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# A 4-DIMENSIONAL JACOBI MANIFOLD WITH NONSINGULAR AND NONSEMICLOSED 2-FORM

Akira IKUSHIMA*, Fumiyo FUJIWARA** and Fumitake MIMURA**

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## 1. Introduction

Lichnerowicz [6] showed that a Lie algebra structure (so-called Jacobi structure) on a differentiable manifold $\mathfrak{M}$ is equivalent to the existence of a couple $(\Lambda, \Xi)$ of a bivector (contravariant skew-symmetric 2-tensor) field $\Lambda$ and a vector field $\Xi$ satisfying certain conditions which were given in terms of the Schouten-Nijenhuis bracket for multivector fields (Schouten [9], Nijenhuis [8]). Whenever $\dim \mathfrak{M} = 2m$ and $\Lambda$ is nonsingular, the couple $(\Lambda, \Xi)$ can be transformed into a couple $(\Omega, \omega)$ of nonsingular differential 2-form $\Omega$ and a closed differential 1-form $\omega$ $(d\omega = 0)$ satisfying

$$d\Omega + \omega \wedge \Omega = 0. \tag{1}$$

Such a form $\Omega$ (called a semiclosed form with $\omega$) defines an infinitesimally conformally symplectic structure on $\mathfrak{M}$ (Guédira and Lichnerowicz [2]). Mimura and Nôno [7] gave an alternative approach to the infinite dimensional Lie algebra structure with the couple $(\Omega, \Xi)$ of nonsingular differential 2-form $\Omega$ and a vector field $\Xi$ on $\mathfrak{M}$, where $\Omega$ was assumed to be a semiclosed form with $\omega = \Xi \rfloor \Omega$ (contraction of $\Omega$ by $\Xi$) so that $\Xi(\Omega) = 0$ (Lie derivative of $\Omega$ by $\Xi$). More generally, Ikushima and Mimura [3] investigated the structure with the couple $(\Omega, \omega)$ satisfying (1) where (instead of on $\omega$) the closedness condition was imposed on $\Xi(\omega)$:

$$d\Xi(\omega) = 0. \tag{2}$$

Since the form $\omega$ in (1) is always closed if $\dim \mathfrak{M} = 2m \geq 6$ (Libermann [5]), nonsemiclosed 2-form $\Omega$ exists only if $m = 1$ or 2, while (1) is identical for an arbitrary 2-form $\Omega$ if $m = 1$. In fact, Fujiwara, Sakurai and Mimura [1] established a complete class of nonsingular 2-forms $\Omega$ on 4-dimensional differentiable manifold, in which appeared a subclass of nonsemiclosed 2-forms. The presented paper is devoted to find the Jacobi structure associated with the 2-forms in the subclass.

The all objects involved on $\mathfrak{M}$ is assumed to be differentiable of sufficiently high order.

## 2. Jacobi structure

Let $\mathfrak{M}$ be a 4-dimensional differentiable manifold, $\mathfrak{X}$ a set of all vector fields on $\mathfrak{M}$ and $\mathfrak{R}$ a ring of all differentiable functions on $\mathfrak{M}$. Our discussion begins with a brief review of [3] for introducing Jacobi structure associated with a nonsingular 2-form $\Omega$ on $\mathfrak{M}$. For the form $\Omega$, since $\dim \mathfrak{M} = 4$, there exists a unique 1-form $\omega$ satisfying (1) (Lee [4]); and for the form $\omega$, also a unique vector field $\Xi$ satisfying

$$\omega = \Xi \rfloor \Omega. \tag{3}$$

For an arbitrary $f \in \mathfrak{R}$, set a unique vector field $X_f$ on $\mathfrak{M}$:

$$X_f \rfloor \Omega = df + f\omega, \tag{4}$$

and define the product $\{f, g\}$ on $\mathfrak{R}$:

$$\{f, g\} = X_f(g) - g\Xi(f). \tag{5}$$

Then, whenever the condition (2) is imposed on the couple $(\Omega, \omega)$, the subset $\mathfrak{X}_\Xi$:

$$\mathfrak{X}_\Xi = \{X_f \in \mathfrak{X} \mid [\Xi, X_f] = X_{\Xi(f)}\}$$

forms a Lie subalgebra of the Lie algebra $\mathfrak{X}$ under the bracket $[\,,]$, and the subset $\mathfrak{R}_\Xi$ (of course, subring of $\mathfrak{R}$):

$$\mathfrak{R}_\Xi = \{f \in \mathfrak{R} \mid X_f \in \mathfrak{X}_\Xi\}$$

forms an infinite dimensional Lie algebra under the product $\{\,,\}$.

Now, in view of (1) and (3), since

$$\Xi \rfloor d\Omega = -\Xi \rfloor (\omega \wedge \Omega) = \omega \wedge (\Xi \rfloor \Omega) = \omega \wedge \omega = 0,$$

the basic identity

$$\Xi(\Omega) = \Xi \rfloor d\Omega + d(\Xi \rfloor \Omega)$$

leads to

$$\Xi(\Omega) = d\omega. \tag{6}$$

The identity (1), i.e., $d\Omega = -\omega \wedge \Omega$ yields $\omega \wedge d\Omega = 0$, and then its exterior derivative leads to $d\omega \wedge \Omega = 0$. Accordingly

$$\Xi(d\omega \wedge \Omega) = \Xi(d\omega) \wedge \Omega + d\omega \wedge \Xi(\Omega) = 0,$$

for which (2) and (6) are substituted to see $d\omega \wedge d\omega = 0$. Hence the following cases occur on nonclosed $\omega$:

$$\text{(i)} \quad d\omega \neq 0, \quad \omega \wedge d\omega = 0, \qquad \text{(ii)} \quad \omega \wedge d\omega \neq 0, \quad d\omega \wedge d\omega = 0.$$

Therefore, within a suitable local coordinate system $(x^1, x^2, y^1, y^2)$, the forms $\omega$ and $\Omega$ have the following appearances according to the cases (see [1]):

$$\omega = x^1 dy^1, \qquad \Omega = e^{-x^1 y^1}\Theta, \qquad \text{for (i);}$$

$$\omega = x^1 dy^1 + dy^2, \qquad \Omega = e^{-x^1 y^1 - y^2}\Theta, \qquad \text{for (ii);}$$

in which $\Theta$ is of the form

$$\Theta = d\alpha \wedge dx^1 - d\beta \wedge dy^1 + y^1 \beta dx^1 \wedge dy^1,$$

where $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are arbitrary differentiable functions satisfying

$$\begin{vmatrix} \dfrac{\partial\alpha}{\partial x^2} & \dfrac{\partial\alpha}{\partial y^2} \\ \dfrac{\partial\beta}{\partial x^2} & \dfrac{\partial\beta}{\partial y^2} \end{vmatrix} \equiv a \neq 0. \tag{7}$$

In both cases, since $d\omega = dx^1 \wedge dy^1$, by putting

$$\Xi = \xi^1(x, y)\frac{\partial}{\partial x^1} + \xi^2(x, y)\frac{\partial}{\partial x^2} + \eta^1(x, y)\frac{\partial}{\partial y^1} + \eta^2(x, y)\frac{\partial}{\partial y^2},$$

it follows that $\Xi \rfloor d\omega = -\eta^1 dx^1 + \xi^1 dy^1$. So the condition (2):

$$d\Xi(\omega) = d(\Xi \rfloor d\omega + d(\Xi \rfloor \omega)) = d(\Xi \rfloor d\omega) = 0$$

leads to $d\eta^1 \wedge dx^1 - d\xi^1 \wedge dy^1 = 0$, i.e.,

$$\frac{\partial\xi^1}{\partial x^1} + \frac{\partial\eta^1}{\partial y^1} = 0, \qquad \frac{\partial\xi^1}{\partial x^2} = \frac{\partial\xi^1}{\partial y^2} = 0, \qquad \frac{\partial\eta^1}{\partial x^2} = \frac{\partial\eta^1}{\partial y^2} = 0.$$

Therefore $\xi^1$ and $\eta^1$ are of the forms

$$\xi^1 = \frac{\partial\Phi(x^1, y^1)}{\partial y^1}, \qquad \eta^1 = -\frac{\partial\Phi(x^1, y^1)}{\partial x^1}. \tag{8}$$

Pay attention here to the identity

$$[\Xi, X_f] \rfloor \Omega = \Xi(X_f \rfloor \Omega) - X_f \rfloor \Xi(\Omega),$$

for which (4) and (6) are substituted to see

$$[\Xi, X_f] \rfloor \Omega = X_{\Xi(f)} \rfloor \Omega + f\Xi(\omega) - X_f \rfloor d\omega.$$

Therefore it is verified that

$$X_f \in \mathfrak{X}_\Xi \qquad \text{if and only if} \qquad f\Xi(\omega) = X_f \rfloor d\omega. \tag{9}$$

We are now in a position to pursue our discussion in each case of (i) and (ii) with the identity

$$\Xi \rfloor \Theta = \left(\xi^2 \frac{\partial \alpha}{\partial x^2} + \eta^2 \frac{\partial \alpha}{\partial y^2} + \eta^1 \gamma\right) dx^1 - \left(\xi^1 \frac{\partial \alpha}{\partial x^2} - \eta^1 \frac{\partial \beta}{\partial x^2}\right) dx^2 \tag{10}$$

$$- \left(\xi^2 \frac{\partial \beta}{\partial x^2} + \eta^2 \frac{\partial \beta}{\partial y^2} + \xi^1 \gamma\right) dy^1 - \left(\xi^1 \frac{\partial \alpha}{\partial y^2} - \eta^1 \frac{\partial \beta}{\partial y^2}\right) dy^2,$$

where $\gamma = \partial\alpha/\partial y^1 + \partial\beta/\partial x^1 - y^1\beta$. For the case (i), since $\omega = x^1 dy^1$ in (3), the coefficients of $dx^2$ and $dy^2$ (also of $dx^1$) in $\Xi \rfloor \Omega = e^{-x^1 y^1} \Xi \rfloor \Theta$, i.e., in (10) vanish. Accordingly

$$\xi^1 \frac{\partial \alpha}{\partial x^2} - \eta^1 \frac{\partial \beta}{\partial x^2} = 0,$$

$$\xi^1 \frac{\partial \alpha}{\partial y^2} - \eta^1 \frac{\partial \beta}{\partial y^2} = 0,$$

so that $\xi^1 = \eta^1 = 0$ by (7). Therefore $\Xi$ is of the form

$$\Xi = \xi^2(x, y) \frac{\partial}{\partial x^2} + \eta^2(x, y) \frac{\partial}{\partial y^2}.$$

Hence $\Xi(\omega) = 0$, i.e., $X_f \rfloor d\omega = 0$ in (9), which concludes that $X_f \in \mathfrak{X}_\Xi$ is of the form

$$X_f = \xi_f^2(x, y) \frac{\partial}{\partial x^2} + \eta_f^2(x, y) \frac{\partial}{\partial y^2}.$$

Consequently, since $X_f \rfloor \Theta = X_f(\alpha) dx^1 - X_f(\beta) dy^1$, (4) implies that $\partial f/\partial x^2 = \partial f/\partial y^2 = 0$, i.e., $f = f(x^1, y^1)$. Thus

$$\mathfrak{R}_\Xi = \{f \mid f = f(x^1, y^1) \in \mathfrak{R}\}, \tag{11}$$

on which the product $\{f, g\}$ of (5) vanishes because of $\Xi(f) = X_f(g) = 0$.

For the case (ii), in view of (8) and (10), since $\omega = x^1 dy^1 + dy^2$, (3) gives in coordinates the system of equations

$$\xi^2 \frac{\partial \alpha}{\partial x^2} + \eta^2 \frac{\partial \alpha}{\partial y^2} = \gamma \frac{\partial \Phi}{\partial x^1}, \tag{12a}$$

$$\frac{\partial \alpha}{\partial x^2} \frac{\partial \Phi}{\partial y^1} + \frac{\partial \beta}{\partial x^2} \frac{\partial \Phi}{\partial x^1} = 0, \tag{12b}$$

$$\xi^2 \frac{\partial \beta}{\partial x^2} + \eta^2 \frac{\partial \beta}{\partial y^2} = -\gamma \frac{\partial \Phi}{\partial y^1} - x^1 e^{x^1 y^1 + y^2}, \tag{12c}$$

$$\frac{\partial \alpha}{\partial y^2} \frac{\partial \Phi}{\partial y^1} + \frac{\partial \beta}{\partial y^2} \frac{\partial \Phi}{\partial x^1} = -e^{x^1 y^1 + y^2}. \tag{12d}$$

Remind that $\Phi = \Phi(x^1, y^1)$ for the integration of (12b):

$$\alpha \frac{\partial \Phi}{\partial y^1} + \beta \frac{\partial \Phi}{\partial x^1} = \psi(x^1, y^1, y^2),$$

whose differentiation by $y^2$ is combined with (12d) to see $\partial\psi/\partial y^2 = -e^{x^1 y^1 + y^2}$, i.e.,

$$\psi(x^1, y^1, y^2) = -e^{x^1 y^1 + y^2} + \varphi(x^1, y^1).$$

Hence (12b) and (12d) are valid if and only if

$$\alpha \frac{\partial \Phi}{\partial y^1} + \beta \frac{\partial \Phi}{\partial x^1} = -e^{x^1 y^1 + y^2} + \varphi(x^1, y^1). \tag{13}$$

Since $a \neq 0$ in (7), the equations (12b) and (12d) are equivalent to

$$a \frac{\partial \Phi}{\partial x^1} = -e^{x^1 y^1 + y^2} \frac{\partial \alpha}{\partial x^2}, \tag{12b)$'$}$$

$$a \frac{\partial \Phi}{\partial y^1} = e^{x^1 y^1 + y^2} \frac{\partial \beta}{\partial x^2}. \tag{12d)$'$}$$

In view of (12b) and (12d), the solutions $\xi^2$ and $\eta^2$ of (12a) and (12c) are given respectively by

$$a\xi^2 = \gamma\left(\frac{\partial \alpha}{\partial y^2}\frac{\partial \Phi}{\partial y^1} + \frac{\partial \beta}{\partial y^2}\frac{\partial \Phi}{\partial x^1}\right) + x^1 e^{x^1 y^1 + y^2}\frac{\partial \alpha}{\partial y^2} = e^{x^1 y^1 + y^2}\left(x^1 \frac{\partial \alpha}{\partial y^2} - \gamma\right),$$

$$-a\eta^2 = \gamma\left(\frac{\partial \alpha}{\partial x^2}\frac{\partial \Phi}{\partial y^1} + \frac{\partial \beta}{\partial x^2}\frac{\partial \Phi}{\partial x^1}\right) + x^1 e^{x^1 y^1 + y^2}\frac{\partial \alpha}{\partial x^2} = x^1 e^{x^1 y^1 + y^2}\frac{\partial \alpha}{\partial x^2}.$$

Consequently $\Xi$ is determined as

$$\Xi = \frac{\partial \Phi}{\partial y^1}\frac{\partial}{\partial x^1} + a^{-1} e^{x^1 y^1 + y^2}\left(x^1 \frac{\partial \alpha}{\partial y^2} - \gamma\right)\frac{\partial}{\partial x^2} - \frac{\partial \Phi}{\partial x^1}\frac{\partial}{\partial y^1} - a^{-1} x^1 e^{x^1 y^1 + y^2}\frac{\partial \alpha}{\partial x^2}\frac{\partial}{\partial y^2},$$

in which the coefficient of $\partial/\partial y^2$ is replaced by (12b)$'$ with $x^1 \partial\Phi/\partial x^1$ to see $\Xi(\omega) = d\Phi$. Therefore, by putting

$$X_f = \xi_f^1(x, y)\frac{\partial}{\partial x^1} + \xi_f^2(x, y)\frac{\partial}{\partial x^2} + \eta_f^1(x, y)\frac{\partial}{\partial y^1} + \eta_f^2(x, y)\frac{\partial}{\partial y^2},$$

$\xi_f^1$ and $\eta_f^1$ are determined by (9) respectively as $\xi_f^1 = f\partial\Phi/\partial y^1$ and $\eta_f^1 = -f\partial\Phi/\partial x^1$. Accordingly $X_f$ is of the form

$$X_f = f\frac{\partial \Phi}{\partial y^1}\frac{\partial}{\partial x^1} + \xi_f^2\frac{\partial}{\partial x^2} - f\frac{\partial \Phi}{\partial x^1}\frac{\partial}{\partial y^1} + \eta_f^2\frac{\partial}{\partial y^2}.$$

So, similary as (10), $X_f \rfloor \Theta$ is written in coordinate. And then, in the result, (12b) and (12d) is used to derive

$$\begin{aligned} X_f \rfloor \Theta = &\left(\xi_f^2 \frac{\partial \alpha}{\partial x^2} + \eta_f^2 \frac{\partial \alpha}{\partial y^2} - f \frac{\partial \Phi}{\partial x^1} \gamma\right) dx^1 - f\left(\frac{\partial \Phi}{\partial y^1}\frac{\partial \alpha}{\partial x^2} + \frac{\partial \Phi}{\partial x^1}\frac{\partial \beta}{\partial x^2}\right) dx^2 \\ &- \left(\xi_f^2 \frac{\partial \beta}{\partial x^2} + \eta_f^2 \frac{\partial \beta}{\partial y^2} + f \frac{\partial \Phi}{\partial y^1} \gamma\right) dy^1 - f\left(\frac{\partial \Phi}{\partial y^1}\frac{\partial \alpha}{\partial y^2} + \frac{\partial \Phi}{\partial x^1}\frac{\partial \beta}{\partial y^2}\right) dy^2 \\ = &\left(\xi_f^2 \frac{\partial \alpha}{\partial x^2} + \eta_f^2 \frac{\partial \alpha}{\partial y^2} - f \frac{\partial \Phi}{\partial x^1} \gamma\right) dx^1 \\ &- \left(\xi_f^2 \frac{\partial \beta}{\partial x^2} + \eta_f^2 \frac{\partial \beta}{\partial y^2} + f \frac{\partial \Phi}{\partial y^1} \gamma\right) dy^1 + e^{x^1y^1+y^2} f dy^2, \end{aligned}$$

where $\gamma = \partial\alpha/\partial y^1 + \partial\beta/\partial x^1 - y^1\beta$. Consequently (4) gives in coorinates the system of equations

$$\frac{\partial \alpha}{\partial x^2}\xi_f^2 + \frac{\partial \alpha}{\partial y^2}\eta_f^2 = f\frac{\partial \Phi}{\partial x^1}\left(\frac{\partial \alpha}{\partial y^1} + \frac{\partial \beta}{\partial x^1} - y^1\beta\right) + e^{x^1y^1+y^2}\frac{\partial f}{\partial x^1},$$

$$\frac{\partial \beta}{\partial x^2}\xi_f^2 + \frac{\partial \beta}{\partial y^2}\eta_f^2 = -f\frac{\partial \Phi}{\partial y^1}\left(\frac{\partial \alpha}{\partial y^1} + \frac{\partial \beta}{\partial x^1} - y^1\beta\right) - e^{x^1y^1+y^2}\left(\frac{\partial f}{\partial y^1} + x^1 f\right),$$

and $\partial f/\partial x^2 = \partial f/\partial y^2 = 0$. Hence $\xi_f^2$ and $\eta_f^2$ are determined uniquely by the above equations, while $f = f(x^1, y^1)$. Thus $\mathfrak{R}_\Xi$ is determined also as (11), on which the product $\{f, g\}$ of (5) is written as

$$\{f, g\} = f\left(\frac{\partial g}{\partial x^1}\frac{\partial \Phi}{\partial y^1} - \frac{\partial g}{\partial y^1}\frac{\partial \Phi}{\partial x^1}\right) - g\left(\frac{\partial f}{\partial x^1}\frac{\partial \Phi}{\partial y^1} - \frac{\partial f}{\partial y^1}\frac{\partial \Phi}{\partial x^1}\right); \tag{14}$$

and, by (12b)′ and (12d)′, also as

$$\{f, g\} = a^{-1}e^{x^1y^1+y^2}\frac{\partial \beta}{\partial x^2}\left(f\frac{\partial g}{\partial x^1} - g\frac{\partial f}{\partial x^1}\right) + a^{-1}e^{x^1y^1+y^2}\frac{\partial \alpha}{\partial x^2}\left(f\frac{\partial g}{\partial y^1} - g\frac{\partial f}{\partial y^1}\right), \tag{14$'$}$$

which completes the Jacobi structure in the consideration. In conclusion, the following theorem is deduced (see [3], Theorem 2).

THEOREM. *Let $\Omega$ be a nonsingular and nonsemiclosed differential* 2*-form on* 4*-dimensional differentiable manifold $\mathfrak{M}$ satisfying* (1), *where $\omega$ is a differential* 1*-form whose existence is a matter of course. Then the nonvanishing Jacobi structure in the consideration exists if and only if $\Omega$ has the following appearance within a suitable local coordinate system*:

$$\Omega = e^{-x^1y^1-y^2}(d\alpha \wedge dx^1 - d\beta \wedge dy^1 + y^1\beta dx^1 \wedge dy^1), \tag{15}$$

*where $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are arbitrary differentiable functions satisfying the nonsingularlity condition* (7), *and the relation* (13) *for some differentiable functions*

*$\Phi(x^1, y^1)$ and $\varphi(x^1, y^1)$. For the form $\Omega$ of* (15), *the Jacobi structure on $\mathfrak{R}_{\Xi}$ of* (11) *is given by* (14) *or equivalently by* (14)′.

## References

[1] F. Fujiwara, T. Sakurai and F. Mimura, A complete class of nonsingular 2-form on 4-dimensional differentiable manifold, to appear.

[2] F. Guédira and A. Lichnerowicz, Géométrie des algébres de Lie locales de Kirillov, J. Math. Pures et Appl., **63** (1984), 407–484.

[3] A. Ikushima and F. Mimura, Lie algebra structures on a manifold with a couple of differential forms, to appear in Tensor, N. S., **60** (1998).

[4] H. C. Lee, A kind of even-dimensional differential geometry and its application to exterior calculus, Amer. Mth., **65** (1943), 433–438.

[5] P. Libermann, Sur le probleme d'equivalence de certains structures infinitesimales, Ann. Math. Pura Appl., **36** (1954), 27–120.

[6] A. Lichnerowicz, Les variétés de Jacobi et leurs algébres de Lie associées, J. Math. Pures et Appl., **57** (1978), 453–488.

[7] F. Mimura and Nôno, A generalized Poisson algebra on a manifold I, A geometric formulation with non-degenerate 2-form, Tensor, N. S., **55** (1994), 54–58.

[8] A. Nijenhuis, Jacobi-type identities for bilinear differential concomitants of certain tensor fields, Indag. Math., **17** (1955) 390–403.

[9] J. A. Schouten, Über differentialkomitanten zweier kontravarianter grossen, Proc. Kon. Ned. Akad. Wet. Amsterdam, **43** (1940), 449–452.

**Department of Information Science*
*Yomiuri Institute of Science and Engineering*
*Kokurakita, Kitakyushu, 802-0017, Japan*
*and*
***Department of Mathematics*
*Kyushu Institute of Technology*
*Tobata, Kitakyushu, 804-8550, Japan*