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DYNAMICAL SYMMETRIES VI

By

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1. Introduction

On the basis of differential geometric procedure, a theory was developed of invariant variational principles for dynamical systems under a group of continuous transformations, in the series of our papers [1, 2, 3, 4, 5]. There were obtained the dynamical symmetry equations. Particularly in the papers [1] and [5], a conservative Lagrangian system in continuum mechanics and a non-conservative one with generalized forces were taken respectively as examples of dynamical systems. So, the dynamical symmetry equations were refered to the group of transformations and the Lagrangian density in [1], and moreover, to the generalized forces in [5].

In this paper, although the problem was initiated by J. C. Houard [6], there is given an another investigation for the problem of determining Lagrangian density from given group of transformations under the invariant variational principles. Such an investigation begins with the aid of symmetry equations. In **2**, after a brief review of [1, 5] for obtaining the dynamical symmetry equations and conservation laws, the dynamical symmetry equations are discussed for special groups of transformations. In **3**, illustrative examples of determining the Lagrangian densities are given in Lagrangian particle mechanics and field theories. And moreover, in the conservative case, conservation laws are derived for the determined Lagrangian densities.

The notations and terminologies in this paper are essentially those of the privious one [1, 5]. The summation convention is used throughout.

2. Dynamical symmetry equations of Lagrangian systems

A review of [1, 5] will be given with the setting for manifolds N, M_1 and M_2 with local coordinates (x^i), $(x^i, y^\alpha, z_i^\alpha)$ and $(x^i, y^\alpha, z_i^\alpha, u_{ij}^\alpha)$ respectively, where x^i $(i=1,\dots, n)$ are regarded as the coordinates of continuum and y^α $(\alpha=1,\dots, m)$ the field variables. Let $\phi_2: N \to M_2$ be a submanifold map: $\phi_2(x^i)=(x^i, \partial y^\alpha/\partial x^i, \partial^2 y^\alpha/\partial x^j \partial x^i)$ and ϕ_2^*: $M_2 \to N$ be its pull-back map. Then, for given Lagrangian density $L(x, y, z)$ on M_1, the conservative Lagrangian system is written as $\phi_2^*([L]_\alpha)=0$, and moreover for given generalized forces $F_\alpha(x, y, z)$ on M_1, the non-conservative one is written as $\phi_2^*([L]_\alpha + F_\alpha)=0$, where

$$[L]_\alpha=\frac{\partial L}{\partial y^\alpha}-\frac{1}{f}\frac{d}{dx^i}\left(f\frac{\partial L}{\partial z_i^\alpha}\right),$$

in which d/dx^i denotes the total differentiation: $d/dx^i=\partial/\partial x^i+z_i^\alpha\partial/\partial y^\alpha+u_{ki}^\alpha\partial/\partial z_k^\alpha$.

In this place, we recall that the infinitesimal transformations X_λ^2 ($\lambda=1,\dots, l$) of l-parameter group of transformations defined on M_2 are obtained from

$$X_\lambda^0=\psi_\lambda^i(x, y)\frac{\partial}{\partial x^i}+\xi_\lambda^\alpha(x, y)\frac{\partial}{\partial y^\alpha},$$

by the 2-prolongation method, that is X_λ^2 are given as

$$X_\lambda^2=X_\lambda^0+\eta_{\lambda i}^\alpha(x, y, z)\frac{\partial}{\partial z_i^\alpha}+\zeta_{\lambda ij}^\alpha(x, y, z, u)\frac{\partial}{\partial u_{ij}^\alpha},$$

where the coefficients $\eta_{\lambda i}^\alpha$ and $\zeta_{\lambda ij}^\alpha$ are

$$\eta_{\lambda i}^\alpha=\frac{d\xi_\lambda^\alpha}{dx^i}-z_i^\alpha\frac{d\psi_\lambda^k}{dx^i},\quad \zeta_{\lambda ij}^\alpha=\frac{d\eta_{\lambda i}^\alpha}{dx^j}-u_{ik}^\alpha\frac{d\psi_\lambda^k}{dx^j}.$$

Then the following theorem was obtained (partially, the descriptions of the theorems in [1, 5] are modified for familiarity).

THEOREM. *Let the conservative Lagrangian system be invariant under the infinitesimal transformations X_λ^2, i.e., $\phi_2^*X_\lambda^2[L]_\alpha=0$. Then it follows the symmetry equatons: $\phi_2^*[N_\lambda]_\alpha=0$, where by using of*

$$N_\lambda=X_\lambda^2(L)+\frac{L}{f}\frac{d}{dx^i}(f\psi_\lambda^i),$$

$[N_\lambda]_\alpha$ is defined by

$$[N_\lambda]_\alpha=\frac{\partial N_\lambda}{\partial y^\alpha}-\frac{1}{f}\frac{d}{dx^i}\left(f\frac{\partial N_\lambda}{\partial z_i^\alpha}\right);$$

and also, the conservation laws: $\partial\phi_2^*(fK_\lambda^i)/\partial x^i-\phi_2^*(fN_\lambda)=0$, where K_λ^i is defined by

$$K_\lambda^i=(\xi_\lambda^\alpha-z_j^\alpha\psi_\lambda^j)\frac{\partial L}{\partial z_i^\alpha}+\psi_\lambda^i L. \tag{1}$$

Moreover, let the non-conservative Lagrangian system be given as the above, i.e., $\phi_2^*X_\lambda^2([L]_\alpha+F_\alpha)=0$. Then it follows the symmetry equations: $\phi_2^*([N_\lambda]_\alpha+\Phi_{\lambda\alpha})=0$, where $\Phi_{\lambda\alpha}$ is defined by

$$\Phi_{\lambda\alpha}=X_\lambda^2(F_\alpha)+F_\beta\left(\frac{\partial\xi_\lambda^\beta}{\partial y^\alpha}-z_i^\alpha\frac{\partial\psi_\lambda^i}{\partial y^\alpha}\right)+\frac{F_\alpha}{f}\frac{d}{dx^i}(f\psi_\lambda^i). \tag{2}$$

And all the converses are valid in the infinitesimal version.

We shall now give further calculations of $[N_\lambda]_\alpha$ in the following special cases, letting $f=1$ for brevity.

In the first place, let X_λ^2 be the 2-prolongation of $X_\lambda^0=\xi_\lambda^\alpha(x)\partial/\partial y^\alpha$:

$$X_\lambda^2=\xi_\lambda^\alpha\frac{\partial}{\partial y^\alpha}+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\frac{\partial}{\partial z_i^\alpha}+\eta_{\lambda ij}^\alpha\frac{\partial}{\partial u_{ij}^\alpha}.$$

Then, since N_λ is written as

$$N_\lambda=\xi_\lambda^\alpha\frac{\partial L}{\partial y^\alpha}+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\frac{\partial L}{\partial z_i^\alpha},\tag{3}$$

it follows that

$$\frac{\partial N_\lambda}{\partial y^\beta}=\xi_\lambda^\alpha\frac{\partial^2 L}{\partial y^\beta\partial y^\alpha}+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\frac{\partial^2 L}{\partial y^\beta\partial z_i^\alpha},$$

$$\frac{\partial N_\lambda}{\partial z_j^\beta}=\xi_\lambda^\alpha\frac{\partial^2 L}{\partial z_j^\beta\partial y^\alpha}+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha},$$

and moreover

$$\frac{d}{dx^j}\left(\frac{\partial N_\lambda}{\partial z_j^\beta}\right)=\frac{\partial\xi_\lambda^\alpha}{\partial x^j}\frac{\partial^2 L}{\partial z_j^\beta\partial y^\alpha}+\xi_\lambda^\alpha\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z_j^\beta\partial y^\alpha}\right)$$
$$+\frac{\partial^2\xi_\lambda^\alpha}{\partial x^j\partial x^i}\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}\right).$$

Hence, $[N_\lambda]_\beta$ becomes

$$[N_\lambda]_\beta=\xi_\lambda^\alpha\left[\frac{\partial^2 L}{\partial y^\beta\partial y^\alpha}-\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z_j^\beta\partial y^\alpha}\right)\right]$$
$$+\frac{\partial\xi_\lambda^\alpha}{\partial x^i}\left[\frac{\partial^2 L}{\partial y^\beta\partial z_i^\alpha}-\frac{\partial^2 L}{\partial z_i^\beta\partial y^\alpha}-\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}\right)\right]$$
$$-\frac{\partial^2\xi_\lambda^\alpha}{\partial x^j\partial x^i}\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}.\tag{4}$$

In the second place, let X_λ^2 be the 2-prolongation of $X_\lambda^0=\psi_\lambda^i(x)\partial/\partial x^i$:

$$X_\lambda^2=\psi_\lambda^i\frac{\partial}{\partial x^i}-z_k^\alpha\frac{\partial\psi_\lambda^k}{\partial x^i}\frac{\partial}{\partial z_i^\alpha}+\zeta_{\lambda ij}^\alpha\frac{\partial}{\partial u_{ij}^\alpha}.$$

Then, since N_λ is written as

$$N_\lambda=\psi_\lambda^i\frac{\partial L}{\partial x^i}-z_k^\alpha\frac{\partial\psi_\lambda^k}{\partial x^i}\frac{\partial L}{\partial z_i^\alpha}+\frac{\partial\psi_\lambda^i}{\partial x^i}L,\tag{5}$$

it follows that

$$\frac{\partial N_\lambda}{\partial y^\beta}=\psi^i_\lambda\frac{\partial^2 L}{\partial y^\beta\partial x^i}-z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial y^\beta\partial z^\alpha_i}+\frac{\partial\psi^i_\lambda}{\partial x^i}\frac{\partial L}{\partial y^\beta},$$

$$\frac{\partial N_\lambda}{\partial z^\beta_j}=\psi^i_\lambda\frac{\partial^2 L}{\partial z^\beta_j\partial x^i}-z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}-\frac{\partial\psi^j_\lambda}{\partial x^i}\frac{\partial L}{\partial z^\beta_j}+\frac{\partial\psi^i_\lambda}{\partial x^i}\frac{\partial L}{\partial z^\beta_j}.$$

Moreover, in the differentiation of the third term $(\partial\psi^j_\lambda/\partial x^i)(\partial L/\partial z^\beta_j)$, observing that $d(\partial L/\partial z^\beta_j)/dx^i=\partial^2 L/\partial x^i\partial z^\beta_j+z^\alpha_i\partial^2 L/\partial y^\alpha\partial z^\beta_j+u^\alpha_{ki}\partial^2 L/\partial z^\alpha_k\partial z^\beta_j$, the differentiation is written as

$$\begin{aligned}\frac{d}{dx^j}\left(\frac{\partial N_\lambda}{\partial z^\beta_j}\right)=&\psi^i_\lambda\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z^\beta_j\partial x^i}\right)+\frac{\partial\psi^i_\lambda}{\partial x^i}\frac{d}{dx^j}\left(\frac{\partial L}{\partial z^\beta_j}\right)\\&-z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial y^\alpha\partial z^\beta_i}-z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}\right)\\&-z^\alpha_k\frac{\partial^2\psi^k_\lambda}{\partial x^j\partial x^i}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}-u^\alpha_{kj}\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}-u^\alpha_{kj}\frac{\partial\psi^j_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial z^\alpha_k\partial z^\beta_i}.\end{aligned}$$

Hence, $[N_\lambda]_\beta$ becomes

$$\begin{aligned}[N_\lambda]_\beta=&\psi^i_\lambda\left[\frac{\partial^2 L}{\partial y^\beta\partial x^i}-\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z^\beta_j\partial x^i}\right)\right]+\frac{\partial\psi^i_\lambda}{\partial x^i}\left[\frac{\partial L}{\partial y^\beta}-\frac{d}{dx^j}\left(\frac{\partial L}{\partial z^\beta_j}\right)\right]\\&+z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial x^i}\left[\frac{\partial^2 L}{\partial y^\alpha\partial z^\beta_i}-\frac{\partial^2 L}{\partial y^\beta\partial z^\alpha_i}+\frac{d}{dx^j}\left(\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}\right)\right]\\&+z^\alpha_k\frac{\partial^2\psi^k_\lambda}{\partial x^j\partial x^i}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}+u^\alpha_{kj}\frac{\partial\psi^k_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\alpha_i}+u^\alpha_{kj}\frac{\partial\psi^j_\lambda}{\partial x^i}\frac{\partial^2 L}{\partial z^\alpha_k\partial z^\beta_i}.\end{aligned}\tag{6}$$

3. Examples of determining the Lagrangian densities

With the aid of symmetry equations, Lagrangian densities can be determined in the examples of I: harmonic oscillator and II: field theory.

I: As a special case, let $n=1$ and $m=1$. So skip off the indices $i=1$, $\alpha=1$ and write $t=x^1$, $y=y^1$, $z=z^1_1$, $u=u^1_{11}$ and so forth for the indices. On this situation, we shall first take a review of [6] in our formulation.

I–1: Let X^2_1 and X^2_2 be the 2-prolongations of infinitesimal generators of 2-dimensional translation group: $X^0_1=\cos\omega t\,\partial/\partial y$ and $X^0_2=\sin\omega t\,\partial/\partial y$ respectively, where ω $(\neq 0)$ is the given constant. Then, since $\psi_1=0$, $\psi_2=0$, $\xi_1=\cos\omega t$ and $\xi_2=\sin\omega t$, from (4) it follows that

$$\begin{aligned}[N_1]&=\left[\frac{\partial^2 L}{\partial y^2}-\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z\partial y}\right)+\omega^2\frac{\partial^2 L}{\partial z^2}\right]\cos\omega t+\omega\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right)\sin\omega t,\\ [N_2]&=\left[\frac{\partial^2 L}{\partial y^2}-\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z\partial y}\right)+\omega^2\frac{\partial^2 L}{\partial z^2}\right]\sin\omega t-\omega\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right)\cos\omega t.\end{aligned}\tag{7}$$

Therefore, since $\det\begin{pmatrix}\cos\omega t & \sin\omega t\\ \sin\omega t & -\cos\omega t\end{pmatrix}\neq 0$, the equations $\phi_2^*[N_1]=0$ and $\phi_2^*[N_2]=0$ for arbitrary maps ϕ_2 are satisfied if and only if

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right)=0,\quad \frac{\partial^2 L}{\partial y^2}-\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z\partial y}\right)+\omega^2\frac{\partial^2 L}{\partial z^2}=0.$$

The first equation shows that $\partial^2 L/\partial z^2=m$ (const.) and hence the Lagrangian density L should be of the form:

$$L=\frac{1}{2}mz^2+g(t,\ y)z+h(t,\ y).$$

So, by substituting this into the second equation of the above, it follows: $\partial^2 h/\partial y^2=-m\omega^2+\partial^2 g/\partial t\partial y$, which determine the function h as

$$h=-\frac{1}{2}m\omega^2 y^2+k(t)y+\int\frac{\partial g}{\partial t}dy+s(t).$$

Hence, the Lagrangian density L becomes

$$L=\frac{1}{2}mz^2-\frac{1}{2}m\omega^2 y^2+k(t)y+gz+\int\frac{\partial g}{\partial t}dy+s(t).$$

Here note the identity:

$$gz+\int\frac{\partial g}{\partial t}dy+s=\frac{d}{dt}\int g\,dy+\frac{d}{dt}\int s\,dt.$$

The right expression is understood as a null class [7, p. 78] which satisfies the Lagrangian system with respect to it identically. Thus, the Lagrangian density L is determined as

$$L=\frac{1}{2}mz^2-\frac{1}{2}m\omega^2 y^2+k(t)y,$$

up to the null class.

For the determined Lagrangian density L, from (1) it follows that

$$K_1=\frac{\partial L}{\partial z}\cos\omega t=mz\cos\omega t,$$

$$K_2=\frac{\partial L}{\partial z}\sin\omega t=mz\sin\omega t,$$

and moreover from (3) that

$$N_1=\frac{\partial L}{\partial y}\cos\omega t-\frac{\partial L}{\partial z}\omega\sin\omega t=(-m\omega^2 y+k)\cos\omega t-m\omega z\sin\omega t,$$

$$N_2=\frac{\partial L}{\partial y}\sin\omega t+\frac{\partial L}{\partial z}\omega\cos\omega t=(-m\omega^2 y+k)\sin\omega t+m\omega z\cos\omega t.$$

Hence, the conservation laws are written as

$$\frac{d}{dt}(my'\cos\omega t)+(m\omega^2 y-k)\cos\omega t+m\omega y'\sin\omega t$$

$$=\frac{d}{dt}(my'\cos\omega t+m\omega y\sin\omega t)-k\cos\omega t=0,$$

$$\frac{d}{dt}(my'\cos\omega t)+(m\omega^2 y-k)\sin\omega t-m\omega y'\cos\omega t$$

$$=\frac{d}{dt}(mt'\sin\omega t-m\omega y\cos\omega t)-k\sin\omega t=0,$$

where $y=y(t)$ and $y'=dy/dt$. So, the integration yields

$$my'\cos\omega t+m\omega y\sin\omega t-\int^t k(\tau)\cos\omega\tau d\tau=Q_1 \quad (\text{const.}),$$

$$my'\sin\omega t-m\omega y\cos\omega t-\int^t k(\tau)\sin\omega\tau d\tau=Q_2 \quad (\text{const.}),$$

and which gives the solution

$$m\omega y=Q_1\sin\omega t-Q_2\cos\omega t+\int^t k(\tau)\sin\omega(t-\tau)d\tau,$$

$$my'=Q_1\cos\omega t+Q_2\sin\omega t+\int^t k(\tau)\cos\omega(t-\tau)d\tau.$$

I-2: More generally, let us assume that the second order differential equation

$$h''(t)+h'(t)u(t)+h(t)v(t)=0 \tag{8}$$

has two independent real solutions $h_\lambda(t)$, where $\lambda=1, 2$. And for the solutions h_λ, let X_λ^2 be the 2-prolongations of infinitesimal generators of 2-dimensional translation group: $X_\lambda^0=h_\lambda\partial/\partial y$. Then, from (4) it follows that

$$[N_\lambda]=h_\lambda\left[\frac{\partial^2 L}{\partial y^2}-\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z\partial y}\right)\right]-h'_\lambda\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right)-h''_\lambda\frac{\partial^2 L}{\partial z^2}.$$

So that, since h_λ are the independent solutions of (8), the equations $\phi_2^*[N_\lambda]=0$ for arbitrary maps ϕ_2 yield

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right)=\frac{\partial^2 L}{\partial z^2}u,\quad \frac{d}{dt}\left(\frac{\partial^2 L}{\partial z\partial y}\right)-\frac{\partial^2 L}{\partial y^2}=\frac{\partial^2 L}{\partial z^2}v.$$

From the first equation it follows: $\partial^2 L/\partial z^2=m\exp\int^t u(\tau)d\tau$ (m:const.). So, by putting

$$U(t)=\int^t u(\tau)d\tau,$$

the Lagrangian density L should be of the form:

$$L=\frac{1}{2}mUz^2+g(t, y)z+h(t, y).$$

Hence, similarly as the above, by substituting this into the second equation, it follows: $\partial^2 h/\partial y^2=-mUv+\partial^2 g/\partial t\partial y$. So that h is of the form

$$h=-\frac{1}{2}mUvy^2+k(t)y+\int\frac{\partial g}{\partial t}\,dy+s(t),$$

and hence the Lagrangian density is determined as

$$L=U(t)\Big(\frac{1}{2}mz^2-\frac{1}{2}mvy^2+k(t)y\Big),$$

up to the null class.

Further, for the determined Lagrangian density, from (1) it follows that

$$K_\lambda=\frac{\partial L}{\partial z}h_\lambda=mUzh_\lambda,$$

and from (3) that

$$N_\lambda=\frac{\partial L}{\partial y}h_\lambda+\frac{\partial L}{\partial z}h'_\lambda=U(-mvy+k)h_\lambda+mUzh'_\lambda.$$

So that, since $h''_\lambda+h'_\lambda u+h_\lambda v=0$, the conservation laws are written as

$$\frac{d}{dt}(mUy'h_\lambda)+mUvyh_\lambda-Ukh_\lambda-mUy'h'_\lambda$$

$$=\frac{d}{dt}(mUy'h_\lambda-mUyh'_\lambda)-Ukh_\lambda=0.$$

Therefore, the integration yields

$$mU(y'h_\lambda-yh'_\lambda)-\int^t U(\tau)k(\tau)h_\lambda(\tau)d\tau=Q_\lambda \quad (\text{const.}),$$

and so, from which the solution is obtained:

$$mUW(h_1, h_2)y=Q_1h_2-Q_2h_1+\int^t U(\tau)k(\tau)[h_1(\tau)h_2(t)-h_1(t)h_2(\tau)]d\tau,$$

$$mUW(h_1, h_2)y'=Q_1h'_2-Q_2h'_1+\int^t U(\tau)k(\tau)[h_1(\tau)h'_2(t)-h'_1(t)h_2(\tau)]d\tau.$$

Here note that, since h_1 and h_2 are assumed to be independent, the Wronskian $W(h_1, h_2)=h_1h'_2-h'_1h_2$ does not vanish.

The above examples were already undertaken by J. C. Houard in [6]. But, here we have observed that the same results can be obtained from our formulation.

I–3: We shall now take our attention for non-conservative Lagrangian system with generalized force $F(t, y, z)$. Let X_1^2 and X_2^2 be the same defined as in I–1. In this case, from (2) it follows that

$$\Phi_1 = X_1^2(F) = \frac{\partial F}{\partial y}\cos\omega t - \frac{\partial F}{\partial z}\omega\sin\omega t,$$

$$\Phi_2 = X_2^2(F) = \frac{\partial F}{\partial y}\sin\omega t + \frac{\partial F}{\partial z}\omega\cos\omega t.$$

So that, in view of (7), the equations $\phi_2^*([N_1]+\Phi_1)=0$ and $\phi_2^*([N_2]+\Phi_2)=0$ for arbitrary maps ϕ_2 are satisfied if and only if

$$\frac{d}{dt}\left(\frac{\partial^2 L}{\partial z^2}\right) = \frac{\partial F}{\partial z}, \quad \frac{\partial^2 L}{\partial y^2} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial z \partial y}\right) + \omega^2 \frac{\partial^2 L}{\partial z^2} = -\frac{\partial F}{\partial y}.$$

The first equation is rewritten as $\partial^3 L/\partial t\partial z^2 + z\partial^3 L/\partial y\partial z^2 + u\partial^3 L/\partial z^3 = \partial F/\partial z$ in which L and F are functions of t, y and z. So, $\partial^3 L/\partial z^3 = 0$ and hence, by putting $\partial^2 L/\partial z^2 = \mu(t, y)$, the first equation is rewritten as $\partial F/\partial z = d\mu/dt$. Hence F is of the form

$$F = \frac{\partial\mu}{\partial y}z^2 + \frac{\partial\mu}{\partial t}z + \nu(t, y).$$

Moreover, since $\partial^2 L/\partial z^2 = \mu(t, y)$, the Lagrangian density L is of the form

$$L = \frac{1}{2}\mu(t, y)z^2 + g(t, y)z + h(t, y).$$

Therefore, the second equation of the above is rewritten as

$$\frac{1}{2}\frac{\partial^2\mu}{\partial y^2}z^2 - \frac{\partial\mu}{\partial y}u - \frac{\partial^2 g}{\partial t\partial y} + \frac{\partial^2 h}{\partial y^2} + \omega^2\mu + \frac{\partial\nu}{\partial y} = 0,$$

in which it is concluded: $\partial\mu/\partial y = 0$, i.e., $\mu = \mu(t)$. And so, by integrating, h is determined as

$$h = -\frac{1}{2}\omega^2\mu(t)y^2 + k(t)y - \int\nu(t, y)dy + \int\frac{\partial g}{\partial t}dy + s(t).$$

Thus, the generalized force F is determined as

$$F = \mu'(t)z + \nu(t, y),$$

and corresponding to F, the Lagrangian density L is determined as

$$L = \frac{1}{2}\mu(t)z^2 - \frac{1}{2}\omega^2\mu(t)y^2 + k(t)y - \int\nu(t, y)dy,$$

up to the null class.

II: As another considerable case of $n=4$, let x^1, x^2, x^3 be the space coordinates and $x^4 = \sqrt{-1}\,ct$ the time coordinate, where c is the velocity of light in vacuo. In the coordinates, let X^2_{jk} be the 2-prolongations of infinitesimal generators of rotation or Lorentz group: $X^0_{jk}=\delta^i_{[j}g_{k]s}x^s\partial/\partial x^i$, where the bracket of indices denotes the skew-symmetric part. Here $g_{ij}=0$ for $i\neq j$ and $g_{11}, g_{22}, g_{33}, g_{44}$ or $g_{11}, g_{22}, g_{33}, -g_{44}$ are equal to constants corresponding to the rotation or Lorentz transformation, respectively. Then, since $\psi^i_{jk}=\delta^i_{[j}g_{k]s}x^s$ and $\xi^\alpha_{jk}=0$, from (6) it follows that

$$\begin{aligned}
[N_{jk}]_\beta &= \delta_{[j}^{\,i}g_{k]t}\left[\frac{\partial^2 L}{\partial y^\beta\partial x^i}-\frac{d}{dx^s}\left(\frac{\partial^2 L}{\partial z^\beta_s\partial x^i}\right)\right]x^t \\
&+\delta_{[j}^{\,s}g_{k]i}\left[\frac{\partial^2 L}{\partial y^\alpha\partial z^\beta_i}-\frac{\partial^2 L}{\partial y^\beta\partial z^\alpha_i}+\frac{d}{dx^t}\left(\frac{\partial^2 L}{\partial z^\beta_t\partial z^\alpha_i}\right)\right]z^\alpha_s \\
&+\delta_{[j}^{\,s}g_{k]i}\left(\frac{\partial^2 L}{\partial z^\alpha_t\partial z^\beta_i}u^\alpha_{ts}+\frac{\partial^2 L}{\partial z^\beta_t\partial z^\alpha_i}u^\alpha_{st}\right) \\
&=\delta_{[j}^{\,i}g_{k]t}\left(\frac{\partial^2 L}{\partial y^\beta\partial x^i}-\frac{\partial^3 L}{\partial x^s\partial z^\beta_s\partial x^i}\right)x^t \\
&+\delta_{[j}^{\,s}g_{k]i}\left(\frac{\partial^2 L}{\partial y^\alpha\partial z^\beta_i}-\frac{\partial^2 L}{\partial y^\beta\partial z^\alpha_i}+\frac{\partial^3 L}{\partial x^t\partial z^\beta_t\partial z^\alpha_i}\right)z^\alpha_s \\
&+\left(\delta_{[j}^{\,s}g_{k]i}\frac{\partial^2 L}{\partial z^\alpha_t\partial z^\beta_i}+\delta_{[j}^{\,t}g_{k]i}\frac{\partial^2 L}{\partial z^\beta_s\partial z^\alpha_i}\right)u^\alpha_{ts} \\
&-\delta_{[j}^{\,i}g_{k]t}\frac{\partial^3 L}{\partial y^\alpha\partial z^\beta_s\partial x^i}x^t z^\alpha_s-\delta_{[j}^{\,i}g_{k]t}\frac{\partial^3 L}{\partial z^\alpha_l\partial z^\beta_s\partial x^i}x^t u^\alpha_{ls} \\
&+\delta_{[j}^{\,t}g_{k]i}\frac{\partial^3 L}{\partial y^\gamma\partial z^\beta_s\partial z^\alpha_i}z^\alpha_t z^\gamma_s+\delta_{[j}^{\,t}g_{k]i}\frac{\partial^3 L}{\partial z^\gamma_l\partial z^\beta_s\partial z^\alpha_i}z^\alpha_t u^\gamma_{ls}.
\end{aligned}$$

Our goal is to determine the Lagrangian density L by the equation $\phi_2^*[N_{jk}]_\beta=0$ for arbitrary maps ϕ_2. But here we shall assume that each of the coefficients of x^t, z^α_s, u^α_{ts}, $x^t z^\alpha_s$, $x^t u^\alpha_{ls}$, $z^\alpha_t z^\gamma_s$ and $z^\alpha_t u^\gamma_{ls}$ in $[N_{jk}]_\beta$ vanishes identically. Then, by putting $g_{ki}\partial^2 L/\partial z^\alpha_t\partial z^\beta_i=G^t_{k\alpha\beta}$, from the coefficient of u^α_{ts} it follows that $\delta_{[j}^{\,s}G^t_{k]\alpha\beta}+\delta_{[j}^{\,t}G^s_{k]\alpha\beta}=0$. This becomes $G^j_{k\alpha\beta}=0$ for $j=s=t\neq k$ and $G^j_{j\alpha\beta}-G^k_{k\alpha\beta}=0$ (no summation for j, k) for $j=s\neq k=t$, respectively. Hence $G^t_{k\alpha\beta}=c_{\alpha\beta}\delta^t_k$, i.e., $g_{ki}\partial^2 L/\partial z^\alpha_t\partial z^\beta_i=c_{\alpha\beta}\delta^t_k$ follows, where $c_{\alpha\beta}$ are constants. Therefore $\partial^2 L/\partial z^\alpha_i\partial z^\beta_j=c_{\alpha\beta}g^{ij}$ where $(g^{ij})=(g_{ij})^{-1}$ and $c_{\alpha\beta}=c_{\beta\alpha}$ may be assumed in the following since $g^{ij}=g^{ji}$, and so, the Lagrangian density L is of the form

$$L=\frac{1}{2}c_{\alpha\beta}g^{ij}z^\alpha_i z^\beta_j+h^i_\alpha(x, y)z^\alpha_i+s(x, y),$$

by which the coefficients of $x^t u^\alpha_{ls}$, $z^\alpha_t z^\gamma_s$ and $z^\alpha_t u^\gamma_{ls}$ are vanished. Hence, by putting $i=k\neq s=j$ in the coefficient of z^α_s, it follows: $\partial h^i_\beta/\partial y^\alpha=\partial h^i_\alpha/\partial y^\beta$, and also, by putting $i=j\neq k=t$ in the coefficient of $x^t z^\alpha_s$: $\partial^2 h^s_\beta/\partial y^\alpha\partial x^i=0$. So, h^i_α is written as

$$h^i_\alpha(x, y)=\overset{1}{h}{}^i_\alpha(x)+\frac{\partial\overset{2}{h}{}^i(y)}{\partial y^\alpha}.$$

Moreover, by putting $i=j\neq k=t$ in the coefficient of x^t, it follows: $\partial^2 s/\partial y^\beta \partial x^i = \partial^2 \overset{1}{h}{}^s_\beta/\partial x^s \partial x^i$, and so, $\partial s/\partial y^\beta = \partial \overset{1}{h}{}^s_\beta/\partial x^s + k^\beta(y)$. This relation yields $\partial k^\beta/\partial y^\alpha = \partial k^\alpha/\partial y^\beta$, since $\overset{1}{h}{}^s_\beta = \overset{1}{h}{}^s_\beta(x)$. Hence k^α is written as $k^\alpha = \partial k(y)/\partial y^\alpha$. Therefore s is determined as

$$s(x, y) = \frac{\partial \overset{1}{h}{}^i_\alpha(x)}{\partial x^i} y^\alpha + k(y) + t(x).$$

So that the Lagrangian density L is written as

$$L = \frac{1}{2} c_{\alpha\beta} g^{ij} z^\alpha_i z^\beta_j + k(y) + \left(\overset{1}{h}{}^i_\alpha(x) + \frac{\partial \overset{2}{h}{}^i(y)}{\partial y^\alpha}\right) z^\alpha_i + \frac{\partial \overset{1}{h}{}^i_\alpha(x)}{\partial x^i} y^\alpha + t(x),$$

in which is contained the null class

$$\left(\overset{1}{h}{}^i_\alpha(x) + \frac{\partial \overset{2}{h}{}^i(y)}{\partial y^\alpha}\right) z^\alpha_i + \frac{\partial \overset{1}{h}{}^i_\alpha(x)}{\partial x^i} y^\alpha + t(x)$$

$$= \frac{d}{dx^i}\left(\overset{1}{h}{}^i_\alpha(x) y^\alpha + \overset{2}{h}{}^i(y) + \frac{1}{4}\int t(x) dx^i\right).$$

Thus, the Lagrangian density L is determined as

$$L = \frac{1}{2} c_{\alpha\beta} g^{ij} z^\alpha_i z^\beta_j + k(y),$$

up to the null class.

Further and finally, for the determined Lagrangian density L, from (1) it follows that

$$K^i_{jk} = -z^\alpha_s \psi^s_{jk} \frac{\partial L}{\partial z^\alpha_i} + L\psi^i_{jk}$$

$$= -c_{\alpha\beta} g^{it} z^\beta_t z^\alpha_{[j} g_{k]s} x^s + L\delta^i_{[j} g_{k]t} x^t,$$

and from (5) that

$$N_{jk} = \delta^i_{[j} g_{k]t} x^t \frac{\partial L}{\partial x^i} - \delta^s_{[j} g_{k]i} z^\alpha_s \frac{\partial L}{\partial z^\alpha_i}$$

$$= -c_{\alpha\beta} \delta^s_{[j} g_{k]i} g^{it} z^\alpha_s z^\beta_t,$$

which vanishes since $c_{\alpha\beta} = c_{\beta\alpha}$ and $g_{ki} g^{it} = \delta^t_k$. Hence the conservation laws are derived:

$$\frac{\partial}{\partial x^i}\left(c_{\alpha\beta} g^{it} \frac{\partial y^\beta(x)}{\partial x^t} \frac{\partial y^\alpha(x)}{\partial x^{[j}} g_{k]s} x^s - L\delta^i_{[j} g_{k]s} x^s\right) = 0,$$

which is similar to that in [1, p. 23] derived from the Lagrangian density $L = \frac{1}{2}(g^{ij} z^\alpha_i z^\alpha_j + k^2 y^\alpha y^\alpha)$ where k is a constant.

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