6)X_6=0,$$
$$((a_2^1)^2-a_2^6a_5^1-a_2^3a_4^6+a_2^6a_4^3)X_1+(a_2^1a_2^3-a_2^6a_5^3+a_2^1a_4^6-a_2^6a_4^1)X_3+a_2^6(a_2^1-a_5^6)X_6=0,$$
$$(a_2^1(a_4^3+a_5^1)-a_4^1a_4^6-a_5^1a_5^6+a_4^3a_5^6-a_4^6a_5^3)X_1+(a_2^3(a_4^3+a_5^1)-a_4^3a_4^6-a_5^3a_5^6-a_4^1a_5^6+a_4^6a_5^1)X_3$$
$$+(a_2^6(a_4^3+a_5^1)-(a_4^6)^2-(a_5^6)^2)X_6=0.$$

Since $a_2^6\neq0$, from which we see that $a_2^1-a_5^6=a_2^3-a_4^6=0$, and that

$$(a_2^1)^2-(a_2^3)^2=a_2^6(a_5^1-a_4^3),$$
$$(a_2^1)^2+(a_2^3)^2=a_2^6(a_4^1-a_5^3),$$
$$2a_2^1a_2^3=a_2^6(a_4^1+a_5^3),$$
$$2a_2^1a_4^3=a_2^3(a_4^1+a_5^3),$$
$$2a_2^3a_5^1=a_2^1(a_4^1+a_5^3).$$

We can see that the above equations are equivalent to the first four equations of them, and then we have $a_4^3=cb^2$, $a_5^1=ca^2$ and $a_4^1+a_5^3=2abc$ where $a_2^1=a$, $a_2^3=b$ and $a_2^6=1/c$. In this case u satisfies (1),..., (15), and it is written as the following form:

$$u_{19}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & b & 0 & 0 & 1/c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & cb^2 & 0 & 0 & b \\ ca^2 & 0 & 2abc-d & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ where } abc\neq d.$$

The condition $abc\neq d$ follows from $\mathbf{dim}\ uG=3$.

Let $uG=G_3^3$. Then putting $uX_i=a_i^1X_1+a_i^2X_2+a_i^3X_3$; since $(uG, uG)=0$, from (5) and (6) we have $a_2^3uX_1-a_2^1uX_3=0$ and $a_3^2uX_1-a_3^1uX_2=0$. Assume that $uX_i\neq0$ $(i=1, 2, 3)$. Then if $a_2^1a_3^1\neq0$ we can put $uX_2=kuX_1$ and $uX_3=suX_1$. Hence from the above we have

$$a_2^3uX_1-a_2^1uX_3=k(a_1^3-sa_1^1)uX_1=0,$$
$$a_3^2uX_1-a_3^1uX_2=s(a_1^2-ka_1^1)uX_1=0,$$

and then since $uX_1\neq0$, we have $a_1^3=sa_1^1$ and $a_1^2=ka_1^1$. Moreover since $u^pX_1=(a_1^1+ka_1^2+su_1^3)^{p-1}uX_1=0$, we have $a_1^1+ka_1^2+sa_1^3=0$; and then since $a_1^1+ka_1^2+sa_1^3=a_1^1(1+k^2+s^2)=0$, we have $a_1^1=0$ and $a_1^3=a_1^2=0$. This contradicts to $uX_1\neq0$. Therfore we have $a_2^1a_3^1=0$. If $a_2^1=0$, we have $a_2^3uX_1=0$ i. e., $a_2^3=0$. Hence we have $uX_2=a_2^2X_2$; then since $u^pX_2=0$, from which we have $uX_2=0$. This contradicts to $uX_2\neq0$. If $a_3^1=0$; we have $a_3^2=0$ i. e., $uX_3=a_3^3X_3$, then by the same way as the above we have $uX_3=0$. This contradicts to $uX_3\neq0$. Therefore we have $uX_1=0$ or $uX_2=0$ or $uX_3=0$. Let $uX_1=0$; then from (7) and (10) we have $u^2X_2=u^2X_3=0$, and hence from (8) and (9) we have $a_3^3uX_2-a_3^2uX_3=0$ and $a_2^3uX_2-a_2^2uX_3=0$ i. e.,

$$(a_3^3a_2^2-(a_3^2)^2)X_2+a_3^3(a_2^3-a_3^2)X_3=0 \text{ and } a_2^2(a_2^3-a_3^2)X_2+((a_2^3)^2-a_2^2a_3^3)X_3=0.$$

Hence we have $a_2^3-a_3^2=0$, and then since $u^p X_i=0$ $(i=2, 3)$, we have $a_2^2=a_2^3=a_3^2=a_3^3=0$. By the same way as this; if $uX_2=0$ from (9), (12), (10) and (11) we have $uX_1=uX_3=0$, and if $uX_3=0$ from (8), (11), (7) and (12) we have $uX_1=uX_2=0$. Therefore we have $uX_1=uX_2=uX_3=0$. In this case we can see that u satisfies (1),..., (15), and it is written as the following form:

$$u_{20}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ a&d&g&0&0&0\\ b&e&h&0&0&0\\ c&f&i&0&0&0 \end{bmatrix}, \text{ where } \textbf{rank} \begin{bmatrix} a&d&g\\ b&e&h\\ c&f&i \end{bmatrix}=3.$$

(d). Let **dim** $uG=4$. Let $uG=G_4$. Then putting $uX_i=a_i^1X_1+a_i^2X_2+a_i^3X_3+a_i^4X_4$; from (1) we have $a_1^4uX_4+a_2^4uX_2-(uX_1, uX_2)=0$, and then since $(uX_1\ uX_2)\in[X_1, X_2]$ we have $((a_1^4)^2+(a_2^4)^2)X_4=0$ i. e., $a_1^4=a_2^4=0$. Hence from (3) and (2) we have $a_3^4uX_1+(uX_2, uX_3)=0$ and $a_3^4uX_2+(uX_3, uX_1)=0$ i. e.,

$$a_3^4(a_1^1-a_2^2)X_1+a_3^4(a_1^2+a_2^1)X_2+a_3^4a_1^3X_3=0,$$
$$a_3^4(a_1^2+a_2^1)X_1-a_3^4(a_1^1-a_2^2)X_2+a_3^4a_2^3X_3=0.$$

If $a_3^4=0$; since $uX_i\in[X_1, X_2, X_3]$ $(i=1, 2, 3)$ and $(uG, uG)\subset[X_1, X_2]$, from (15) we have $((a_4^4)^2+(a_5^4)^2+(a_6^4)^2)X_4=0$ i. e., $a_4^4=a_5^4=a_6^4=0$. Hence we have $uG\subset[X_1, X_2, X_3]$, but this contradicts to **dim** $uG=4$. Therefore $a_3^4\neq0$; then we have $a_1^3=a_2^3=0$ and $a_1^1-a_2^2=a_1^2+a_2^1=0$. Since $u^pX_i=0$ $(i=1, 2)$, we have $a_1^1=a_1^2=a_2^1=a_2^2=0$ i. e., $uX_1=uX_2=0$. Therefore from (9) we have $u^2X_3=0$ i. e.,

$$(a_3^1a_3^3+a_3^4a_4^1)X_1+(a_3^2a_3^3+a_3^4a_4^2)X_2+((a_3^3)^2+a_3^4a_4^3)X_3+a_3^4(a_3^3+a_4^4)X_4=0.$$

Since $a_3^4\neq0$ we have $a_3^3+a_4^4=0$, and then we see

$$a_3^1a_4^4-a_3^4a_4^1=0,\ a_3^2a_4^4-a_3^4a_4^2=0 \text{ and } (a_4^4)^2+a_3^4a_4^3=0.$$

From the first and the last of the above we have $a_4^4(a_4^1a_4^4-a_3^1a_4^3)=0$. If $a_4^4=0$, since $a_3^4\neq0$ we have $a_4^1=a_4^2=a_4^3=0$ i. e., $uX_4=0$; and if $a_4^4\neq0$, we have $a_4^1a_4^4-a_3^1a_3^4=0$. Therefore we have $a_3^1:a_3^2:a_3^3:a_3^4=a_4^1:a_4^2:a_4^3:a_4^4$ i. e., uX_3 and uX_4 are linearly dependent, and then we have $uG\subset[uX_3, uX_5, uX_6]$. This contradicts to **dim** $uG=4$. Thus in this case u does not exist.

7. Relations in u_j. If there exists a nonsingular matrix A such as $uA=v$ and if u and v satisfy (1.3), we shall call *u is equivalent to v*. We see that if we drop off the second condition of $u_{10},\dots, u_{14}$, then $u_1,\dots, u_5$ become special cases of $u_{10},\dots, u_{14}$ respectively. Since there exist nonsingular matrices A and B such as

$$\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&a&0&0 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&a_1&0&0\\ 0&0&0&a_2&0&0\\ 0&0&0&a_3&0&0 \end{bmatrix}$$

$$u_{17}A=\begin{vmatrix} 0&0&0&-a&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{vmatrix},\quad u_{11}B=\begin{vmatrix} 0&0&0&a_4&0&0\\ b&c&0&a_5&0&0\\ -c&b&0&a_6&0&0 \end{vmatrix},$$

u_{17} is equivalent to the spesial case of u_{10}, and if $a_4\neq 0$ in u_{11}, u_{11} is equivalent to the special case of u_{10}. If $a_4=0$ in u_{11}, we shall denote $u_{11}=u'_{11}$. We know that the contracted algebra with respect to u such as $uG=u^2G\neq[0]$ is isomorphic to the İnönü-Wigner contracted algebra ([2], pp. 7-8). This result can be seen as following. Let $X, X'\in G_R$ and $Y, Y'\in G_N$; then since $uG_N=0$, $uG=G_R$ and $(uG_R, G_R)=(G_R, G_R)\subset G_R$, from (1.4) we have

$$(X, X')_\circ=u^{-1}(uX, uX'),\ (X, Y)_\circ=(uX, Y)_N \text{ and } (Y, Y')_\circ=0.$$

Consider the linear transformation F such as $F=u$ on G_R and $F=identity$ on G_N; then since u is nonsingular on G_R, we can see that F transforms isomorphically the above contracted products to the followings:

$$(X, X')_\circ=(X, X'),\ (X, Y)_\circ=(X, Y)_N \text{ and } (Y, Y')_\circ=0. \tag{7.1}$$

This is the contracted products with respect to v such as $v=identity$ on G_R and $v=0$ on G_N. Therefore instead of $u_6, \dots, u_9$ we may take, $v_1, \dots, v_4$ of the following forms respectively:

$$v_1=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},\ v_2=\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix},\ v_3=\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix},\ v_4=\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}.$$

There exists nonsingular matrix A such that $u'_{11}A$ is equal to

$$v_5=\begin{bmatrix} 0&0&a_1&0&0&0\\ 0&0&a_2&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ b&c&a_5&0&0&0\\ -c&b&a_6&0&0&0 \end{bmatrix}.$$

Then since $v_5X_3=X_3$, $v_5Y_j=0$ $(j=1, 2, 4)$, and $v_5Y_k\in[Y_1, Y_2, X_3]$ $(k=5, 6)$ where $Y_j=X_j-a_jX_3$, $Y_4=X_4$, $Y_5=X_5-(ba_1+ca_2+a_5)X_3$ and $Y_6=X_6+(ca_1-ba_2-a_6)X_3$, we can see that v_5 satisfies (1.3). Therefore u_{15} is equivalent to the special case of v_5 and u'_{11} is equivalent to v_5. There exist nonsingular matrices A_i such as $u_{10}A_1$, $u_{13}A_2$, $u_{12}A_3$, $u_{14}A_4$ and $u_{19}A_5$ are equal to the followings respectively:

$$
v_6=\begin{bmatrix} 0&0&0&a_1&0&0\\ 0&0&0&a_2&0&0\\ 0&0&0&a_3&0&0\\ 0&0&0&1&0&0\\ b&c&0&a_5&0&0\\ -c&b&0&a_6&0&0 \end{bmatrix},\quad
v_7=\begin{bmatrix} 0&0&a_1&b_1&0&0\\ 0&0&a_2&b_2&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ c&d&a_5&b_5&0&0\\ -d&c&a_6&b_6&0&0 \end{bmatrix},\quad
v_8=\begin{bmatrix} 1&0&0&0&0&0\\ a_2&0&0&0&0&0\\ a_3&0&0&0&0&0\\ a_4&b&e&0&0&0\\ a_5&c&f&0&0&0\\ a_6&d&g&0&0&0 \end{bmatrix}
$$

$$
v_9=\begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ a_3&b_3&0&0&0&0\\ a_4&b_4&c&0&0&0\\ a_5&b_5&d&0&0&0\\ a_6&b_6&e&0&0&0 \end{bmatrix},\quad
v_{10}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}.
$$

Since the above matrices are special cases of u_{10}, u_{13}, u_{12}, u_{14} and u_{19} respectively, they satisfy (1.3). Therefore u_{10}, u_{13}, u_{12}, u_{14} and u_{19} are equivalent to $v_6, \dots, v_{10}$ respectively. Since even if we drop off the condition of u_{16}, u_{18} and u_{20}, they satisfy (1.3); u_{16} and u_{18} become special case of u_{20}, i. e.,

$$
v_{11}=\begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ a&d&g&0&0&0\\ b&e&h&0&0&0\\ c&f&i&0&0&0 \end{bmatrix}.
$$

8. Contracted Lie algebras. In this section we shall compute the contracted Lie products with respect to each v_i. By straightforward computations of (7.1), we can obtain the contracted products with respect to v_i $(i=1, \dots, 4)$. We have that

$$
\text{if } v=v_5;\ \begin{cases} vX_3=X_3,\ vY_j=0\ (j=1,2,4), \\ vY_5=bY_1+cY_2, \\ vY_6=-cY_1+bY_2, \end{cases}
\quad \text{where } \begin{cases} Y_j=X_j-a_jX_3\ \ (i=1,2),\ \ Y_4=X_4, \\ Y_5=X_5-(ba_1+ca_2+a_5)X_3, \\ Y_6=X_6+(ca_1-ba_2-a_6)X_3, \end{cases}
$$

$$
\text{if } v=v_6;\ \begin{cases} vX_4=X_4,\ vY_j=0\ (j=1,2,3), \\ vY_5=bY_1+cY_2, \\ vY_6=-cY_1+bY_2, \end{cases}
\quad \text{where } \begin{cases} Y_j=X_j-a_jX_4\ \ (j=1,2,3), \\ Y_5=X_5-(ba_1+ca_2+a_5)X_4, \\ Y_6=X_6+(ca_1-ba_2-a_6)X_4, \end{cases}
$$

$$
\text{if } v=v_7;\ \begin{cases} vX_i=X_i\ (i=1,2),\ vY_j=0\ (j=1,2), \\ vY_5=cY_1+dY_2, \\ vY_6=-dY_1+cY_2, \end{cases}
\quad \text{where } \begin{cases} Y_j=X_j-a_jX_3-b_jX_4\ \ (j=1,2), \\ Y_5=X_5-(ca_1+da_2+a_5)X_3 \\ \qquad -(cb_1+db_2+b_5)X_4, \\ Y_6=X_6+(da_1-ca_2-a_6)X_3 \\ \qquad +(db_1-cb_2-b_6)X_4, \end{cases}
$$

$$\text{if } v=v_8;\ \begin{cases} vX_1=X_1,\ vY_j=0\ (j=2,\ 3), \\ vY_4=bY_2+eY_3, \\ vY_5=cY_2+fY_3, \\ vY_6=dY_2+gY_3, \end{cases} \quad \text{where} \begin{cases} Y_j=X_j-a_jX_1\ (j=2,\ 3), \\ Y_4=X_4-(ba_2+ea_3+a_4)X_1, \\ Y_5=X_5-(ca_2+fa_3+a_5)X_1, \\ Y_6=X_6-(da_2+ga_3+a_6)X_1, \end{cases}$$

$$\text{if } v=v_9;\ \begin{cases} vX_i=X_i\ (i=1,\ 2),\ vY_3=0, \\ vY_4=cY_3, \\ vY_5=dY_3, \\ vY_6=eY_3, \end{cases} \quad \text{where} \begin{cases} Y_3=X_3-a_3X_1-b_3X_2, \\ Y_4=X_4-(ca_3+a_4)X_1-(cb_3+b_4)X_2, \\ Y_5=X_5-(da_3+a_5)X_1-(db_3+b_5)X_2, \\ Y_6=X_6-(ea_3+a_6)X_1-(eb_3+b_6)X_2. \end{cases}$$

From the above, by straightforward computations of (1.4) we can obtain the contracted products with respect to v_i $(i=5, \ldots, 9)$. If $v=v_{10}$ or $v=v_{11}$; since $G_R=0$ and $G_N=G$, from (1.4) we have

$$(X,\ Y)_\circ=(vX,\ Y)+(X,\ vY)-v(X,\ Y). \tag{8.1}$$

By straitforward computations of (8.1) we can obtain the contracted products with respect to v_{10} and to v_{11}. Thus as the main result we have obtained the following theorem.

THEOREM. *With a suitable linear transformation of variables in 3-dimensional Euclidean space E_3, the singular matrix u of the form* (1.2) *which can contract the Lie algebra of motion group of E_3 is v_1 or v_2 or ... or v_4 (this is the I-W cases) or is equivalent to the matrix v_5 or v_6 or ... or v_{11} whose contracted Lie algebra is the following:*

v_1	*the contracted products are given by* (2.2)
v_2	$(X_1, X_4)_\circ=-(X_3, X_5)_\circ=X_2$, $(X_1\ X_6)_\circ=-X_3$, $(X_3, X_6)_\circ=X_1$, $(X_5, X_6)_\circ=-X_4$, $(X_6, X_4)_\circ=-X_5$, *other products*$=0$
v_3	*all of the products*$=0$
v_4	$(X_1, X_4)_\circ=X_2$, $(X_2, X_4)_\circ=-X_1$, $(X_4, X_5)_\circ=-X_6$, $(X_6, X_4)_\circ=-X_5$, *other products*$=0$
v_5	$(X_3, Y_5)_\circ=-Y_2$, $(X_3, Y_6)_\circ=Y_1$, $(Y_5, Y_6)_\circ=-(ba_1+ca_2)Y_1+(ca_1-ba_2)Y_2$, *other products*$=0$
v_6	$(Y_1, Y_5)_\circ=-(ba_1+ca_2)Y_2$, $(Y_2, Y_6)_\circ=-(ca_1-ba_2)Y_1$, $(Y_1, Y_4)_\circ=Y_2$, $(Y_2, Y_5)_\circ=(ba_1+ca_2)Y_1$, $(Y_1, Y_6)_\circ=(ca_1-ba_2)Y_2$, $(Y_5, Y_6)_\circ=-2bY_3-(ba_1+ca_2)Y_5+(ca_1-ba_2)Y_6$, *other products*$=0$
v_7	$(Y_1, Y_5)_\circ=-(cb_1+db_2)Y_2$, $(Y_2, Y_6)_\circ=-(db_1-cb_2)Y_1$, $(Y_1, Y_4)_\circ=-(X_3, Y_5)_\circ=Y_2$, $(Y_2, Y_5)_\circ=(Y_2, Y_5)_\circ=(cb_1+db_2)Y_1$, $(Y_1, Y_6)_\circ=(db_1-cb_2)Y_2$, $(X_3, Y_6)_\circ=-(Y_2, X_4)_\circ=Y_1$, $(X_4, Y_5)_\circ=-Y_6$, $(Y_6, X_4)_\circ=-Y_5$, $(Y_5, Y_6)_\circ=-2cY_3-(ca_1+da_2)Y_1+(da_1-ca_2)Y_2-(cb_1+db_2)Y_5-(db_1+cb_2)Y_6$, *other products*$=0$

ν_8	$(X_1, Y_4)_\circ = Y_2$, $(X_1, Y_6)_\circ = -Y_3$, $(Y_4, Y_5)_\circ = (ca_2+fa_3+d-e)Y_2+(b+g)Y_3$, $(Y_5, Y_6)_\circ = (b+g)Y_2+(ca_2+fa_3-d+e)Y_3$, $(Y_6, Y_4)_\circ = -(da_2+ga_3-c)Y_2-(ba_2+ea_3-f)Y_3$, *other products*$=0$
ν_9	$(X_2, Y_5)_\circ = -(X_1, Y_6)_\circ = Y_3$, $(Y_4, Y_5)_\circ = -(cb_3-e)Y_3$, $(Y_5, Y_6)_\circ = (da_3+ea_3+c)Y_3$, $(Y_6, Y_4)_\circ = -(ca_3-d)Y_3$, *other products*$=0$
ν_{10}	$(X_1, X_2)_\circ = -X_3$, $(X_2, X_3)_\circ = -X_1$, $(X_1, X_4)_\circ = -(X_3, X_5)_\circ = X_6$, $(X_2, X_5)_\circ = X_4$, $(X_2, X_4)_\circ = -X_5$, *other products*$=0$
ν_{11}	$(X_4, X_5)_\circ = (c+e)X_1-(b+g-f)X_2+(d+i)X_3$, $(X_5, X_6)_\circ = (a+b)X_1+(d+i)X_2-(b+f-g)X_3$, $(X_6, X_4)_\circ = (b-f-g)X_1+(c+e)X_2+(a+b)X_3$, *other products*$=0$

Acknowledgement. I would like to express my hearty thanks to Prof. T. Nôno for his valuable abvices and encouragements.

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