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A THEOREM ON THE MALCEV STRUCTURE OF AN ALTERNATIVE RING

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1. The Malcev structure of an alternative ring is defined by introducing the commutator of two elements as a new multiplication [2]. This is a natural generalization of the Lie structure of an associative ring. In our previous paper [1] we proved a theorem concerning the Lie structure of an associative ring. The purpose of this paper is to extend the theorem to the Malcev structure of an alternative ring.

2. For brevity we use definitions and identities in [1], [3] and [4] without explaining or proving them here again. In what follows we always denote an alternative ring by A. For x, y and z in A we use the following notations:

$$\text{Commutator,} \qquad [x, y]=xy-yx;$$

$$\text{Associator,} \qquad (x, y, z)=(xy)z-x(yz).$$

For a pair (x, y) of elements in A we define a linear operator $D(x, y)$ (or briefly D) by the equality

$$D(x, y)z=Dz=[x, [y, z]]-[y, [x, z]]+[[x, y], z].$$

Then we have

$$Dz=2[[x, y], z]-6(x, y, z).$$

It is wellknown that the operator $D(x, y)$ is a derivation of A [4]. From this fact and from the theorem of Artin [4, Th. 3.1] we have the next lemma.

LEMMA 1. *For any u, v and w in A,*

$$D^n(uv)=\sum_{k=0}^{n}\binom{n}{k}(D^{n-k}u)(D^k v),$$

$$D^n[u, v]=\sum_{k=0}^{n}\binom{n}{k}[D^{n-k}u, D^k v],$$

$$D(u, v, w)=(Du, v, w)+(u, Dv, w)+(u, v, Dw),$$

$$D[x, y]=D(x, y)[x, y]=0.$$

Using this lemma we have the following two lemmas.

LEMMA 2. *For any u in A*

$$D^n(u[x, y])=(D^n u)[x, y].$$

LEMMA 3. *For any u in A*

$$D^n\{6(x, y, u)\}=6(x, y, D^n u).$$

PROOF.

$$\begin{aligned} D^2u &= D(Du)=D\{2[[x, y], u]-6(x, y, u)\} \\ &=2[[x, y], Du]-6(Dx, y, u)-6(x, Dy, u)-6(x, y, Du) \\ &=D^2u-6(Dx, y, u)-6(x, Dy, u). \end{aligned}$$

This equality shows that

$$6(Dx, y, u)+6(x, Dy, u)=0.$$

Therefore we have

$$D\{6(x, y, u)\}=6(x, y, Du).$$

Continuing this process we have

$$D^n\{6(x, y, u)\}=6(x, y, D^n u).$$

3. Now we proceed to extend the theorem to the Malcev structure of an alternative ring.

THEOREM. *Let A be a semiprime, 3-torsion-free alternative ring and let U be a Malcev ideal of A. Then for a derivation* $D=D(x, y)$, $D^2U\subset Z$ *(the center of A) implies* $D^2U=(0)$.

PROOF. Suppose that $D^2U\neq(0)$, then there exists an element $u\in U$ such that $D^2u\neq 0$. Since U is a Malcev ideal of A, $[u[x, y], u]\in U$. Therefore, from the assumption $D^2U\subset Z$ and from the lemma 1 and the lemma 2 we have

$$\begin{aligned} 0 &= D^3[u[x, y], u] \\ &=[D^3u[x, y], u]+3[D^2u[x, y], Du]+3[Du[x, y], D^2u]+[u[x, y], D^3u] \\ &=3[D^2u[x, y], Du]=3[[x, y], Du]D^2u. \end{aligned}$$

On the other hand, it follows from the Moufang identity

$$x(uyu)=[(xu)y]u$$

that

$$(x,\ uy,\ u)=(x,\ y,\ u)u\,.$$

The left hand side of the equality is in U, therefore the right hand side is in U too. Then from the assumption $D^2U\subset Z$ and from the lemma 1 and the lemma 3 we have

$$\begin{aligned}0&=D^3\{6(x,\ y,\ u)u\}\\&=6\{(x,\ y,\ D^3u)u+3(x,\ y,\ D^2u)Du+3(x,\ y,\ Du)D^2u+(x,\ y,\ u)D^3u\}\\&=3\times 6(x,\ y,\ Du)D^2u\,.\end{aligned}$$

Combining these equalities we obtain

$$\begin{aligned}0&=D^3\{2[u[x,\ y],\ u]-6(x,\ y,\ u)u\}\\&=3\{2[[x,\ y],\ Du]-6(x,\ y,\ Du)\}D^2u\\&=3(D^2u)^2\,.\end{aligned}$$

Since A is 3-torsion-free, we obtain $(D^2u)^2=0$. Hence we have

$$((D^2u)A)^2\subset(D^2u)^2=(0)\,.$$

Therefore $(D^2u)A$ is a nilpotent ideal. Since the ring A is semiprime, we get $D^2u=0$. This contradicts the assumption $D^2u\neq 0$. Thus the proof is completed.

References

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