Bull. Kyushu Inst. Tech.
(M. & N. S.) No. 24, 1977, pp. 23–27

NOTE ON AN EXPONENTIAL GENERATING FUNCTION OF BELL NUMBERS

By

Kyôichi YOSHINAGA and Masato MORI

(Received Oct. 23, 1976)

Let B_n denote the number of partitions of a set of n distinct objects. B_n are sometimes called exponential numbers or Bell numbers and it is known by E. T. Bell and by several other authors, e.g., [1], [2], [3], [4, pp. 37–38], [5] and [7], that the following formula

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \exp(\exp t - 1)$$

is true for each complex number t, $|t| < \infty$, where we have set $B_0 = 1$ by convention. The elegant proof given by G.-C. Rota [5], [4, pp. 37–38] to this identity is seen to be rather too formal and so a certain logical gap is unavoidable.

The purpose of the present note is to give a complete proof to this identity by filling up such logical gap. In doing so, it will be found that the theorem on summation by packets relative to a summable series [6, p. 172] plays an essential role.

1. Preliminaries

For any pair (x, n) of a real number x and a positive integer n, let us write

$$[x]_n = x(x-1)\cdots(x-n+1).$$

S_n^m is the Stirling number (of second kind), that is, the number of partitions of a set of n distinct objects into m classes, where m and n, $n \geq m$, are positive integers. The Bell number B_n, the number of partitions of a set of n distinct objects, is then given by the identity

(1) $$B_n = S_n^1 + S_n^2 + \cdots + S_n^n.$$

Another formula requisite for what follows is the next identity, whose proof presenting few difficulties will be found in [4, p. 36].

(2) $$x^n = S_n^1 [x]_1 + S_n^2 [x]_2 + \cdots + S_n^n [x]_n.$$

LEMMA 1. *Put for* $k = 1, 2, \ldots,$

$$M_k(t)=\sum_{l=0}^{\infty}\frac{S_{k+l}^k}{(k+l)!}t^{k+l}.$$

Then, for any t, $0\le t<1/e$, *the series* $M_k(t)$ *is convergent with*

$$M_k(t)\le\frac{(et)^k}{\sqrt{2\pi}k!\sqrt{k}}\frac{1}{1-et}$$

PROOF. Owing to the identity (2), it follows that

$$S_{k+l}^k\le\frac{(k+l)^{k+l}}{[k+l]_k}=\frac{(k+l)^{k+l}l!}{(k+l)!}.$$

By means of Stirling's formula:

$$n!=\sqrt{2\pi}n^{n+1/2}e^{-n}e^{\mu(n)},\quad \mu(n)=\frac{\theta}{12n}\qquad(0<\theta<1),$$

it is not difficult to see

$$\frac{S_{k+l}^k t^{k+l}}{(k+l)!}\le\frac{(et)^{k+l}l!}{(k+l)!\sqrt{2\pi}\sqrt{k+l}\,e^{\mu(k+l)}}$$

$$<\frac{(et)^k}{\sqrt{2\pi}}\frac{(et)^l l!}{(k+l)!\sqrt{k+l}}$$

$$\le\frac{(et)^k}{\sqrt{2\pi}k!\sqrt{k}}(et)^l.$$

Therefore for each t, $0\le t<1/e$, one obtains

$$M_k(t)\le\frac{(et)^k}{\sqrt{2\pi}k!\sqrt{k}}\sum_{l=0}^{\infty}(et)^l=\frac{(et)^k}{\sqrt{2\pi}k!\sqrt{k}}\frac{1}{1-et}$$

as desired. This completes the proof.

LEMMA 2. *Put*

$$h(t,x)=\sum_{k=1}^{\infty}M_k(|t|)|[x]_k|.$$

Then for any t, $|t|<1/e$ *and for any real number x,* $|x|<\infty$, *one observes that the series* $h(t,x)$ *is convergent.*

PROOF. By means of Lemma 1, it follows that

$$M_k(|t|)|[x]_k|\le\frac{1}{\sqrt{2\pi}(1-e|t|)}\frac{(e|t|)^k}{\sqrt{k}k!}|[x]_k|$$

$$\le \frac{1}{\sqrt{2\pi(1-e|t|)}}\left|\binom{x}{k}\right|(e|t|)^k$$

for $k=1, 2,\ldots$. Because of the absolute convergence of the binomial series

$$\textstyle\sum_{k=0}^{\infty}\binom{x}{k}s^k \qquad (|s|<1),$$

it is not difficult to get the result as desired. This completes the proof.

2. Formula of Bell and formula of Dobinski

The formula of Bell in the following proposition tells us that the entire function $\exp(\exp t-1)$ is a generating function of Bell numbers. We here now give a complete proof of this formula.

PROPOSITION 1. (E. T. Bell) *For any complex number t, $|t|<\infty$, it holds that*

$$1+\textstyle\sum_{n=1}^{\infty}\frac{B_n}{n!}t^n=\exp(\exp t-1).$$

PROOF. Owing to the convergence of the series $h(t, x)$, it follows that the double series

$$H(t, x)=1+\textstyle\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{S_k^l}{k!}t^k[x]_l$$

is absolutely and hence unconditionally convergent, that is to say $H(t, x)$ is summable and so the summation by packets may be allowed [6, pp. 172–173]. Therefore according to the identity (2) one obtains

$$H(t, x)=1+\textstyle\sum_{k=1}^{\infty}\frac{t^k}{k!}(S_k^1[x]_1+\cdots+S_k^k[x]_k)$$

$$=1+\textstyle\sum_{k=1}^{\infty}\frac{(tx)^k}{k!}=\exp(tx),$$

where $|t|<1/e$ and $|x|<\infty$. On the other hand by virtue of the binomial expansion, one observes that

$$\exp(tx)=(1+(\exp t-1))^x$$

$$=1+\textstyle\sum_{l=1}^{\infty}\frac{(\exp t-1)^l}{l!}[x]_l$$

for each real number t, $|\exp t-1|<1$, that is $-\infty<t<\log 2$. Thus for each t and x, $|t|<1/e=\min(1/e, \log 2)$, it follows that

$$1+\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{S_k^l}{k!}t^k[x]_l=H(t,x)=\exp(tx)$$

$$=1+\sum_{l=1}^{\infty}\frac{(\exp t-1)^l}{l!}[x]_l,$$

and therefore for $l=1, 2,\dots$, one obtains the next identity

$$\frac{(\exp t-1)^l}{l!}=\sum_{k=l}^{\infty}\frac{S_k^l}{k!}t^k=M_l(t)$$

for t, $|t|<1/e$. Owing to the summability of the series $H(t, x)$, it is not difficult to see the absolute convergence and hence the summability of the double series

$$\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{S_k^l}{k!}t^k.$$

Thus one obtains for $|t|<1/e$ that

$$\exp(\exp t-1)=1+\sum_{l=1}^{\infty}\frac{(\exp t-1)^l}{l!}$$

$$=1+\sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\frac{S_k^l}{k!}t^k$$

$$=1+\sum_{k=1}^{\infty}\sum_{l=1}^{k}\frac{S_k^l}{k!}t^k$$

$$=1+\sum_{k=1}^{\infty}\frac{B_k}{k!}t^k,$$

where the last equality is seen to be a consequence of the identity (1). Since $\exp(\exp t-1)$ is an entire function of t, it is not difficult to get

$$\exp(\exp t-1)=1+\sum_{k=1}^{\infty}\frac{B_k}{k!}t^k$$

for t, $|t|<\infty$, as desired. This completes the proof.

We next give a complete proof of the formula of Dobinski.

PROPOSITION 2. (G. Dobinski) *For any $m=1, 2,\dots$, it holds that*

$$B_m=\frac{1}{e}\left(1+\frac{2^{m-1}}{1!}+\frac{3^{m-1}}{2!}+\frac{4^{m-1}}{3!}+\cdots\right).$$

PROOF. We first note that

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}=\sum_{k=n}^{\infty}\frac{[k]_n}{k!}$$

and so

$$1=\frac{1}{e}\sum_{k=n}^{\infty}\frac{[k]_n}{k!}$$

is obtained for every $n=1, 2,\ldots$. Therefore it follows that

$$\begin{aligned}B_m&=S_m^1+S_m^2+\cdots+S_m^m\\&=\sum_{n=1}^{m}S_m^n\frac{1}{e}\sum_{k=n}^{\infty}\frac{[k]_n}{k!}\\&=\frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}(S_m^1[k]_1+S_m^2[k]_2+\cdots+S_m^m[k]_m)\\&=\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^m}{k!}\\&=\frac{1}{e}\left(1+\frac{2^{m-1}}{1!}+\frac{3^{m-1}}{2!}+\frac{4^{m-1}}{3!}+\cdots\right).\end{aligned}$$

This completes the proof.

References

[1] H. W. Becker and J. Riordan, *The arithmetic of Bell and Stirling numbers,* Amer. J. Math., **70** (1948) 385–394.

[2] E. T. Bell, *Exponential polynomials,* Ann. of Math., **35** (1934) 258–277.

[3] ————, *The iterated exponential integers,* Ann. of Math., **39** (1938) 539–557.

[4] C. Berge, *Principes de combinatoire,* Dunod, Paris, 1968.

[5] G.-C. Rota, *The number of partitions of a set,* Amer. Math. Monthly, **71** (1964) 498–504.

[6] L. Schwartz, *Topologie générale et analyse fonctionnelle,* Hermann, Paris, 1970.

[7] J. Touchard, *Nombres exponentiels et nombres de Bernoulli,* Canad. J. Math., **8** (1956) 305–320.

Department of Computer Science
Kyushu Institute of Technology