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EQUIVALENT LAGRANGIAN DENSITIES IN CONTINUUM MECHANICS ASSOCIATED WITH DYNAMICAL SYMMETRIES

By

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1. Introduction

E. Noether's theorem [1] for invariant variational principle under continuous group of transformations has established a correspondence between conservation laws and arbitrary differential equations obtained from the principle. In the theorem, conservation laws may be derived from equivalent differential equations to the original one. So that a considerable problem is appeared in Lagrangian (or Hamiltonian) dynamics to determine a class of equivalent Lagrangians (or Hamiltonians) which yields equivalent Euler-Lagrange (or Hamilton) equations. Such a class which yields the same Euler-Lagrange equation was first obtained as a sum of original Lagrangian for a given Euler-Lagrange equation and a divergence term in particle mechanics or null term (class) in continuum mechanics by E. Bessel-Hagen [2], E. L. Hill [3] or D. G. B. Edelen [4], T. Nôno and F. Mimura [5], respectively. Recently, a new class of Lagrangians which yields equivalent (not exactly the same) Euler-Lagrange equations was obtained by D. G. Currie and E. J. Saletan [6] in one-dimensional Lagrangian and Hamiltonian particle mechanics and, more generally, by S. Hojman and H. Harleston [7] or F. Mimura and T. Nôno [8] in multi-dimensional Lagrangian or Hamiltonian particle mechanics respectively, while there was no consideration for the symmetries (invariances) of EulerLagrange equations. Moreover, M. Lutzky [9, 10, 11] obtained conservation laws in particle Lagrangian mechanics under symmetries of Euler-Lagrange equations (dynamical symmetries) which do not leave invariant the action integral of Larangian in the consideration (he called that non-Noether conservation laws). His conservation laws can be derived from S. Hojman and H. Harleston's ones [7] (this was shown in [8]; see also [11]), but involved new consideration relating to dynamical symmetries.

Our program is to generalize these concepts into continuum mechanics (see F. Mimura and T. Nôno [12], which is a generalization of [6, 7]). Further, in this paper, we generalize [11] under dynamical symmetries of generalized Lagrangian system in continuum mechanics [13]. So in **2**, a brief review of [13] is given for dynamical symmetries in terms of generating differential form of generalized Lagrangian system. In **3**, linear transformation laws of generalized Lagrangian systems are obtained by imposing some conditions for dynamical symmetries. These linear transformation laws are discussed on a viewpoint of conservation laws in continuum mechanics. Finally in **4**, the non-Noether conservation laws in [11] are reviewed in a special case of our generalization. And, illustrative examples are given for one-dimensional harmonic oscillator [9, p. 88] and a field of two-dimensional space-time.

2. Dynamical symmetries of generalized Lagrangian systems

First of all, we shall review the differential geometric treatment for dynamical symmetries of generalized Lagrangian system in continuum mechanics [13].

On a setting for manifolds N and M_1 with local coordinates (x^i) and $(x^i, y^\alpha, z_i^\alpha)$ respectively, the motion of continuums (fields) can be regarded as a (submanifold) mapping ϕ_1 from N into M_1:

$$\phi_1(x^i)=\left(x^i,\ y^\alpha(x),\ \frac{\partial y^\alpha(x)}{\partial x^i}\right),$$

where $i, j=1,\dots, n$ and $\alpha=1,\dots, m$. For a given Lagrangian density $L(x, y, z)$ or its exterior derivative

$$dL=L_i dx^i+L_\alpha dy^\alpha+L_\alpha^i dz_i^\alpha,$$

where $L_i=\partial L/\partial x^i$, $L_\alpha=\partial L/\partial y^\alpha$ and $L_\alpha^i=\partial L/\partial z_i^\alpha$; the generalized Lagrangian system is written as

$$\phi_1^*(L_\alpha)=\frac{1}{f}\frac{\partial}{\partial x^i}\phi_1^*(fL_\alpha^i)=0\quad (f=f(x)\text{: volume density on } N), \tag{1}$$

(here and in the following, the asterisk $*$ denotes the pull-back of considering map). Extending the manifold M_1 toward M_2 with local coordinates $(x^i, y^\alpha, z_i^\alpha, u_{ij}^\alpha)$, and using a (submanifold) mapping ϕ_2 from N into M_2:

$$\phi_2(x^i)=\left(x^i,\ y^\alpha(x),\ \frac{\partial y^\alpha(x)}{\partial x^i},\ \frac{\partial^2 y^\alpha(x)}{\partial x^j\partial x^i}\right),$$

the generalized Lagrangian system (1) is rewritten as

$$\phi_2^*([L]_\alpha)=0, \tag{2}$$

where $[L]_\alpha$ id defined as

$$[L]_\alpha=L_\alpha-\frac{1}{f}\frac{d}{dx^i}(fL_\alpha^i),$$

in which d/dx^i denotes the total differentiation:

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + z_i^\alpha \frac{\partial}{\partial y^\alpha} + u_{ki}^\alpha \frac{\partial}{\partial z_k^\alpha}.$$

An l-parameter (ε^λ: $\lambda = 1,\ldots, l$) group of transformations

$$T: \begin{cases} \bar{x}^i = \bar{x}^i(x^j, y^\beta; \varepsilon^\lambda) = \bar{x}^i(x, y; \varepsilon), \\ \bar{y}^\alpha = \bar{y}^\alpha(x^j, y^\beta; \varepsilon^\lambda) = \bar{y}^\alpha(x, y; \varepsilon), \\ \bar{z}_i^\alpha = \bar{z}_i^\alpha(x^j, y^\beta, z_j^\beta; \varepsilon^\lambda) = \bar{z}_i^\alpha(x, y, z; \varepsilon), \\ \bar{u}_{ij}^\alpha = \bar{u}_{ij}^\alpha(x^j, y^\beta, z_k^\beta, u_{ks}^\beta; \varepsilon^\lambda) = \bar{u}_{ij}^\alpha(x, y, z, u; \varepsilon), \end{cases}$$

determined by the 2-prolongation into M_2 from a group of transformations T_0: $\bar{x}^i = \bar{x}^i(x^j, y^\beta; \varepsilon^\lambda) = \bar{x}^i(x, y; \varepsilon)$, $\bar{y}^\alpha = \bar{y}^\alpha(x^j, y^\beta; \varepsilon^\lambda) = \bar{y}^\alpha(x, y; \varepsilon)$, is said to be a dynamical symmetry (invariance) group of the generalized Lagrangian system $\phi_2^*([L]_\alpha) = 0$ if $\phi_2^*([L]_\alpha) = 0$ implies

$$\phi_2^*(T_2^*[L]_\alpha) = 0. \tag{3}$$

We shall now pass on to the differential geometric reformulation of the above statement, introducing the generating differential n-form Θ on M_1 of the generalized Lagrangian system:

$$\Theta = L_\alpha^i \theta^\alpha \wedge \omega_i + L\omega,$$

where $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ (volume form on N), ω_i is its contraction by $\partial/\partial x^i$, i.e., $\omega_i = \partial/\partial x^i \lrcorner\, \omega$ and $\theta^\alpha = dy^\alpha - z_i^\alpha dx^i$. Since [13, p. 14]

$$d\Theta \equiv [L]_\alpha \theta^\alpha \wedge \omega \quad (\text{mod}\, \theta^\alpha \wedge \theta^\beta\text{'s}, \theta^\alpha \wedge \theta_j^\beta\text{'s}),$$

where $\theta^\alpha = dy^\alpha - z_i^\alpha dx^i$, it follows for arbitrary vector fields Y on M_2:

$$Y \lrcorner\, d\Theta \equiv (Y \lrcorner\, \theta^\alpha)[L]_\alpha \omega \quad (\text{mod}\, \theta^\alpha\text{'s}, \theta_i^\alpha\text{'s}). \tag{4}$$

So that the generalized Lagrangian system (1) or equivalently (2) is expressed as (note ϕ_2^* vanishes θ^α and θ_i^α)

$$\phi_2^*(Y \lrcorner\, d\Theta) = 0 \quad \text{for arbitrary vector fields } Y, \tag{5}$$

which can reformulate the dynamical symmetry (3) of the generalized Lagrangian system [13, p. 13].

DEFINITION. The generating form Θ is said to have dynamical symmetry under the l-parameter (group of) transformations T_2 if

$$\phi_2^*(Y \lrcorner\, d\Theta) = 0 \quad \text{for arbitrary vector fields } Y \tag{5}$$

implies

$$\phi_2^*[T_2^*(Y \lrcorner d\Theta)] = 0 \quad \text{for arbitrary vector fields } Y. \tag{6}$$

This definition yields the following version of dynamical symmetry in terms of the Lie derivative of $Y \lrcorner d\Theta$ by infinitesimal transformations X_λ^2 ($\lambda = 1, \ldots, l$) generating T_2:

$$X_\lambda^2 = \psi_\lambda^i(x, y)\frac{\partial}{\partial x^i} + \xi_\lambda^\alpha(x, y)\frac{\partial}{\partial y^\alpha} + \eta_{\lambda i}^\alpha(x, y, z)\frac{\partial}{\partial z_i^\alpha} + \zeta_{\lambda ij}^\alpha(x, y, z, u)\frac{\partial}{\partial u_{ij}^\alpha},$$

where the coefficients $\eta_{\lambda i}^\alpha$ and $\zeta_{\lambda ij}^\alpha$ are given as [13, p. 13]

$$\eta_{\lambda i}^\alpha = \frac{d\xi_\lambda^\alpha}{dx^i} - z_k^\alpha \frac{d\psi_\lambda^k}{dx^i}, \quad \zeta_{\lambda ij}^\alpha = \frac{d\eta_{\lambda i}^\alpha}{dx^j} - u_{ik}^\alpha \frac{d\psi_\lambda^k}{dx^j}. \tag{7}$$

DEFINITION. The generating form Θ is said to have dynamical symmetry under the infinitesimal l-parameter (group of) transformations X_λ^2 (X_λ^2 are said to be infinitesimal dynamical symmetries of $\phi_2^*([L]_\alpha) = 0$) if

$$\phi_2^*(Y \lrcorner d\Theta) = 0 \quad \text{for arbitrary vector fields } Y \tag{5}$$

implies ($X_\lambda^2(\cdot)$ denotes the Lie derivative by X_λ^2)

$$\phi_2^*[X_\lambda^2(Y \lrcorner d\Theta)] = 0 \quad \text{for arbitrary vector fields } Y. \tag{8}$$

REMARK. By the expansion for ε^λ:

$$T_2^*(Y \lrcorner d\Theta) = Y \lrcorner d\Theta + \varepsilon^\lambda X_\lambda^2(Y \lrcorner d\Theta) + (\text{higher order terms}),$$

from (6) it follows both of (5) and (8). In [13], the dynamical symmetries were studied by starting from the postulation of original generalized Lagrangian system (5). So, (5) vanished and only (6) (or also (8)) appeared in the definition for symmetries in [13, p. 13] (see also [14, 15]). But in this paper, the Lagrangian system (5) is assumed after discussing equivalent Lagrangian densities under dynamical symmetries defined as the above.

3. Generalized Lagrangian systems under dynamical symmetries

Before assuming the dynamical symmetries under X_λ^2, we shall derive an identity for generalized Lagrangian systems by virtue of the fundamental relation of differential form $d\Theta$ with respect to the vector fields X_λ^2 and Y:

$$Y \lrcorner X_\lambda^2(d\Theta) = [Y, X_\lambda^2] \lrcorner d\Theta + X_\lambda^2(Y \lrcorner d\Theta). \tag{9}$$

In this relation, first note that [13, p. 17]

$$X_\lambda^2(d\Theta) \equiv d\Xi_\lambda \quad (\text{mod } \theta^\alpha \wedge \theta^\beta\text{'s}, \theta^\alpha \wedge d\theta^\beta\text{'s});$$

where, by using of the transformed Lagrangian density N_λ:

$$N_\lambda = X_\lambda^2(L) + L\left(\frac{d\psi_\lambda^i}{dx^i} + \frac{1}{f} X_\lambda^2(f)\right),$$

and its derivatives $N^i_{\lambda\alpha}=\partial N_\lambda/\partial z^\alpha_i$, the form Ξ_λ is defined as

$$\Xi_\lambda=N^i_{\lambda\alpha}\theta^\alpha\wedge\omega_i+N_\lambda\omega.$$

This form has a similar relation as (4):

$$\begin{aligned} Y\lrcorner X^?_\lambda(d\Theta)&\equiv Y\lrcorner d\Xi_\lambda \qquad (\text{mod } \theta^\alpha\text{'s, } d\theta^\alpha\text{'s})\\ &\equiv(Y\lrcorner\theta^\alpha)[N_\lambda]_\alpha\omega \qquad (\text{mod } \theta^\alpha\text{'s, } \theta^\alpha_i\text{'s}), \end{aligned} \tag{10}$$

and so, is a generating differential form of new generalized Lagrangian system $\phi^*_?([N_\lambda]_\alpha)=0$. Since the infinitesimal transformations $X^?_\lambda$ satisfy [13, p. 13]

$$X^?_\lambda(\theta^\alpha)\equiv 0 \ (\text{mod } \theta^\alpha\text{'s, } \theta^\alpha_i\text{'s}), \quad X^?_\lambda(\theta^\alpha_i)\equiv 0 \ (\text{mod } \theta^\alpha\text{'s, } \theta^\alpha_i\text{'s}),$$

from (4) it follows that

$$X^?_\lambda(Y\lrcorner d\Theta)=X^?_\lambda[(Y\lrcorner\theta^\alpha)[L]_\alpha\omega] \quad (\text{mod } \theta^\alpha\text{'s, } \theta^\alpha_i\text{'s}). \tag{11}$$

Moreover, since

$$\begin{aligned} [Y, X^?_\lambda]\lrcorner d\Theta&\equiv([Y, X^?_\lambda]\lrcorner\theta^\alpha)[L]_\alpha\omega \quad (\text{mod } \theta^\alpha\text{'s, } \theta^\alpha_i\text{'s})\\ &=[Y\lrcorner X^?_\lambda(\theta^\alpha)-X^?_\lambda(Y\lrcorner\theta^\alpha)][L]_\alpha\omega, \end{aligned} \tag{12}$$

by substituting (10), (11) and (12) into (9), it is obtained:

$$\begin{aligned} (Y\lrcorner\theta^\alpha)[N_\lambda]_\alpha\omega\equiv&[Y\lrcorner X^?_\lambda(\theta^\alpha)][L]_\alpha\omega+(Y\lrcorner\theta^\alpha)[L]_\alpha X^?_\lambda(\omega)\\ &+(Y\lrcorner\theta^\alpha)X^?_\lambda([L]_\alpha)\omega \quad (\text{mod } \theta^\alpha\text{'s, } d\theta^\alpha\text{'s, } \theta^\alpha_i\text{'s}). \end{aligned} \tag{13}$$

So that, since $X^?_\lambda(\omega)\equiv\Phi_\lambda\omega$ (mod θ^α's) and $X^?_\lambda(\theta^\alpha)=A^\alpha_{\lambda\beta}\theta^\beta$ [13, p. 16], where

$$\Phi_\lambda=\frac{d\psi^i_\lambda}{dx^i}+\frac{\psi^i_\lambda}{f}\frac{\partial f}{\partial x^i}, \quad A^\alpha_{\lambda\beta}=\frac{\partial\xi^\alpha_\lambda}{\partial y^\beta}-z^\alpha_k\frac{\partial\psi^k_\lambda}{\partial y^\beta}, \tag{14}$$

the following theorem is obtained by putting Y as $Y_\alpha=\partial/\partial y^\alpha$ in (13) (note: $Y_\alpha\lrcorner\theta^\beta=\delta^\beta_\alpha$) and operating $\phi^*_?$.

THEOREM 1. *For arbitrary $\phi^*_?$, the following relations are satisfied between the original Lagrangian density L and its transformed Lagrangian densities N_λ under infinitesimal (group of) transformations $X^?_\lambda$:*

$$\phi^*_?([N_\lambda]_\alpha)=\phi^*_?(A^\beta_{\lambda\alpha}[L]_\beta+\Phi_\lambda[L]_\alpha+X^?_\lambda([L]_\alpha)). \tag{15}$$

REMARK. Let assume the dynamical symmetry equation [12, p. 4] $\phi^*_?[X^?_\lambda(Y\lrcorner d\Theta)]=0$ of generalized Lagrangian system $\phi^*_?(Y\lrcorner d\Theta)=0$, where Y is an arbitrary vector field. Then, by putting Y as $Y_\alpha=\partial/\partial y^\alpha$, it follows from (11) that [13, p. 15]

$$\phi^*_?(\Phi_\lambda[L]_\alpha+X^?_\lambda([L]_\alpha))=0.$$

Therefore (15) takes more simple form

$$\phi_2^*([N_\lambda]_\alpha)=\phi_2^*(A_{\lambda\alpha}^\beta[L]_\beta),$$

which is the linear transformation law of generalized Lagrangian system obtained in our previous paper [12, p. 4].

Now recall the derivatives $\partial N_\lambda/\partial z_i^\alpha$ $(=N_{\lambda\alpha}^i)$ can be given as [13, p. 17]

$$\frac{\partial N_\lambda}{\partial z_i^\alpha}=A_{\lambda\alpha}^\gamma\frac{\partial L}{\partial z_i^\gamma}+\Phi_\lambda\frac{\partial L}{\partial z_i^\alpha}+X_\lambda^2\left(\frac{\partial L}{\partial z_i^\alpha}\right)-\frac{d\psi_\lambda^i}{dx^k}\frac{\partial L}{\partial z_k^\alpha}+\frac{\partial\psi_\lambda^i}{\partial y^\alpha}L. \tag{16}$$

By substituting the identities (note: $\psi_\lambda^i=\psi_\lambda^i(x,y)$, $\xi_\lambda^\alpha=\xi_\lambda^\alpha(x,y)$ and $\eta_{\lambda i}^\alpha=d\xi^\alpha/dx^i-z_k^\alpha d\psi^k/dx^i$)

$$\frac{\partial\Phi_\lambda}{\partial z_j^\beta}=\frac{\partial}{\partial z_j^\beta}\left(\frac{d\psi_\lambda^i}{dx^i}+\frac{\psi_\lambda^i}{f}\frac{\partial f}{\partial x^i}\right)=\frac{\partial\psi_\lambda^j}{\partial y^\beta},$$

$$\frac{\partial A_{\lambda\alpha}^\gamma}{\partial z_j^\beta}=\frac{\partial}{\partial z_j^\beta}\left(\frac{\partial\xi_\lambda^\alpha}{\partial y^\beta}-z_k^\alpha\frac{\partial\psi_\lambda^k}{\partial y^\beta}\right)=-\delta_\beta^\gamma\frac{\partial\psi_\lambda^j}{\partial y^\alpha},$$

$$\frac{\partial}{\partial z_j^\beta}X_\lambda^2\left(\frac{\partial L}{\partial z_i^\alpha}\right)=X_\lambda^2\left(\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}\right)+\frac{\partial\eta_{\lambda k}^\gamma}{\partial z_j^\beta}\frac{\partial^2 L}{\partial z_k^\gamma\partial z_i^\alpha}$$

$$=X_\lambda^2\left(\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}\right)+\delta_j^k A_{\lambda\beta}^\gamma-\delta_\beta^\gamma\frac{d\psi_\lambda^j}{dx^k},$$

into the differentiation of (16) with respect to z_j^β, the relations are obtained:

$$\frac{\partial^2 N_\lambda}{\partial z_j^\beta\partial z_i^\alpha}=A_{\lambda\alpha}^\gamma\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\gamma}+A_{\lambda\beta}^\gamma\frac{\partial^2 L}{\partial z_j^\gamma\partial z_i^\alpha}+\Phi_\lambda\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}+X_\lambda^2\left(\frac{\partial^2 L}{\partial z_j^\beta\partial z_i^\alpha}\right)$$
$$-\frac{d\psi_\lambda^i}{dx^k}\frac{\partial^2 L}{\partial z_j^\beta\partial z_k^\alpha}-\frac{d\psi_\lambda^j}{dx^k}\frac{\partial^2 L}{\partial z_k^\beta\partial z_i^\alpha}+4\frac{\partial\psi_\lambda^{[i}}{\partial y^{[\alpha}}\frac{\partial L}{\partial z_{j]}^{\beta]}}, \tag{17}$$

where the brackets denote respectively the skew symmetric parts for the corresponding indices i, j and α, β.

We shall here assume that there exist some functions $C_{\lambda\alpha}^\beta(x, y, z)$ and the group of infinitesimal transformations X_λ^2 satisfies the following linear relations for arbitrary ϕ_2^*:

$$\phi_2^*(X_\lambda^2([L]_\alpha))=\phi_2^*(C_{\lambda\alpha}^\beta[L]_\beta). \tag{18}$$

Under these X_λ^2, the generating form $d\Theta$ has a dynamical symmetry (of course, X_λ^2 are infinitesimal dynamical symmetries of $\phi_2^*([L]_\alpha)=0$), because (4) leads

$$X_\lambda^2(Y\lrcorner d\Theta)\equiv X_\lambda^2[(Y\lrcorner\theta^\alpha)\omega]+(Y\lrcorner\theta^\alpha)X_\lambda^2([L]_\alpha)\omega \quad (\text{mod}\ \theta^\alpha\text{'s},\ \theta_i^\alpha\text{'s}).$$

Since the terms containing u_{ij}^α in $\zeta_{\lambda ij}^\alpha$ of X_λ^2 are

$$u_{ij}^\alpha\text{'s term in }\zeta_{\lambda ij}^\alpha=A_{\lambda\beta}^\alpha u_{ij}^\beta-\frac{d\psi_\lambda^k}{dx^i}u_{kj}^\alpha-\frac{d\psi_\lambda^k}{dx^j}u_{ik}^\alpha,$$

by separating both sides of (18) into two independent terms according to $\phi_2^*(u^\beta_{ji})=\partial^2 y^\beta(x)/\partial x^j\partial x^i$ (note ϕ_2^* is arbitrary), the relations are obtained:

$$\phi_2^*\Big(C^\gamma_{\lambda\alpha}\frac{\partial^2 L}{\partial z^\beta_{(j}\partial z^\gamma_{i)}}\Big)=\phi_2^*\Big(\Lambda^\gamma_{\lambda\beta}\frac{\partial^2 L}{\partial z^\gamma_{(j}\partial z^\alpha_{i)}}+X^2_\lambda\Big(\frac{\partial^2 L}{\partial z^\beta_{(j}\partial z^\alpha_{i)}}\Big)$$
$$-\frac{d\psi^{(i}_\lambda}{dx^k}\frac{\partial^2 L}{\partial z^\beta_{j)}\partial z^\alpha_k}-\frac{d\psi^{(j}_\lambda}{dx^k}\frac{\partial^2 L}{\partial z^\beta_k\partial z^\alpha_{i)}}\Big), \tag{19}$$

where the parenthesis denote the symmetric parts for the corresponding indices. Thus the follwing theorem is obtained by substituting (18) and (19) into (15) and the symmetric part of (17) for i, j, respectively.

THEOREM 2. *The l-parameter (group of) infinitesimal dynamical symmetries X^2_λ satisfying linear relations for arbitrary ϕ_2^*:*

$$\phi_2^*(X^2_\lambda[L]_\alpha)=\phi_2^*(C^\beta_{\lambda\alpha}[L]_\beta) \tag{18}$$

yield the following linear relations:

$$\phi_2^*([N_\lambda]_\alpha)=\phi_2^*(\Delta^\beta_{\lambda\alpha}[L]_\beta),$$
$$\phi_2^*\Big(\frac{\partial^2 N_\lambda}{\partial z^\beta_{(j}\partial z^\alpha_{i)}}\Big)=\phi_2^*\Big(\Delta^\gamma_{\lambda\alpha}\frac{\partial^2 L}{\partial z^\beta_{(j}\partial z^\gamma_{i)}}\Big),$$

where $\Delta^\beta_{\lambda\alpha}$ are defined as

$$\Delta^\beta_{\lambda\alpha}=\Lambda^\beta_{\lambda\alpha}+\delta^\beta_\alpha\Phi_\lambda+C^\beta_{\lambda\alpha}. \tag{20}$$

Moreover the following theorem is obtained from (17) and (19) with some further conditions for the Lagrangian density L.

THEOREM 3. *Let the Lagrangian density L satisfies*

$$\frac{\partial\psi^{[i}_\lambda}{\partial y^{[\alpha}}\frac{\partial L}{\partial z^{\beta]}_{j]}}=0,\qquad \frac{\partial^2 L}{\partial z^\beta_{[j}\partial z^\alpha_{i]}}=0. \tag{21}$$

Then the l-parameter (group of) infinitesimal dynamical symmetries X^2_λ satisfying linear relations for arbitrary ϕ_2^*:

$$\phi_2^*(X^2_\lambda([L]_\alpha))=\phi_2^*(C^\beta_{\lambda\alpha}[L]_\beta) \tag{18}$$

yield the following linear relations:

$$\phi_2^*([N_\lambda]_\alpha)=\phi_2^*(\Delta^\beta_{\lambda\alpha}[L]_\beta),$$
$$\phi_2^*\Big(\frac{\partial^2 N_\lambda}{\partial z^\beta_j\partial z^\alpha_i}\Big)=\phi_2^*\Big(\Delta^\gamma_{\lambda\alpha}\frac{\partial^2 L}{\partial z^\beta_j\partial z^\gamma_i}\Big).$$

By virtue of the theorem in [12, Theorem 3], the theorem 3 yields immediately the following theorem.

THEOREM 4. *Let the Lagrangian density satisfies*

$$\frac{\partial\psi^{[i}_{\lambda}}{\partial y^{[\alpha}}\frac{\partial L}{\partial z^{\beta]}_{j]}}=0, \qquad \frac{\partial^2 L}{\partial z^{\beta}_{[j}\partial z^{\alpha}_{i]}}=0. \tag{21}$$

Then the l-parameter (group of) infinitesimal dynamical symmetries X^2_λ satisfying linear relations for arbitrary ϕ^*_2:

$$\phi^*_2(X^2_\lambda[L]_\alpha)=\phi^*_2(C^\beta_{\lambda\alpha}[L]_\beta) \tag{18}$$

yield the following relations:

(i) *if $\Delta^\beta_{\lambda\alpha}$ are symmetric for α, β:*

$$\phi^*_2\left(\Delta^\alpha_{\lambda\gamma}\frac{d\Delta^\beta_{\lambda\alpha}}{dx^i}\frac{\partial^2 L}{\partial z^\gamma_j\partial z^\beta_i}+\Delta^\alpha_{\lambda\gamma}\frac{\partial\Delta^\beta_{\lambda\alpha}}{\partial z^\gamma_j}[L]_\beta\right)=0,$$

(ii) *if $\Delta^\beta_{\lambda\alpha}$ are symmetric for α, β and nonsingular:*

$$\phi^*_2\left((\Delta^{-1}_\lambda)^\alpha_\gamma\frac{d\Delta^\beta_{\lambda\alpha}}{dx^i}\frac{\partial^2 L}{\partial z^\gamma_j\partial z^\beta_i}+(\Delta^{-1}_\lambda)^\alpha_\gamma\frac{\partial\Delta^\beta_{\lambda\alpha}}{\partial z^\alpha_j}[L]_\beta\right)=0,$$

(iii) *if $\Delta^\gamma_{\lambda\alpha}=\underset{1}{\Delta}^\beta_{\lambda\alpha}\underset{2}{\Delta}^\gamma_{\lambda\beta}$ where $\underset{1}{\Delta}^\beta_{\lambda\alpha}$ and $\underset{2}{\Delta}^\beta_{\lambda\alpha}$ are both symmetric for α, β:*

$$\phi^*_2\left(\underset{2}{\Delta}^\alpha_{\lambda\gamma}\frac{d\Delta^\beta_{\lambda\alpha}}{dx^i}\frac{\partial^2 L}{\partial z^\gamma_j\partial z^\beta_i}+\underset{2}{\Delta}^\alpha_{\lambda\gamma}\frac{\Delta^\beta_{\lambda\alpha}}{\partial z^\gamma_j}[L]_\beta\right)=0.$$

4. Illustrative examples

I. We shall first review the M. Lutzky's conservation laws ([11], see also [9]) as a special case of our generalized formulation in continuum mechanics (field theory): dim· $N=1$ (denote $x^1=t$ and $f=1$ for brevity). This is the case of particle mechanics and the Lagrangian density L on M_1 is given as $L(t, y, z)$. Here is assumed the nonsingularity of $(W_{\alpha\beta})=(\partial^2L/\partial z^\alpha\partial z^\beta)$: det $(W_{\alpha\beta})\neq 0$, and $(Z^{\alpha\beta})=(W_{\alpha\beta})^{-1}$. Then, by using of

$$(L)_\alpha=\frac{\partial L}{\partial y^\alpha}-\frac{\partial^2 L}{\partial z^\alpha\partial t}-\frac{\partial^2 L}{\partial z^\alpha\partial z^\beta}z^\beta,$$

$[L]_\alpha$ is rewritten as

$$[L]_\alpha=W_{\alpha\beta}(F^\beta-u^\beta) \quad \text{where} \quad F^\alpha=Z^{\alpha\beta}(L)_\beta. \tag{22}$$

So the Lagrangian system (Euler-Lagrange equation) $\phi^*_2([L]_\alpha)=0$ is equivalent to

$$\phi^*_2(u^\alpha)=\phi^*_2(F^\alpha), \quad \text{i.e.,} \quad \ddot{y}^\alpha(t)=F^\alpha(t, y(t), \dot{y}(t)). \tag{23}$$

M. Lutzky's starting point for deriving conservation laws is that the one-parameter infinitesimal group of transformations leaves invariant the equivalent form of Euler-

Lagrange equation (23). In terms of Lutzky's notation $\Gamma=\partial/\partial t+z^{\alpha}\partial/\partial y^{\alpha}+F^{\alpha}\partial/\partial z^{\alpha}$ [11, p. 88], the invariance condition is given as [9, p. 86]

$$\phi_2^*[X^2(F^{\alpha})]=\phi_2^*[\Gamma(\xi^{\alpha})-z^{\alpha}\Gamma(\psi)-2\Gamma(\psi)F^{\alpha}], \tag{24}$$

where X^2 is of the form (assume $l=1$ and put $X_1^2=X^2$ in (11))

$$X^2=\psi(t, y)\frac{\partial}{\partial t}+\xi^{\alpha}(t, y)\frac{\partial}{\partial y^{\alpha}}+\eta^{\alpha}\frac{\partial}{\partial z^{\alpha}}+\zeta^{\alpha}\frac{\partial}{\partial u^{\alpha}}.$$

Since $\eta^{\alpha}=d\xi^{\alpha}/dt-z^{\alpha}d\psi/dt$ and $\zeta^{\alpha}=d\eta^{\alpha}/dt-u^{\alpha}d\psi/dt$, further calculation of (24) leads

$$\phi_2^*[X^2(F^{\alpha})]=\phi_2^*\left[\left(\frac{\partial\xi^{\alpha}}{\partial y^{\beta}}-z^{\alpha}\frac{\partial\psi}{\partial y^{\beta}}-2\delta_{\beta}^{\alpha}\frac{d\psi}{dt}\right)(F^{\beta}-u^{\beta})+\zeta^{\alpha}\right],$$

which yields the relation by (22):

$$\phi_2^*[X^2(F^{\alpha})-\zeta^{\alpha}]=\phi_2^*[(\Lambda_{\beta}^{\alpha}-2\delta_{\beta}^{\alpha}\Phi)Z^{\beta\gamma}[L]_{\gamma}],$$

where $\Phi=d\psi/dt$ and $\Lambda_{\beta}^{\alpha}=\partial\xi^{\alpha}/\partial y^{\beta}-z^{\alpha}\partial\psi/\partial y^{\beta}$ [see (23)]. Therefore, since

$$\begin{aligned}X^2([L]_{\alpha})&=X^2[W_{\alpha\beta}(F^{\beta}-u^{\beta})]\\&=X^2(W_{\alpha\beta})(F^{\beta}-u^{\beta})+W_{\alpha\beta}[X^2(F^{\beta})-\zeta^{\beta}],\end{aligned}$$

the invariance condition (24) is rewritten as

$$\phi_2^*[X^2([L]_{\alpha})]=\phi_2^*(C_{\alpha}^{\beta}[L]_{\beta}),\quad\text{where}$$
$$C_{\alpha}^{\beta}=X^2(W_{\alpha\gamma})Z^{\gamma\beta}+W_{\alpha\gamma}(\Lambda_{\sigma}^{\gamma}-2\delta_{\sigma}^{\gamma}\Phi)Z^{\sigma\beta},$$

which yields the same identities in [11, p. 88] [see (20)]:

$$\begin{aligned}\Delta_{\alpha}^{\beta}&=\Lambda_{\alpha}^{\beta}+\delta_{\alpha}^{\beta}\Phi+C_{\alpha}^{\beta}\\&=\Lambda_{\alpha}^{\beta}+W_{\alpha\gamma}\Lambda_{\sigma}^{\gamma}Z^{\sigma\beta}+X^2(W_{\alpha\gamma})Z^{\gamma\beta}-\delta_{\alpha}^{\beta}\Phi.\end{aligned}$$

Now in the theorem 3, since the conditions of (21) are always satisfied (note i and j take only one value 1), the following relations are verified:

$$\phi_2^*(N_{\alpha\beta})=\phi_2^*(\Delta_{\alpha}^{\gamma}W_{\gamma\beta}),\quad\text{i.e.,}\quad\phi_2^*(\Delta_{\alpha}^{\beta})=\phi_2^*(N_{\alpha\gamma}Z^{\gamma\beta}),$$

where $N_{\alpha\beta}=\partial^2N/\partial z^{\alpha}\partial z^{\beta}$ correspond to $\underset{1}{\Delta}{}_{\alpha}^{\beta}$ in the theorem 4 [case (iii)]. Thus the theorem yields the following conservation law derived by Lutzky in [11] (see also [9]):

$$\phi_2^*\left(Z^{\alpha\gamma}\frac{d\Delta_{\alpha}^{\beta}}{dt}W_{\gamma\beta}\right)=\phi_2^*\left(\frac{d\Delta_{\alpha}^{\alpha}}{dt}\right)=0.$$

EXAMPLE 1. Let $m=1$, i.e., α takes only one value 1 (so denote $y^1=y$, $z^1=z$, $u^1=u$) and in this case consider the Lagrangian density for one-dimensional harmonic oscillator [9, p. 88]:

$$L=\frac{1}{2}(y^2-z^2) \quad ([L]=y+u).$$

From given $\psi(t, y)$ and $\xi(t, y)$, the (one-parameter) infinitesimal transformation X^2 is determined by [see (7)]

$$\eta=\frac{d\xi}{dt}-\frac{d\psi}{dt}z=\frac{\partial\xi}{\partial t}+\left(\frac{\partial\xi}{\partial y}-\frac{\partial\psi}{\partial t}\right)z+\frac{\partial\psi}{\partial y}z^2,$$

$$\zeta=\frac{d\eta}{dt}-\frac{d\psi}{dt}u=\frac{\partial^2\xi}{\partial t^2}+\left(2\frac{\partial^2\xi}{\partial y\partial z}-\frac{\partial^2\psi}{\partial t^2}\right)z+\left(\frac{\partial^2\xi}{\partial y^2}-2\frac{\partial^2\psi}{\partial y\partial t}\right)z^2$$
$$+\frac{\partial^2\psi}{\partial y^2}z^3+\left(\frac{\partial\xi}{\partial y}-2\frac{\partial\psi}{\partial t}-3\frac{\partial\psi}{\partial y}z\right)u.$$

So the condition: $\phi_2^*(X^2[L])=\phi_2^*(C(t, y, z)[L])$, i.e., $\phi_2^*(\xi+\zeta)=\phi_2^*(C(y+u))$ is satisfied for arbitrary ϕ_2 if and only if

$$\left(\xi+\frac{\partial^2\xi}{\partial t^2}\right)+\left(2\frac{\partial^2\xi}{\partial y\partial z}-\frac{\partial^2\psi}{\partial t^2}\right)z+\left(\frac{\partial^2\xi}{\partial y^2}-2\frac{\partial^2\psi}{\partial y\partial t}\right)z^2+\frac{\partial^2\psi}{\partial y^2}z^3=Cy, \tag{25}$$

$$\frac{\partial\xi}{\partial y}-2\frac{\partial\psi}{\partial t}-3\frac{\partial\psi}{\partial y}z=C. \tag{26}$$

The equation (26) is substituted into (25) to obtain

$$\left(\xi+\frac{\partial^2\xi}{\partial t^2}-y\frac{\partial\xi}{\partial y}+2y\frac{\partial\psi}{\partial t}\right)+\left(2\frac{\partial^2\xi}{\partial y\partial t}-\frac{\partial^2\psi}{\partial t^2}+3y\frac{\partial\psi}{\partial y}\right)z$$
$$+\left(\frac{\partial^2\xi}{\partial y^2}-2\frac{\partial^2\psi}{\partial y\partial t}\right)z^2+\frac{\partial^2\psi}{\partial y^2}z^3=0,$$

which leads the equations:

$$-2y\frac{\partial\psi}{\partial t}=\xi+\frac{\partial^2\xi}{\partial t^2}-y\frac{\partial\xi}{\partial y}, \tag{27}$$

$$\frac{\partial^2\psi}{\partial t^2}-3y\frac{\partial\psi}{\partial y}=2\frac{\partial^2\xi}{\partial y\partial t}, \tag{28}$$

$$2\frac{\partial^2\psi}{\partial y\partial t}=\frac{\partial^2\xi}{\partial y^2}, \tag{29}$$

$$\frac{\partial^2\psi}{\partial y^2}=0. \tag{30}$$

Equation (30) gives the form of ψ:

$$\psi=\alpha(t)y+\beta(t),$$

which rewrites (29) as

$$\frac{\partial^2 \xi}{\partial y^2}=2\frac{\partial \alpha}{\partial t},$$

and determine the form of ξ:

$$\xi=\frac{\partial \alpha}{\partial t}y^2+\varphi(t)y+\gamma(t).$$

So that, by substituting these ψ and ξ, (28) becomes

$$\left(2\frac{\partial \varphi}{\partial t}-\frac{\partial^2 \beta}{\partial t^2}\right)+3\left(\frac{\partial^2 \alpha}{\partial t^2}+\alpha\right)y=0,$$

$$\text{i.e., } 2\frac{\partial \varphi}{\partial t}-\frac{\partial^2 \beta}{\partial t^2}=0, \quad \frac{\partial^2 \alpha}{\partial t^2}+\alpha=0.$$

Hence φ and α are of the form:

$$\varphi=\frac{1}{2}\frac{\partial \beta}{\partial t}+b \qquad (b\text{: const.}),$$

$$\alpha=\alpha_1 \sin t+\alpha_2 \cos t \qquad (\alpha_1,\ \alpha_2\text{: const.}),$$

and so

$$\psi=(\alpha_1 \sin t+\alpha_2 \cos t)y+\beta(t),$$

$$\xi=(\alpha_1 \cos t-\alpha_2 \sin t)y^2+\left(\frac{1}{2}\frac{\partial \beta}{\partial t}+b\right)y+\gamma(t).$$

By substituting these ψ and ξ into (27), it is obtained:

$$\left(\frac{\partial^2 \gamma}{\partial t^2}+\gamma\right)+\left(\frac{1}{2}\frac{\partial^3 \beta}{\partial t^3}+2\frac{\partial \beta}{\partial t}\right)y=0,$$

$$\text{i.e., } \frac{\partial^2 \gamma}{\partial t^2}+\gamma=0, \quad \frac{\partial^3 \beta}{\partial t^3}+4\frac{\partial \beta}{\partial t}=0,$$

So, γ is determined as

$$\gamma=\gamma_1 \sin t+\gamma_2 \cos t \quad (\gamma_1,\ \gamma_2\text{: const.}),$$

and moreover the β's equation:

$$\frac{\partial^2 \beta}{\partial t^2}+4\beta=a, \quad \text{i.e., } \frac{\partial^2(\beta-a)}{\partial t^2}+4(\beta-a)=0 \quad (a\text{: const.})$$

determines β as

$$\beta=\beta_1 \sin 2t+\beta_2 \cos 2t \quad (\beta_1,\ \beta_2\text{: const.}).$$

Therefore the complete forms of ψ and ξ are given as

$$\psi=(\alpha_1 \sin t+\alpha_2 \cos t)y+\beta_1 \sin 2t+\beta_2 \cos 2t+a,$$

$$\xi=(\alpha_1 \cos t-\alpha_2 \sin t)y^2+(\beta_1 \cos 2t-\beta_2 \sin 2t+b)y+\gamma_1 \sin t+\gamma_2 \cos t,$$

$$(a,\ b:\ \alpha_1,\ \alpha_2;\ \beta_1,\ \gamma_2;\ \gamma_1,\ \gamma_2:\ \text{const.}).$$

Now, since [see (14) and (26)]

$$\Phi=\frac{\partial\psi}{\partial t}+\frac{\partial\psi}{\partial y}z, \quad \Lambda=\frac{\partial\xi}{\partial y}-\frac{\partial\psi}{\partial y}z, \quad C=\frac{\partial\xi}{\partial y}-2\frac{\partial\psi}{\partial t}-3\frac{\partial\psi}{\partial y}z,$$

the conserved quantity (constant of motion) $\Delta=\Lambda+\Phi+C$ [see (20)] is derived:

$$\Delta=3\alpha_1(y \cos t-z \sin t)-3\alpha_2(y \sin t+z \cos t)+2b.$$

Here note that Lutzky obtained only the non-Noether conserved quantity: $\Delta=3(y \cos t-z \sin t)$ in his method [9, p. 88].

II. Finally we can give an example for a field of two-dimensional space-time in our formulation.

EXAMPLE 2. Let $n=2$ (i.e., $\dim N=2$), $m=1$ and denote $(x^i)=(x^1, x^2)=(t, x)$; $y^1=y$, $z^1_i=z_i$, $u^1_{ij}=u_{ij}$ $(i, j=1, 2)$. In this case, consider the Lagrangian density

$$L=\frac{1}{2}(x^2y^2-z_1^2) \quad ([L]=x^2y+u_{11}),$$

which satisfies the condition of (21). First, η_1 and ζ_{11} are determined from ψ^k $(k=1, 2)$ and ξ:

$$\eta_1=\frac{d\xi}{dt}-\frac{d\psi^k}{dt}z_k=\frac{\partial\xi}{\partial t}+\frac{\partial\xi}{\partial y}z_1-\frac{\partial\psi^k}{\partial t}z_k-\frac{\partial\psi^k}{\partial y}z_1z_k,$$

$$\begin{aligned}\zeta_{11}=\frac{d\eta_1}{dt}-\frac{d\psi^k}{dt}u_{1k}=&\frac{\partial^2\xi}{\partial t^2}+2\frac{\partial^2\xi}{\partial y\partial t}z_1-\frac{\partial^2\psi^k}{\partial t^2}z_k-\frac{\partial^2\xi}{\partial y^2}z_1^2\\&-2\frac{\partial^2\psi^k}{\partial y\partial t}z_1z_k+\frac{\partial^2\psi^k}{\partial y^2}z_1^2z_k\\&+\left(\frac{\partial\xi}{\partial y}-2\frac{\partial\psi^1}{\partial t}-3\frac{\partial\psi^1}{\partial y}z_1-2\frac{\partial\psi^2}{\partial y}z_2\right)u_{11}\\&-2\left(\frac{\partial\psi^2}{\partial t}+\frac{\partial\psi^2}{\partial y}z_1\right)u_{12};\end{aligned}$$

where $u_{12}=u_{21}$ is assumed for brevity, since $\phi_2^*(u_{12})=\phi_2^*(u_{21})$. So that from the condition $\phi_2^*(X^2[L])=\phi_2^*(C(t, x, y, z_1, z_2)[L])$, i.e., $\phi_2^*(2xy\psi^2+x^2\xi+\zeta_{11})=\phi_2^*(C(x^2y+u_{11}))$ it follows:

$$\frac{\partial \psi^2}{\partial t} + \frac{\partial \psi^2}{\partial y} z_1 = 0, \quad \text{i.e.,} \quad \frac{\partial \psi^2}{\partial t} = \frac{\partial \psi^2}{\partial y} = 0,$$

and so ($\psi^2 = c = \text{const.}$) it follows moreover:

$$\left(2cxy + x^2\xi + \frac{\partial^2 \xi}{\partial t^2}\right) + \left(2\frac{\partial^2 \xi}{\partial y \partial t} - \frac{\partial^2 \psi^1}{\partial t^2}\right) z_1 + \left(\frac{\partial^2 \xi}{\partial y^2} - 2\frac{\partial^2 \psi^1}{\partial y \partial t}\right) z_1^2$$
$$+ \left(\frac{\partial^2 \xi}{\partial y^2} - 2\frac{\partial^2 \psi^1}{\partial y \partial t}\right) z_1^2 + \frac{\partial^2 \psi^1}{\partial y^2} z_1^3 = Cx^2 y, \tag{31}$$

$$\frac{\partial \xi}{\partial y} - 2\frac{\partial \psi^1}{\partial t} - 3\frac{\partial \psi^1}{\partial y} z_1 = C. \tag{32}$$

The equation (32) is substituted into (31) to obtain

$$\left(2cxy + x^2\xi - x^2 y\frac{\partial \xi}{\partial y} + \frac{\partial^2 \xi}{\partial t^2} + 2x^2 y\frac{\partial \psi^1}{\partial t}\right)$$
$$+ \left(2\frac{\partial^2 \xi}{\partial y \partial t} - \frac{\partial^2 \psi^1}{\partial t^2} + 3x^2 y\frac{\partial \psi^1}{\partial y}\right) z_1 + \left(\frac{\partial^2 \xi}{\partial y^2} - 2\frac{\partial^2 \psi^1}{\partial y \partial t}\right) z_1^2 + \frac{\partial^2 \psi^1}{\partial y^2} z_1^3 = 0,$$

which leads the equations:

$$-2x^2 y\frac{\partial \psi^1}{\partial t} = 2cxy + x^2\xi - x^2 y\frac{\partial \xi}{\partial y} + \frac{\partial^2 \xi}{\partial y^2}, \tag{33}$$

$$\frac{\partial^2 \psi^1}{\partial t^2} - 3x^2 y\frac{\partial \psi^1}{\partial y} = 2\frac{\partial^2 \xi}{\partial y \partial t}, \tag{34}$$

$$2\frac{\partial^2 \psi^1}{\partial y \partial t} = \frac{\partial^2 \xi}{\partial y^2}, \tag{35}$$

$$\frac{\partial^2 \psi^1}{\partial y^2} = 0. \tag{36}$$

Equation (36) gives the form of ψ^1:

$$\psi^1 = \alpha(t, x)y + \beta(t, x),$$

and hence (35) becomes

$$\frac{\partial^2 \xi}{\partial y^2} = 2\frac{\partial^2 \psi^1}{\partial y \partial t} = 2\frac{\partial \alpha}{\partial t},$$

which determines the form of ξ:

$$\xi = \frac{\partial \alpha}{\partial t} y^2 + \varphi(t, x)y + \gamma(t, x).$$

By substituting these ψ^1 and ξ into (34), the following equation is obtained:

$$\left(2\frac{\partial\varphi}{\partial t}-\frac{\partial^2\beta}{\partial t^2}\right)+3\left(\frac{\partial^2\alpha}{\partial t^2}+x^2\alpha\right)y=0,$$

$$\text{i.e.,}\quad 2\frac{\partial\varphi}{\partial t}-\frac{\partial^2\beta}{\partial t^2}=0,\quad \frac{\partial^2\alpha}{\partial t^2}+x^2\alpha=0,$$

and φ, α are determined as

$$\varphi=\frac{1}{2}\frac{\partial\beta}{\partial t}+\tau(x),$$

$$\alpha=\alpha_1(x)\sin tx+\alpha_2(x)\cos tx,$$

and so

$$\psi^1=(\alpha_1\sin tx+\alpha_2\cos tx)y+\beta,$$

$$\xi=(\alpha_1\cos tx-\alpha_2\sin tx)xy^2+\left(\frac{1}{2}\frac{\partial\beta}{\partial t}+\tau\right)y+\gamma.$$

By substituting these ψ^1 and ξ into (33), it is reduced:

$$\left(\frac{\partial^2\gamma}{\partial t^2}+x^2\gamma\right)+\left(\frac{1}{2}\frac{\partial^3\alpha}{\partial t^3}+2x^2\frac{\partial\alpha}{\partial t}+2cx\right)y=0,$$

$$\text{i.e.,}\quad \frac{\partial^2\gamma}{\partial t^2}+x^2\gamma=0,\quad \frac{\partial^3\beta}{\partial t^3}+4x^2\frac{\partial\beta}{\partial t}+4cx=0.$$

So, γ is determined as

$$\gamma=\gamma_1(x)\sin tx+\gamma_2(x)\cos tx;$$

and moreover, since the β's equation is rewritten as

$$\frac{\partial^2\beta}{\partial t^2}+4x^2\beta+4cxt=b(x),\quad \text{i.e.,}$$

$$\frac{\partial^2}{\partial t^2}\left(\beta+\frac{ct}{x}-\frac{b}{x^2}\right)+4x^2\left(\beta+\frac{ct}{x}-\frac{b}{x^2}\right)=0,$$

β is determined as (put $b/x^2=\sigma$)

$$\beta=\beta_1(x)\sin 2tx+\beta_2(x)\cos 2tx-\frac{ct}{x}+\sigma(x).$$

Therefore ψ^1, ψ^2 and ξ are determined completely:

$$\psi^2=c=\text{const.},$$

$$\psi^1=[\alpha_1(x)\sin tx+\alpha_2(x)\cos tx]y+\beta_1(x)\sin 2tx+\beta_2(x)\cos 2tx-\frac{ct}{x}+\sigma(x),$$

$$\xi=[\alpha_1(x)\cos tx+\alpha_2(x)\sin tx]xy^2+[\beta_1(x)\cos 2tx-\beta_2(x)\sin 2tx]xy$$

$$-\left[\frac{c}{2x}-\tau(x)\right]y+\gamma_1(x)\sin tx+\gamma_2(x)\cos tx.$$

Thus, in terms of [see (14) and (32)]

$$\Phi=\frac{\partial\psi^1}{\partial t}+\frac{\partial\psi^1}{\partial y}z_1,\quad \Lambda=\frac{\partial\xi}{\partial y}-\frac{\partial\psi^1}{\partial y}z_1,\quad C=\frac{\partial\xi}{\partial y}-2\frac{\partial\psi^1}{\partial t}-3\frac{\partial\psi^1}{\partial y}z_1,$$

the quantity $\Delta=\Lambda+\Phi+C$ is derived:

$$\Delta=3\alpha_1(x)\,(xy\cos tx-z_1\sin tx)+3\alpha_2(x)\,(xy\sin tx+z_1\cos tx)+2\tau(x).$$

From the Lagrangian density $L=(x^2y^2-z_1^2)/2$ if follows

$$\frac{d\Delta}{dx^i}\frac{\partial^2 L}{\partial z_1\partial z_i}=-\frac{d\Delta}{dt};$$

and so we conclude that the quantity Δ is conserved, in the case of (i) of theorem 4. Of course, this is observed by direct calculation:

$$-\frac{d}{dt}(xy\cos tx-z_1\sin tx)=(x^2y+u_{11})\sin tx=[L]\sin tx,$$

$$\frac{d}{dt}(xy\sin tx-z_1\cos tx)=(x^2y+u_{11})\cos tx=[L]\cos tx.$$

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