Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 49, 2002, pp. 1–18

CONSERVATION LAWS AND OPTIMAL PATHS IN EXTERNAL THREE-SECTOR GROWTH MODEL

Fumiyo FUJIWARA*, Fumitake MIMURA* and Takayuki NÔNO**

(Received October 24, 2001)

1. Introduction

Noether theorem (Noether [11]) concerning with symmetries of the action integral or its generalization (Bessel-Hagen [1]) with those up to divergence plays an effective role for discovering conservation laws from the Lagrangian or the Hamiltonian structures of considering problem.

In contrast with Noether theorem, we built up a new operative procedure for the derivation of conservation laws (Mimura and Nôno [7]) and applied it to various economic growth models (Mimura and Nôno [8]; Mimura, Fujiwara and Nôno [9], [10]; Fujiwara, Mimura and Nôno [3]–[6]) to discover new economic conservation laws including non-Noether ones. Particularly in [3], the procedure was so reformed as to make an effective application to more general neoclassical optimal growth models. And in (Fujiwara, Mimura and Nôno [4]), by a reduction of the theorem 1 in [3], the application was pursued to a one sector model of Ramsey type (Ramsey [12]) with a constant discount rate relative to a utility (welfare) of consumption, and then the model was generalized in an external two-sector version with linear technologies. The growth process relative to the technologies were characterized by a matrix of second order. By the reduced theorem, we found three types of conservation laws according as the discriminant of the characteristic equation of the matrix is positive, zero or negative. And in (Fujiwara, Mimura and Nôno [5]), optimal paths were determined completely through the three types of conservation laws, while the utility is assumed to be of second order polynomial of consumptions.

In this paper, also in (Fujiwara, Mimura and Nôno [6]), more application of the reduced theorem can be made to an external three-sector growth model with linear technology. In **2**, we first sketch the six types of triple conservation laws discovered in [6]. In **3**, through the conservation laws, optimal paths are determined completely for finite horizon and then detailed for infinite horizon under a given utility of second order polynomial of consumptions.

For convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

2. Conservation laws in external three-sector growth model

The external two-sector growth model of Ramsey type [4] can be extended to a three-sector version. In [6], we have discussed the objective of society to maximize the following integration (the social welfare functional) over a finite ($0 < T < \infty$) or an infinite ($T = \infty$) period of time:

$$(1) \qquad \int_0^T e^{-\rho t} U(c^1, c^2, c^3) dt,$$

under constraints (external growth process with respect to the consumption c^μ and the capital-labour ratio x^μ in μ-th ($\mu = 1, 2, 3$) sector):

$$\dot{x}^\mu = g^\mu(x^1, x^2, x^3) - n^\mu_\nu x^\nu - c^\mu \qquad (n^\mu_\nu\text{: const., } n^\mu_\nu > 0;\ \mu, \nu = 1, 2, 3),$$

where U is a utility (welfare) function provided with the concavity (see, e.g., [13]), i.e., the successive principal minors D_k ($k = 1, 2, 3$) of Hessian matrix of U satisfy $D_1 < 0$, $D_2 > 0$ and

$$(2) \qquad D_3 = \det\left(\frac{\partial^2 U}{\partial c^\mu \partial c^\nu}\right) < 0;$$

and g^μ are assumed to be linear production technologies

$$g^\mu = \alpha^\mu_\nu x^\nu + \beta^\mu \qquad (\alpha^\mu_\nu, \beta^\mu\text{: const.}),$$

so that the growth process are written as

$$(3) \qquad \dot{x}^\mu = a^\mu_\nu x^\nu - c^\mu + \beta^\mu \qquad (a^\mu_\nu = \alpha^\mu_\nu - n^\mu_\nu\text{: const.}).$$

In the multiplier technique to the problem, the Lagrangian is given by (π_μ are the multipliers):

$$L = e^{-\rho t} U + \pi_\mu(\dot{x}^\mu - a^\mu_\nu x^\nu + c^\mu - \beta^\mu),$$

whose Euler-Lagrange equations consist of (3) and

$$(4a) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0: \qquad \dot{\pi}_\mu + a^\nu_\mu \pi_\nu = 0,$$

$$(4b) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{c}^\mu}\right) - \frac{\partial L}{\partial c^\mu} = 0: \qquad e^{-\rho t}\frac{\partial U}{\partial c^\mu} + \pi_\mu = 0.$$

A conserved quantity (first integral) for the maximizing problem is a quantity Ξ of the variables $\dot{\pi}_\mu, \dot{x}^\mu, \dot{c}^\mu, \pi_\mu, x^\mu, c^\mu$ ($\mu = 1, 2, 3$) and t whose total time derivative vanishes ($\dot{\Xi} = 0$: conservation law) on the optimal paths, i.e., on solutions to the relating Euler-Lagrange equations (3), (4a) and (4b).

When $A = (a^\mu_\nu)$ has real characteristic values λ_μ ($\mu = 1, 2, 3$), set the linearly independent vectors $\boldsymbol{p}_\mu$ ($\mu = 1, 2, 3$) in the following three cases (i), (ii) and (iii), respectively.

(i) The case of $\lambda_\mu \neq \lambda_\nu$ $(\mu \neq \nu)$: The characteristic vectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ of A with respective characteristic values λ_1, λ_2 and λ_3.

(ii) The case of $\lambda_1 \neq \lambda_2 = \lambda_3$ is devided into two subcases.

(ii-1) $\operatorname{rank}(A - \lambda_2 E) = 1$: The characteristic vector $\boldsymbol{p}_1$ of A with characteristic value λ_1 and linearly independent characteristic vectors $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ of A with characteristic value λ_2.

(ii-2) $\operatorname{rank}(A - \lambda_2 E) = 2$: The characteristic vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of A with respective characterisitic values λ_1 and λ_2, and a vector $\boldsymbol{p}_3$ satisfying $A\boldsymbol{p}_3 = \boldsymbol{p}_2 + \lambda_2 \boldsymbol{p}_3$, i.e., $(A - \lambda_2 E)\boldsymbol{p}_3 = \boldsymbol{p}_2$. Such a vector $\boldsymbol{p}_3$ exists by $\operatorname{rank}(A - \lambda_2 E) = 2$.

(iii) The case of $\lambda \equiv \lambda_1 = \lambda_2 = \lambda_3$ is devided into two subcases with an assumption that A is not a constant multiple of the identity E. Here remark that whenever A is a constant multiple of E, there is no externality, i.e., each of three sectors behaves independently of the others.

(iii-1) $\operatorname{rank}(A - \lambda E) = 1$: Linearly independent characteristic vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ with characteristic value λ, and a vecter $\boldsymbol{p}_3$ satisfying $A\boldsymbol{p}_3 = \boldsymbol{p}_2 + \lambda \boldsymbol{p}_3$, i.e., $(A - \lambda E)\boldsymbol{p}_3 = \boldsymbol{p}_2$.

(iii-2) $\operatorname{rank}(A - \lambda E) = 2$: The characteristic vecter $\boldsymbol{p}_1$ with characteristic value λ, and vectors $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ satisfying $A\boldsymbol{p}_2 = \boldsymbol{p}_1 + \lambda \boldsymbol{p}_2$ and $A\boldsymbol{p}_3 = \boldsymbol{p}_2 + \lambda \boldsymbol{p}_3$, i.e., $(A - \lambda E)\boldsymbol{p}_2 = \boldsymbol{p}_1$ and $(A - \lambda E)\boldsymbol{p}_3 = \boldsymbol{p}_2$, respectively.

In fact, the vectors $\boldsymbol{p}_3$ in (iii-1), $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ in (iii-2) exist by the respective condition of the rank of the matrix $A - \lambda E$.

When A has real characteristic value λ_1 and complex characteristic values $\lambda \pm i\theta$ (λ, θ: real, $\theta \neq 0$), set the linearly independent vectors $\boldsymbol{p}_\mu$ $(\mu = 1, 2, 3)$ such that

(iv) The characteristic vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2 \pm i\boldsymbol{p}_3$ ($\boldsymbol{p}_2$, $\boldsymbol{p}_3$: real) of A with respective characteristic values λ_1 and $\lambda \pm i\theta$.

By virtue of the reduction (Theorem 2 in [4]) of (Theorem 1 in [3]), we have obtained the following result (Theorem in [6]):

In the maximizing problem of (1), *let the consumptions* $c^\mu = g^\mu(x^1, x^2, x^3) - n^\mu_\nu x^\nu - \dot{x}^\mu$ *grow externally under the linear production technologies* $g^\mu = \alpha^\mu_\nu x^\nu + \beta^\mu$. *Then, in the setting of* (i), (ii-1), (ii-2), (iii-1), (iii-2) *and* (iv) *according to the classification of the characteristic values of the matrix* $A = (\alpha^\mu_\nu - n^\mu_\nu)$, *there exist the following respective conserved quantities* (5a)–(5f) *in which* $U_\mu = \partial U / \partial c^\mu$ $(\mu = 1, 2, 3)$; *while in* (5d)–(5e), *the matrix* A *is assumed not to be a constant multiple of* E:

$$\text{(5a)} \qquad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} & 0 & 0 \\ 0 & e^{(\lambda_2 - \rho)t} & 0 \\ 0 & 0 & e^{(\lambda_3 - \rho)t} \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \qquad \text{for (i),}$$

$$\text{(5b)} \qquad \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = \begin{pmatrix} e^{(\lambda_1 - \rho)t} & 0 & 0 \\ 0 & e^{(\lambda_2 - \rho)t} & 0 \\ 0 & 0 & e^{(\lambda_2 - \rho)t} \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \qquad \text{for (ii-1),}$$

(5c) $$\begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{(\lambda_2-\rho)t} \begin{pmatrix} e^{(\lambda_1-\lambda_2)t} & 0 & 0 \\ 0 & t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (ii-2)},$$

(5d) $$\begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{(\lambda-\rho)t} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-1)},$$

(5e) $$\begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{(\lambda-\rho)t} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iii-2)},$$

(5f) $$\begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 \end{pmatrix} = e^{(\lambda-\rho)t} \begin{pmatrix} e^{(\lambda_1-\lambda)t} & 0 & 0 \\ 0 & \sin\theta t & \cos\theta t \\ 0 & \cos\theta t & -\sin\theta t \end{pmatrix} \begin{pmatrix} {}^t\boldsymbol{p}_1 \\ {}^t\boldsymbol{p}_2 \\ {}^t\boldsymbol{p}_3 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \quad \text{for (iv)}.$$

3. Determination of optimal paths

Since the matrix $P=(\boldsymbol{p}_1\ \boldsymbol{p}_2\ \boldsymbol{p}_3)$ is nonsingular, (5a)–(5f) lead respectively to

(6a) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix} \quad \text{for (i)},$$

(6b) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = {}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_2)t} \end{pmatrix} \quad \text{for (ii-1)},$$

(6c) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{(\rho-\lambda_2)t}\,{}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\lambda_2-\lambda_1)t} \\ \Xi_2 \\ \Xi_2 - \Xi_3 t \end{pmatrix} \quad \text{for (ii-2)},$$

(6d) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{(\rho-\lambda)t}\,{}^tP^{-1} \begin{pmatrix} \Xi_1 \\ \Xi_2 \\ \Xi_3 - \Xi_2 t \end{pmatrix} \quad \text{for (iii-1)},$$

(6e) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{(\rho-\lambda)t}\,{}^tP^{-1} \begin{pmatrix} \Xi_1 \\ \Xi_2 - \Xi_1 t \\ \Xi_3 - \Xi_2 t + \frac{1}{2}\Xi_1 t^2 \end{pmatrix} \quad \text{for (iii-2)},$$

(6f) $$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = e^{(\rho-\lambda)t}\,{}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\lambda-\lambda_1)t} \\ \Xi_2 \sin\theta t + \Xi_3 \cos\theta t \\ \Xi_2 \cos\theta t - \Xi_3 \sin\theta t \end{pmatrix} \quad \text{for (iv)}.$$

Therefore, since $\det(\partial U_\mu/\partial c^\nu) = D_3 \neq 0$ by (2), the optimal path $\boldsymbol{c}(t) = {}^t(c^1(t)\ c^2(t)\ c^3(t))$ is implicitly determined as $c^\mu(t) = F^\mu(\psi_1(t), \psi_2(t), \psi_3(t))$, where ${}^t(\psi_1(t)\ \psi_2(t)\ \psi_3(t))$ is the right hand side of (6a)–(6f), respectively. And then $c^\mu(t) = F^\mu$ are substituted for the growth process (3) to have the first order linear differential equations with respect to x^μ, i.e.,

$$(3)' \qquad \dot{\boldsymbol{x}} = A\boldsymbol{x} - \boldsymbol{F} + \boldsymbol{\beta},$$

where $\boldsymbol{x} = {}^t(x^1\ x^2\ x^3)$, $\boldsymbol{F} = {}^t(F^1\ F^2\ F^3)$ and $\boldsymbol{\beta} = {}^t(\beta^1\ \beta^2\ \beta^3)$. The general solution $\boldsymbol{x}$ of the subsidiary equation $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ of $(3)'$ are give as a linear conbination of the independent solutions $\boldsymbol{\varphi}_\mu$ with constant coefficients G^μ, i.e., $\boldsymbol{x} = G^1\boldsymbol{\varphi}_1 + G^2\boldsymbol{\varphi}_2 + G^3\boldsymbol{\varphi}_3 = \boldsymbol{\Phi G}$ where $\boldsymbol{\Phi} = {}^t(\boldsymbol{\varphi}_1\ \boldsymbol{\varphi}_2\ \boldsymbol{\varphi}_3)$ and $\boldsymbol{G} = {}^t(G^1\ G^2\ G^3)$. And then, after replacing the constants G^μ with arbitrary functions $G^\mu(t)$, the solution $\boldsymbol{x}$ is substituted for $(3)'$ to have the equation $\boldsymbol{\Phi}\dot{\boldsymbol{G}} = -\boldsymbol{F} + \boldsymbol{\beta}$, i.e.,

$$\dot{\boldsymbol{G}} = -\boldsymbol{\Phi}^{-1}(\boldsymbol{F} - \boldsymbol{\beta}).$$

Thus the optimal path $\boldsymbol{x}(t)$ is implicitly determined as

$$(7) \qquad \boldsymbol{x}(t) = -\boldsymbol{\Phi}\int \boldsymbol{\Phi}^{-1}(\boldsymbol{F} - \boldsymbol{\beta})dt.$$

To go into detail, let the utility function $U(c^1, c^2, c^3)$ be a second order polynomial of consumptions of the form

$$(8) \quad U(c^1, c^2, c^3) = -\tfrac{1}{2}[(c^1)^2 + (c^2)^2 + (c^3)^2] - \alpha c^1c^2 - \beta c^2c^3 - \gamma c^3c^1 \quad (\alpha, \beta, \gamma\text{: const.});$$

where in view of

$$\left(\frac{\partial^2 U}{\partial c^\mu \partial c^\nu}\right) = \begin{pmatrix} -1 & -\alpha & -\gamma \\ -\alpha & -1 & -\beta \\ -\gamma & -\beta & -1 \end{pmatrix} \equiv M,$$

the constants α, β and γ are assumed for the concavity to satisfy

$$D_2 = \begin{vmatrix} -1 & -\alpha \\ -\alpha & -1 \end{vmatrix} = 1 - \alpha^2 > 0,$$

$$D_3 = \begin{vmatrix} -1 & -\alpha & -\gamma \\ -\alpha & -1 & -\beta \\ -\gamma & -\beta & -1 \end{vmatrix} = -1 - 2\alpha\beta\gamma + \alpha^2 + \beta^2 + \gamma^2 < 0,$$

while $D_1 = -1 < 0$. Then the explicit appearances of (7) can be determined as follows.

The case (i): By (6a), i.e.,

$$Mc(t) = {}^tP^{-1}\begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix},$$

the optimal path $c(t) = F(t)$ leads to

$$c(t) = M^{-1}\,{}^tP^{-1}\begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_3)t} \end{pmatrix}, \tag{9}$$

where Ξ_μ $(\mu = 1, 2, 3)$ are some constants, while

$$M^{-1} = \frac{1}{D_3}\begin{pmatrix} 1-\beta^2 & \beta\gamma-\alpha & \alpha\beta-\gamma \\ \beta\gamma-\alpha & 1-\gamma^2 & \gamma\alpha-\beta \\ \alpha\beta-\gamma & \gamma\alpha-\beta & 1-\alpha^2 \end{pmatrix}.$$

Accordingly, after substituting the solution Φ in the case (i) for $\Phi^{-1}(F-\beta)$, by putting

$$-P^{-1}M^{-1}\,{}^tP^{-1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \qquad P^{-1}\beta = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix}, \tag{10}$$

it follows that

$$\Phi^{-1}(F-\beta) = -\begin{pmatrix} a_1\Xi_1 e^{(\rho-2\lambda_1)t} + b_1\Xi_2 e^{(\rho-\lambda_1-\lambda_2)t} + c_1\Xi_3 e^{(\rho-\lambda_1-\lambda_3)t} + \kappa_1 e^{-\lambda_1 t} \\ a_2\Xi_1 e^{(\rho-\lambda_1-\lambda_2)t} + b_2\Xi_2 e^{(\rho-2\lambda_2)t} + c_2\Xi_3 e^{(\rho-\lambda_2-\lambda_3)t} + \kappa_2 e^{-\lambda_2 t} \\ a_3\Xi_1 e^{(\rho-\lambda_1-\lambda_3)t} + b_3\Xi_2 e^{(\rho-\lambda_2-\lambda_3)t} + c_3\Xi_3 e^{(\rho-2\lambda_3)t} + \kappa_3 e^{-\lambda_3 t} \end{pmatrix}.$$

The above appearance of $\Phi^{-1}(F-\beta)$ can be integrated immediately, where ρ $(\rho \geq 0)$ is assumed that $\rho \neq \lambda_\mu + \lambda_\nu$ $(\mu, \nu = 1, 2, 3)$, i.e., $\rho \neq 2\lambda_1$, $\rho \neq 2\lambda_2$, $\rho \neq 2\lambda_3$, $\rho \neq \lambda_1 + \lambda_2$, $\rho \neq \lambda_2 + \lambda_3$ and $\rho \neq \lambda_3 + \lambda_1$ (the integration can be made also when $\rho = \lambda_\mu + \lambda_\nu$ $(\mu, \nu = 1, 2, 3)$). Therefore, the optimal path $x(t)$ of (7) is determined completely as

$$x(t) = P\begin{pmatrix} \dfrac{a_1\Xi_1}{\rho-2\lambda_1}e^{(\rho-\lambda_1)t} + \dfrac{b_1\Xi_2}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_2)t} + \dfrac{c_1\Xi_3}{\rho-\lambda_1-\lambda_3}e^{(\rho-\lambda_3)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ \dfrac{a_2\Xi_1}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_1)t} + \dfrac{b_2\Xi_2}{\rho-2\lambda_2}e^{(\rho-\lambda_2)t} + \dfrac{c_2\Xi_3}{\rho-\lambda_2-\lambda_3}e^{(\rho-\lambda_3)t} + \delta_2 e^{\lambda_2 t} - \dfrac{\kappa_2}{\lambda_2} \\ \dfrac{a_3\Xi_1}{\rho-\lambda_1-\lambda_3}e^{(\rho-\lambda_1)t} + \dfrac{b_3\Xi_2}{\rho-\lambda_2-\lambda_3}e^{(\rho-\lambda_2)t} + \dfrac{c_3\Xi_3}{\rho-2\lambda_3}e^{(\rho-\lambda_3)t} + \delta_3 e^{\lambda_3 t} - \dfrac{\kappa_3}{\lambda_3} \end{pmatrix}, \tag{11}$$

where δ_μ $(\mu = 1, 2, 3)$ are some constants.

Similarly, for the cases (ii-1), (ii-2), (iii-1), (iii-2) or (iv), the optimal paths can be determined by (6b), (6c), (6d), (6e) or (6f) respectively.

The case (ii-1):

$$(12)\qquad c(t) = M^{-1}\,{}^t\!P^{-1}\begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

$$(13)\qquad x(t) = P\begin{pmatrix} \dfrac{a_1\Xi_1}{\rho-2\lambda_1}e^{(\rho-\lambda_1)t} + \dfrac{b_1\Xi_2 + c_1\Xi_3}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_2)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ \dfrac{a_2\Xi_1}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_1)t} + \dfrac{b_2\Xi_2 + c_2\Xi_3}{\rho-2\lambda_2}e^{(\rho-\lambda_2)t} + \delta_2 e^{\lambda_2 t} - \dfrac{\kappa_2}{\lambda_2} \\ \dfrac{a_3\Xi_1}{\rho-\lambda_1-\lambda_2}e^{(\rho-\lambda_1)t} + \dfrac{b_3\Xi_2 + c_3\Xi_3}{\rho-2\lambda_2}e^{(\rho-\lambda_2)t} + \delta_3 e^{\lambda_2 t} - \dfrac{\kappa_3}{\lambda_2} \end{pmatrix};$$

where δ_μ $(\mu = 1,2,3)$ are some constants, while $\rho \neq \lambda_i + \lambda_j$ $(i,j = 1,2)$.

The case (ii-2):

$$(14)\qquad c(t) = M^{-1}\,{}^t\!P^{-1}\begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ (\Xi_2 - \Xi_3 t)e^{(\rho-\lambda_2)t} \end{pmatrix},$$

$$(15)\qquad x(t) = P\begin{pmatrix} A_{11}e^{(\rho-\lambda_1)t} + (A_{12} - A_{13}t)e^{(\rho-\lambda_2)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ A_{21}e^{(\rho-\lambda_1)t} + (A_{22} - A_{23}t)e^{(\rho-\lambda_2)t} + (\delta_2 + \delta_3 t)e^{\lambda_2 t} - \dfrac{\kappa_2}{\lambda_2} + \dfrac{\kappa_3}{\lambda_2^2} \\ A_{31}e^{(\rho-\lambda_1)t} + (A_{32} - A_{33}t)e^{(\rho-\lambda_2)t} + \delta_3 e^{\lambda_2 t} - \dfrac{\kappa_3}{\lambda_2} \end{pmatrix};$$

where δ_μ $(\mu = 1,2,3)$ are some constants and $A_{\mu\nu}$ $(\mu,\nu = 1,2,3)$ are the constants of the following form with the constants Ξ_1, Ξ_2 and Ξ_3, while $\rho \neq \lambda_i + \lambda_j$ $(i,j = 1,2)$:

$$A_{11} = \frac{a_1}{\rho-2\lambda_1}\Xi_1,$$

$$A_{12} = \frac{(b_1 + c_1)}{\rho-\lambda_1-\lambda_2}\Xi_2 + \frac{c_1}{(\rho-\lambda_1-\lambda_2)^2}\Xi_3,$$

$$A_{13} = \frac{c_1}{\rho-\lambda_1-\lambda_2}\Xi_3,$$

$$A_{21} = \left(\frac{a_2}{\rho-\lambda_1-\lambda_2} + \frac{a_3}{(\rho-\lambda_1-\lambda_2)^2}\right)\Xi_1,$$

$$A_{22} = \left(\frac{b_2 + c_2}{\rho-2\lambda_2} + \frac{b_3 + c_3}{(\rho-2\lambda_2)^2}\right)\Xi_2 + \left(\frac{c_2}{(\rho-2\lambda_2)^2} + \frac{2c_3}{(\rho-2\lambda_2)^3}\right)\Xi_3,$$

$$A_{23}=\left(\frac{c_2}{\rho-2\lambda_2}+\frac{c_3}{(\rho-2\lambda_2)^2}\right)\Xi_3,$$

$$A_{31}=\frac{a_3}{\rho-\lambda_1-\lambda_2}\Xi_1,$$

$$A_{32}=\frac{b_3+c_3}{\rho-2\lambda_2}\Xi_2+\frac{c_3}{(\rho-2\lambda_2)^2}\Xi_3,$$

$$A_{33}=\frac{c_3}{\rho-2\lambda_2}\Xi_3.$$

The case (iii-1):

(16) $$\boldsymbol{c}(t)=e^{(\rho-\lambda)t}M^{-1}\,{}^tP^{-1}\begin{pmatrix}\Xi_1\\ \Xi_2\\ \Xi_3-\Xi_2 t\end{pmatrix},$$

(17) $$\boldsymbol{x}(t)=P\begin{pmatrix}(B_{11}-B_{12}t)e^{(\rho-\lambda)t}+\delta_1 e^{\lambda t}-\dfrac{\kappa_1}{\lambda}\\ (B_{21}-B_{22}t)e^{(\rho-\lambda)t}+(\delta_2+\delta_3 t)e^{\lambda t}+\dfrac{\kappa_3}{\lambda^2}-\dfrac{\kappa_2}{\lambda}\\ (B_{31}-B_{32}t)e^{(\rho-\lambda)t}+\delta_3 e^{\lambda t}-\dfrac{\kappa_3}{\lambda}\end{pmatrix};$$

where δ_μ $(\mu=1,2,3)$ are some constants and B_{ij} $(i=1,2,3;j=1,2)$ are the constants of the following form with the constants Ξ_1, Ξ_2 and Ξ_3, while $\rho\neq 2\lambda$:

$$B_{11}=\frac{a_1}{\rho-2\lambda}\Xi_1+\left(\frac{b_1}{\rho-2\lambda}+\frac{c_1}{(\rho-2\lambda)^2}\right)\Xi_2+\frac{c_1}{\rho-2\lambda}\Xi_3,$$

$$B_{12}=\frac{c_1}{\rho-2\lambda}\Xi_2,$$

$$B_{21}=\left(\frac{a_2}{\rho-2\lambda}+\frac{a_3}{(\rho-2\lambda)^2}\right)\Xi_1+\left(\frac{b_2}{\rho-2\lambda}+\frac{b_3+c_2}{(\rho-2\lambda)^2}+\frac{2c_3}{(\rho-2\lambda)^3}\right)\Xi_2$$
$$+\left(\frac{c_2}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_3,$$

$$B_{22}=\frac{c_3}{(\rho-2\lambda)^2}\Xi_2,$$

$$B_{31}=\frac{a_3}{\rho-2\lambda}\Xi_1+\left(\frac{b_3}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_2+\frac{c_3}{\rho-2\lambda}\Xi_3,$$

$$B_{32}=\frac{c_3}{\rho-2\lambda}\Xi_2.$$

The case (iii-2):

$$(18)\quad \boldsymbol{c}(t)=e^{(\rho-\lambda)t}M^{-1}\,{}^tP^{-1}\begin{pmatrix}\Xi_1\\ \Xi_2-\Xi_1t\\ \Xi_3-\Xi_2t+\frac{1}{2}\Xi_1t^2\end{pmatrix},$$

$$(19)\quad \boldsymbol{x}(t)=P\begin{pmatrix}(C_{11}-C_{12}t+C_{13}t^2)e^{(\rho-\lambda)t}+\left(\delta_1+\delta_2t+\frac{1}{2}\delta_3t^2\right)e^{\lambda t}-\frac{\kappa_1}{\lambda}+\frac{\kappa_2}{\lambda^2}-\frac{\kappa_3}{\lambda^3}\\ (C_{21}-C_{22}t+C_{23}t^2)e^{(\rho-\lambda)t}+(\delta_2+\delta_3t)e^{\lambda t}-\frac{\kappa_2}{\lambda}+\frac{\kappa_3}{\lambda^2}\\ (C_{31}-C_{32}t+C_{33}t^2)e^{(\rho-\lambda)t}+\delta_3e^{\lambda t}-\frac{\kappa_3}{\lambda}\end{pmatrix};$$

where δ_μ $(\mu=1,2,3)$ are some constants and $C_{\mu\nu}$ $(\mu,\nu=1,2,3)$ are the constants of the following form with the constants Ξ_1, Ξ_2 and Ξ_3, while $\rho\neq 2\lambda$:

$$C_{11}=\left(\frac{a_1}{\rho-2\lambda}+\frac{a_2+b_1}{(\rho-2\lambda)^2}+\frac{a_3+2b_2+c_1}{(\rho-2\lambda)^3}+\frac{3(b_3+c_2)}{(\rho-2\lambda)^4}+\frac{6c_3}{(\rho-2\lambda)^5}\right)\Xi_1$$
$$+\left(\frac{b_1}{\rho-2\lambda}+\frac{b_2+c_1}{(\rho-2\lambda)^2}+\frac{b_3+2c_2}{(\rho-2\lambda)^3}+\frac{3c_3}{(\rho-2\lambda)^4}\right)\Xi_2$$
$$+\left(\frac{c_1}{\rho-2\lambda}+\frac{c_2}{(\rho-2\lambda)^2}+\frac{c_3}{(\rho-2\lambda)^3}\right)\Xi_3,$$

$$C_{12}=\left(\frac{2(a_2+b_1)}{\rho-2\lambda}+\frac{b_2+c_1}{(\rho-2\lambda)^2}+\frac{b_3+2c_2}{(\rho-2\lambda)^3}+\frac{3c_3}{(\rho-2\lambda)^4}\right)\Xi_1$$
$$+\left(\frac{2(b_2+c_1)}{\rho-2\lambda}+\frac{c_2}{(\rho-2\lambda)^2}+\frac{c_3}{(\rho-2\lambda)^3}\right)\Xi_2+\frac{2c_2}{\rho-2\lambda}\Xi_3,$$

$$C_{13}=\frac{1}{2}\left(\frac{c_1}{\rho-2\lambda}+\frac{c_2}{(\rho-2\lambda)^2}+\frac{c_3}{(\rho-2\lambda)^3}\right)\Xi_1,$$

$$C_{21}=\left(\frac{a_2}{\rho-2\lambda}+\frac{a_3+b_2}{(\rho-2\lambda)^2}+\frac{2b_3+c_2}{(\rho-2\lambda)^3}+\frac{3c_3}{(\rho-2\lambda)^4}\right)\Xi_1$$
$$+\left(\frac{b_2}{\rho-2\lambda}+\frac{b_3+c_2}{(\rho-2\lambda)^2}+\frac{2c_3}{(\rho-2\lambda)^3}\right)\Xi_2+\left(\frac{c_2}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_3,$$

$$C_{22}=\left(\frac{a_3+b_2}{\rho-2\lambda}+\frac{b_3+c_2}{(\rho-2\lambda)^2}+\frac{2c_3}{(\rho-2\lambda)^3}\right)\Xi_1+\left(\frac{b_3+c_2}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_2+\frac{c_3}{\rho-2\lambda}\Xi_3,$$

$$C_{23}=\frac{1}{2}\left(\frac{c_2}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_1,$$

$$C_{31}=\left(\frac{a_3}{\rho-2\lambda}+\frac{b_3}{(\rho-2\lambda)^2}+\frac{c_3}{(\rho-2\lambda)^3}\right)\Xi_1+\left(\frac{b_3}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_2+\frac{c_3}{\rho-2\lambda}\Xi_3,$$

$$C_{32}=\left(\frac{b_3}{\rho-2\lambda}+\frac{c_3}{(\rho-2\lambda)^2}\right)\Xi_1+\frac{c_3}{\rho-2\lambda}\Xi_2,$$

$$C_{33}=\frac{c_3}{2(\rho-2\lambda)}\Xi_1.$$

The case (iv):

$$\text{(20)}\qquad \boldsymbol{c}(t)=M^{-1}{}^tP^{-1}\begin{pmatrix}\Xi_1 e^{(\rho-\lambda_1)t}\\ (\Xi_2\sin\theta t+\Xi_3\cos\theta t)e^{(\rho-\lambda)t}\\ (\Xi_2\cos\theta t-\Xi_3\sin\theta t)e^{(\rho-\lambda)t}\end{pmatrix},$$

$$\text{(21)}\qquad \boldsymbol{x}(t)=PR\begin{pmatrix}\dfrac{a_1}{\rho-2\lambda_1}\Xi_1 e^{(\rho-2\lambda_1)t}+\delta_1\\ (B_1\sin^2\theta t+B_2\sin\theta t\cos\theta t+B_3\cos^2\theta t+B_4)e^{(\rho-2\lambda)t}+\delta_2\\ (C_1\sin^2\theta t+C_2\sin\theta t\cos\theta t+C_3\cos^2\theta t+C_4)e^{(\rho-2\lambda)t}+\delta_3\end{pmatrix}$$

$$+P\begin{pmatrix}(A_1\sin\theta t+A_2\cos\theta t)e^{(\rho-\lambda)t}-\dfrac{\kappa_1}{\lambda_1}\\ B_5e^{(\rho-\lambda_1)t}+B_6\\ C_5e^{(\rho-\lambda_1)t}+C_6\end{pmatrix};$$

where R is the matrix:

$$R=\begin{pmatrix}e^{\lambda_1 t} & 0 & 0\\ 0 & e^{\lambda t}\cos\theta t & e^{\lambda t}\sin\theta t\\ 0 & -e^{\lambda t}\sin\theta t & e^{\lambda t}\cos\theta t\end{pmatrix},$$

and moreover δ_μ $(\mu=1,2,3)$ are some constants and A_i, B_j, C_j $(i=1,2; j=1,2,\dots,6)$ are the constants of the following form with the constants Ξ_1, Ξ_2 and Ξ_3, while $\rho\neq 2\lambda$ and $\rho\neq 2\lambda_1$:

$$A_1 = \frac{(\rho - \lambda - \lambda_1)(b_1\Xi_2 - c_1\Xi_3) + \theta(b_1\Xi_3 + c_1\Xi_2)}{(\rho - \lambda - \lambda_1)^2 + \theta^2},$$

$$A_2 = \frac{(\rho - \lambda - \lambda_1)(c_1\Xi_2 + b_1\Xi_3) - \theta(b_1\Xi_2 - c_1\Xi_3)}{(\rho - \lambda - \lambda_1)^2 + \theta^2},$$

$$B_1 = \frac{(\rho - 2\lambda)(c_3\Xi_3 - b_3\Xi_2) + \theta[(b_2 - c_3)\Xi_2 - (b_3 + c_2)\Xi_3]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$B_2 = \frac{(\rho - 2\lambda)[(b_2 - c_3)\Xi_2 - (b_3 + c_2)\Xi_3] + 2\theta[(b_2 - c_3)\Xi_3 + (b_3 + c_2)\Xi_2]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$B_3 = \frac{(\rho - 2\lambda)(c_2\Xi_2 + b_2\Xi_3) - \theta[(b_2 - c_3)\Xi_2 - (b_3 + c_2)\Xi_3]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$B_4 = \frac{4\theta^2/(\rho - 2\lambda)}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$B_5 = \frac{(\rho - \lambda - \lambda_1)a_2\Xi_1 + \theta a_3\Xi_1}{(\rho - \lambda - \lambda_1)^2 + \theta^2},$$

$$B_6 = \frac{\kappa_3\theta - \kappa_2\lambda}{\lambda^2 + \theta^2},$$

$$C_1 = \frac{(\rho - 2\lambda)(b_2\Xi_2 - c_2\Xi_3) + \theta[(b_3 + c_2)\Xi_2 + (b_2 - c_3)\Xi_3]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$C_2 = \frac{(\rho - 2\lambda)[(b_3 + c_2)\Xi_2 + (b_2 - c_3)\Xi_3] + 2\theta[(c_3 - b_2)\Xi_2 + (b_3 + c_2)\Xi_3]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$C_3 = \frac{(\rho - 2\lambda)(c_3\Xi_2 + b_3\Xi_3) - \theta[(b_3 + c_2)\Xi_2 + (b_2 - c_3)\Xi_3]}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$C_4 = \frac{4\theta^2/(\rho - 2\lambda)}{(\rho - 2\lambda)^2 + 4\theta^2},$$

$$C_5 = \frac{(\rho - \lambda - \lambda_1)a_3\Xi_1 - \theta a_2\Xi_1}{(\rho - \lambda - \lambda_1)^2 + \theta^2},$$

$$C_6 = -\frac{\kappa_3\lambda + \kappa_2\theta}{\lambda^2 + \theta^2}.$$

THEOREM 1. *In the maximizing problem of* (1), *let the utility function* $U(c^1, c^2, c^3)$ *with the concavity be given as* (8) *and the consumptions* $c^\mu = g^\mu(x^1, x^2, x^3) - n^\mu_\nu x^\nu - \dot{x}^\mu$ *grow externally under the linear production technologies* $g^\mu = \alpha^\mu_\nu x^\nu + \beta^\mu$. *Then, in the* *case of finite horizon $T < \infty$, the optimal paths $\boldsymbol{c}(t)$ and $\boldsymbol{x}(t)$ are determined completely as* (9) *and* (11) *in the case* (i) *with $\rho \neq \lambda_\mu + \lambda_\nu$ $(\mu, \nu = 1, 2, 3)$; as* (12) *and* (13) *in the case* (ii-1) *with $\rho \neq \lambda_i + \lambda_j$ $(i, j = 1, 2)$; as* (14) *and* (15) *in the case* (ii-2) *with $\rho \neq \lambda_i + \lambda_j$ $(i, j = 1, 2)$; as* (16) *and* (17) *in the case* (iii-1) *with $\rho \neq 2\lambda$; as* (18) *and* (19) *in the case* (iii-2) *with $\rho \neq 2\lambda$; as* (20) *and* (21) *in the case* (iv) *with $\rho \neq 2\lambda$ and $\rho \neq 2\lambda_1$.*

REMARK. The interested reader will find the optimal paths when ρ takes the exceptional values in the theorem 1.

The case (i): To look the case of infinite horizon $T = \infty$, the relation (4b): $\pi_\mu = -e^{-\rho t} U_\mu$ are written by (6a) as

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = -{}^tP^{-1} \begin{pmatrix} \Xi_1 e^{-\lambda_1 t} \\ \Xi_2 e^{-\lambda_2 t} \\ \Xi_3 e^{-\lambda_3 t} \end{pmatrix},$$

and then used, together with (11), for the transversality condition $\lim_{t\to\infty} \pi_\mu x^\mu = 0$. In view of the resulting appearance of $\pi_\mu x^\mu$:

$$\begin{aligned} \pi_\mu x^\mu = &-\frac{a_1 \Xi_1^2}{\rho - 2\lambda_1} e^{(\rho - 2\lambda_1)t} - \frac{b_2 \Xi_2^2}{\rho - 2\lambda_2} e^{(\rho - 2\lambda_2)t} - \frac{c_3 \Xi_3^2}{\rho - 2\lambda_3} e^{(\rho - 2\lambda_3)t} \\ &- \frac{(a_2 + b_1)\Xi_1 \Xi_2}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_1 - \lambda_2)t} - \frac{(b_3 + c_2)\Xi_2 \Xi_3}{\rho - \lambda_2 - \lambda_2} e^{(\rho - \lambda_2 - \lambda_3)t} - \frac{(c_1 + a_3)\Xi_1 \Xi_3}{\rho - \lambda_1 - \lambda_3} e^{(\rho - \lambda_1 - \lambda_3)t} \\ &+ \frac{\kappa_1 \Xi_1}{\lambda_1} e^{-\lambda_1 t} + \frac{\kappa_2 \Xi_2}{\lambda_2} e^{-\lambda_2 t} + \frac{\kappa_3 \Xi_3}{\lambda_3} e^{-\lambda_3 t} - (\delta_1 \Xi_1 + \delta_2 \Xi_2 + \delta_3 \Xi_3), \end{aligned}$$

we can find the optimal paths satisfying the transversality condition. For example, let $0 < 2\lambda_1 \leq \rho < 2\lambda_2$ and $0 < 2\lambda_1 \leq \rho < 2\lambda_3$. Then $\lim_{t\to\infty} \pi_\mu x^\mu = 0$ requires $\Xi_1 = 0$ in the coefficient of $e^{(\rho - 2\lambda_1)t}$; so that $\delta_2 \Xi_2 + \delta_3 \Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (9) and (11) take the forms respectively:

$$(9)' \qquad \boldsymbol{c}(t) = M^{-1}\,{}^tP^{-1} \begin{pmatrix} 0 \\ \Xi_2 e^{(\rho - \lambda_2)t} \\ \Xi_3 e^{(\rho - \lambda_3)t} \end{pmatrix},$$

$$(11)' \qquad \boldsymbol{x}(t) = P \begin{pmatrix} \dfrac{b_1 \Xi_2}{\rho - \lambda_1 - \lambda_2} e^{(\rho - \lambda_2)t} + \dfrac{c_1 \Xi_3}{\rho - \lambda_1 - \lambda_3} e^{(\rho - \lambda_3)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ \dfrac{b_2 \Xi_2}{\rho - 2\lambda_2} e^{(\rho - \lambda_2)t} + \dfrac{c_2 \Xi_3}{\rho - \lambda_2 - \lambda_3} e^{(\rho - \lambda_3)t} - \dfrac{\kappa_2}{\lambda_2} \\ \dfrac{b_3 \Xi_2}{\rho - \lambda_2 - \lambda_3} e^{(\rho - \lambda_2)t} + \dfrac{c_3 \Xi_3}{\rho - 2\lambda_3} e^{(\rho - \lambda_3)t} - \dfrac{\kappa_3}{\lambda_3} \end{pmatrix}.$$

Similarly, we can have the optimal paths satisfying the transversality condition for the case (ii-1), (ii-2), (iii-1), (iii-2) and (iv).

The case (ii-1): By (6b) and (13), $\pi_\mu x^\mu$ is written as

$$\pi_\mu x^\mu = -\frac{a_1 \Xi_1^2}{\rho - 2\lambda_1} e^{(\rho-2\lambda_1)t} - \frac{b_2\Xi_2^2 + c_3\Xi_3^2 + (b_3+c_2)\Xi_2\Xi_3}{\rho - 2\lambda_2} e^{(\rho-2\lambda_2)t}$$
$$- \frac{(a_2+b_1)\Xi_1\Xi_2 + (c_1+a_3)\Xi_1\Xi_3}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_1-\lambda_2)t}$$
$$+ \frac{\kappa_1\Xi_1}{\lambda_1} e^{-\lambda_1 t} + \frac{\kappa_2\Xi_2 + \kappa_3\Xi_3}{\lambda_2} e^{-\lambda_2 t} - (\delta_1\Xi_1 + \delta_2\Xi_2 + \delta_3\Xi_3),$$

in which, let $0 < 2\lambda_1 \le \rho < 2\lambda_2$. Then $\lim_{t\to\infty} \pi_\mu x^\mu = 0$ requires $\Xi_1 = 0$ in the coefficient of $e^{(\rho-2\lambda_1)t}$; so that $\delta_2\Xi_2 + \delta_3\Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (12) and (13) take the forms respectively:

$$(12)' \qquad c(t) = M^{-1}{}^tP^{-1} \begin{pmatrix} 0 \\ \Xi_2 e^{(\rho-\lambda_2)t} \\ \Xi_3 e^{(\rho-\lambda_2)t} \end{pmatrix},$$

$$(13)' \qquad x(t) = P \begin{pmatrix} \dfrac{b_1\Xi_2 + c_1\Xi_3}{\rho-\lambda_1-\lambda_2} e^{(\rho-\lambda_2)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ \dfrac{b_2\Xi_2 + c_2\Xi_3}{\rho - 2\lambda_2} e^{(\rho-\lambda_2)t} - \dfrac{\kappa_2}{\lambda_2} \\ \dfrac{b_3\Xi_2 + c_3\Xi_3}{\rho - 2\lambda_2} e^{(\rho-\lambda_2)t} - \dfrac{\kappa_3}{\lambda_2} \end{pmatrix}.$$

The case (ii-2): By (6c) and (15), $\pi_\mu x^\mu$ is written as

$$\pi_\mu x^\mu = -[A_{11}\Xi_1 + (A_{21}+A_{31})\Xi_2 - A_{31}\Xi_3 t] e^{(\rho-2\lambda_1)t}$$
$$- [A_{12}\Xi_1 + (A_{22}+A_{32})\Xi_2 + (A_{13}\Xi_1 + (A_{23}+A_{33})\Xi_2 - A_{32}\Xi_3)t - A_{33}\Xi_3 t^2] e^{(\rho-2\lambda_2)t}$$
$$+ \frac{\kappa_1\Xi_1}{\lambda_1} e^{-\lambda_1 t} + \left[\left(\frac{\kappa_2+\kappa_3}{\lambda_2} - \frac{\kappa_3}{\lambda_2^2}\right)\Xi_2 - \frac{\kappa_3\Xi_3}{\lambda_2} t\right] e^{-\lambda_2 t}$$
$$- (\delta_3\Xi_2 - \delta_3\Xi_3)t - \delta_1\Xi_1 - (\delta_2+\delta_3)\Xi_2.$$

So, let $0 < \rho < 2\lambda_1$ and $0 < \rho < 2\lambda_2$. Then $\lim_{t\to\infty} \pi_\mu x^\mu = 0$ requires $\delta_3\Xi_2 - \delta_3\Xi_3 = 0$ and $\delta_1\Xi_1 + (\delta_2+\delta_3)\Xi_2 = 0$. Therefore, by putting $\delta_1 = 0$, $\Xi_2 = 0$ and $\Xi_3 = 0$, the optimal paths (14) and (15) take the forms respectively:

$$(14)' \qquad c(t) = M^{-1}{}^tP^{-1} \begin{pmatrix} \Xi_1 e^{(\rho-\lambda_1)t} \\ 0 \\ 0 \end{pmatrix},$$

(15)′
$$x(t)=P\begin{pmatrix} A_{11}e^{(\rho-\lambda_1)t}-\dfrac{\kappa_1}{\lambda_1} \\ A_{21}e^{(\rho-\lambda_1)t}+(\delta_2+\delta_3 t)e^{\lambda_2 t}-\dfrac{\kappa_2}{\lambda_2}+\dfrac{\kappa_3}{\lambda_2^2} \\ A_{31}e^{(\rho-\lambda_1)t}+\delta_3 e^{\lambda_2 t}-\dfrac{\kappa_3}{\lambda_2} \end{pmatrix};$$

where A_{11}, A_{21} and A_{31} are the constants given before, while in which Ξ_2 and Ξ_3 are placed as $\Xi_2=\Xi_3=0$.

The case (iii-1): By (6d) and (17), $\pi_\mu x^\mu$ is written as

$$\pi_\mu x^\mu=-[B_{11}\Xi_1+B_{21}\Xi_2+B_{31}\Xi_3+(B_{12}\Xi_1+(B_{22}-B_{31})\Xi_2+B_{32}\Xi_3)t-B_{32}\Xi_2 t^2]e^{(\rho-2\lambda)t}$$
$$+\frac{1}{\lambda}[\kappa_1\Xi_1+(\kappa_2-\kappa_3)\Xi_2+\kappa_3\Xi_3-\kappa_3\Xi_2 t]e^{-\lambda t}-(\delta_1\Xi_1+\delta_2\Xi_2+\delta_3\Xi_3),$$

in which, let $0<\rho<2\lambda$. Then $\lim_{t\to\infty}\pi_\mu x^\mu=0$ requires $\delta_1\Xi_1+\delta_2\Xi_2+\delta_3\Xi_3=0$. Therefore, by putting $\delta_1=0$, $\Xi_2=0$ and $\Xi_3=0$, the optimal paths (16) and (17) take the forms respectively:

(16)′
$$c(t)=M^{-1}{}^t P^{-1}\begin{pmatrix}\Xi_1 e^{(\rho-\lambda)t}\\ 0\\ 0\end{pmatrix},$$

(17)′
$$x(t)=P\begin{pmatrix} B_{11}e^{(\rho-\lambda)t}-\dfrac{\kappa_1}{\lambda} \\ B_{21}e^{(\rho-\lambda)t}+(\delta_2+\delta_3 t)e^{\lambda_2 t}+\dfrac{\kappa_3}{\lambda^2}-\dfrac{\kappa_2}{\lambda} \\ B_{31}e^{(\rho-\lambda)t}+\delta_3 e^{\lambda t}-\dfrac{\kappa_3}{\lambda} \end{pmatrix};$$

where B_{11}, B_{21} and B_{31} are the constants given before, while in which Ξ_2 and Ξ_3 are placed as $\Xi_2=\Xi_3=0$.

The case (iii-2): By (6e) and (19), $\pi_\mu x^\mu$ is written as

$$\pi_\mu x^\mu=-[\Xi_1(C_{11}+C_{12}t+C_{13}t^2)+(\Xi_2-\Xi_1 t)(C_{21}+C_{22}t+C_{23}t^2)$$
$$+\left(\frac{1}{2}\Xi_1 t^2-\Xi_2 t+\Xi_3\right)(C_{31}+C_{32}t+C_{33}t^2)]e^{(\rho-2\lambda)t}$$
$$-\left[\left(-\frac{\kappa_1}{\lambda}+\frac{\kappa_2}{\lambda^2}-\frac{\kappa_3}{\lambda^3}\right)\Xi_1+\left(-\frac{\kappa_2}{\lambda}+\frac{\kappa_3}{\lambda^2}\right)(\Xi_2-\Xi_1 t)-\frac{\kappa_3}{\lambda}\left(\frac{1}{2}\Xi_1 t^2-\Xi_2 t+\Xi_3\right)\right]e^{-\lambda t}$$
$$-(\delta_1\Xi_1+\delta_2\Xi_2+\delta_3\Xi_3),$$

in which, let $0<\rho<2\lambda$. Then $\lim_{t\to\infty}\pi_\mu x^\mu=0$ requires $\delta_1\Xi_1+\delta_2\Xi_2+\delta_3\Xi_3=0$. Therefore, by putting $\Xi_1=0$, $\delta_2=0$ and $\delta_3=0$, the optimal paths (18) and (19) take the forms respectively:

$$(18)' \qquad c(t) = e^{(\rho-\lambda)t} M^{-1}\,{}^t P^{-1} \begin{pmatrix} 0 \\ \Xi_2 \\ \Xi_3 - \Xi_2 t \end{pmatrix},$$

$$(19)' \qquad x(t) = P \begin{pmatrix} (C_{11} + C_{12}t)e^{(\rho-\lambda)t} + \delta_1 e^{\lambda t} - \dfrac{\kappa_1}{\lambda} + \dfrac{\kappa_2}{\lambda^2} - \dfrac{\kappa_3}{\lambda^3} \\ (C_{21} + C_{22}t)e^{(\rho-\lambda)t} - \dfrac{\kappa_2}{\lambda} + \dfrac{\kappa_3}{\lambda^2} \\ (C_{31} + C_{32}t)e^{(\rho-\lambda)t} - \dfrac{\kappa_3}{\lambda} \end{pmatrix};$$

where C_{ij} $(i = 1, 2, 3; j = 1, 2)$ are the constants given before, while in which Ξ_1 is placed as $\Xi_1 = 0$.

The case (iv): By (6f) and (21), $\pi_\mu x^\mu$ is written as

$$\begin{aligned} \pi_\mu x^\mu = & -\frac{a_1}{\rho - 2\lambda_1} \Xi_1^2 e^{(\rho-2\lambda_1)t} - [(B_1\Xi_3 + C_1\Xi_2) \sin^2 \theta t \\ & + (B_2\Xi_3 + C_2\Xi_2) \sin \theta t \cos \theta t + (B_3\Xi_3 + C_3\Xi_2) \cos^2 \theta t + B_4\Xi_3 + C_4\Xi_2] e^{(\rho-2\lambda)t} \\ & - [(A_1\Xi_1 + B_5\Xi_2 - C_5\Xi_3) \sin \theta t + (A_2\Xi_1 + C_5\Xi_2 + B_5\Xi_3) \cos \theta t] e^{(\rho-\lambda-\lambda_1)t} \\ & + \frac{\kappa_1 \Xi_1}{\lambda_1} e^{-\lambda_1 t} - [(B_6\Xi_2 - C_6\Xi_3) \sin \theta t + (C_6\Xi_2 + B_6\Xi_3) \cos \theta t] e^{-\lambda t} \\ & - (\delta_1\Xi_1 + \delta_3\Xi_2 + \delta_2\Xi_3), \end{aligned}$$

in which, let $0 < \rho < 2\lambda_1$ and $0 < \rho < 2\lambda$. Then $\lim_{t\to\infty} \pi_\mu x^\mu = 0$ requires $\delta_1\Xi_1 + \delta_3\Xi_2 + \delta_2\Xi_3 = 0$. Therefore, by putting $\Xi_1 = 0$, $\delta_2 = 0$ and $\delta_3 = 0$, the optimal paths (20) and (21) take the forms respectively:

$$(20)' \qquad c(t) = M^{-1}\,{}^t P^{-1} \begin{pmatrix} 0 \\ (\Xi_2 \sin \theta t + \Xi_3 \cos \theta t) e^{(\rho-\lambda)t} \\ (\Xi_2 \cos \theta t - \Xi_3 \sin \theta t) e^{(\rho-\lambda)t} \end{pmatrix},$$

$$\begin{aligned} (21)' \qquad x(t) = & PR \begin{pmatrix} 0 \\ (B_1 \sin^2 \theta t + B_2 \sin \theta t \cos \theta t + B_3 \cos^2 \theta t + B_4) e^{(\rho-2\lambda)t} \\ (C_1 \sin^2 \theta t + C_2 \sin \theta t \cos \theta t + C_3 \cos^2 \theta t + C_4) e^{(\rho-2\lambda)t} \end{pmatrix} \\ & + P \begin{pmatrix} (A_1 \sin \theta t + A_2 \cos \theta t) e^{(\rho-\lambda)t} + \delta_1 e^{\lambda_1 t} - \dfrac{\kappa_1}{\lambda_1} \\ B_6 \\ C_6 \end{pmatrix}; \end{aligned}$$

where A_i, B_j and C_j $(i = 1, 2; j = 1, 2, 3, 4, 6)$ are the constants given before, while in which Ξ_1 is placed as $\Xi_1 = 0$.

THEOREM 2. *In the case of infinite horizon* $T = \infty$, *there exist the feasible optimal paths* $(9)'$ *and* $(11)'$ *with* $0 < 2\lambda_1 \le \rho < 2\min(\lambda_2, \lambda_3)$ *for the case* (i); $(12)'$ *and* $(13)'$ *with* $0 < 2\lambda_1 \le \rho < 2\lambda_2$ *for the case* (ii-1); $(14)'$ *and* $(15)'$ *with* $0 < \rho < 2\min(\lambda_1, \lambda_2)$ *for the case* (ii-2); $(16)'$ *and* $(17)'$ *with* $0 < \rho < 2\lambda$ *for the case* (iii-1); $(18)'$ *and* $(19)'$ *with* $0 < \rho < 2\lambda$ *for the case* (iii-2); $(20)'$ *and* $(21)'$ *with* $0 < \rho < 2\min(\lambda, \lambda_1)$ *for the case* (iv). *Paticularly, let* $0 < 2\lambda_1 \le \rho < \min(\lambda_2, \lambda_3)$ *and* $\delta_1 = 0$ *in the determined optimal paths* $\boldsymbol{c}(t)$ *and* $\boldsymbol{x}(t)$ *for the case* (i); $0 < 2\lambda_1 \le \rho < \lambda_2$ *and* $\delta_1 = 0$ *in those for the case* (ii-1); $0 < \rho < \min(\lambda_1, 2\lambda_2)$ *and* $\delta_2 = \delta_3 = 0$ *in those for the case* (ii-2); $0 < \rho < \lambda$ *and* $\delta_2 = \delta_3 = 0$ *in those for the case* (iii-1); $0 < \rho < \lambda$ *and* $\delta_1 = 0$ *in those for the cases* (iii-2) *and* (iv). *Then, as* $t \to \infty$, *the optimal paths* $\boldsymbol{c}(t)$ *and* $\boldsymbol{x}(t)$ *converge to zero and some constant vectors, respectively.*

The feasible optimal path $\boldsymbol{x}(t)$ of $(21)'$ goes into details with $\delta_1 = 0$. In view of the identities $B_3 - B_1 = C_2$ and $C_1 - C_3 = B_2$, since

$$\begin{pmatrix} B_1 \sin^2 \theta t + B_2 \sin \theta t \cos \theta t + B_3 \cos^2 \theta t + B_4 \\ C_1 \sin^2 \theta t + C_2 \sin \theta t \cos \theta t + C_3 \cos^2 \theta t + C_4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} B_3 - B_1 & B_2 \\ C_3 - C_1 & C_2 \end{pmatrix}\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \dfrac{B_1 + B_3}{2} + B_4 \\ \dfrac{C_1 + C_3}{2} + C_4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{pmatrix}\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} \dfrac{B_1 + B_3}{2} + B_4 \\ \dfrac{C_1 + C_3}{2} + C_4 \end{pmatrix},$$

the appearance of $(21)'$ can be arranged as

$$(22) \qquad \begin{cases} y^1 = (A_1 \sin \theta t + A_2 \cos \theta t)e^{(\rho - \lambda)t} \\ \begin{pmatrix} \cos \theta t & -\sin \theta t \\ \sin \theta t & \cos \theta t \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \end{pmatrix} = \dfrac{1}{2} e^{(\rho - \lambda)t}\left[\begin{pmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{pmatrix}\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} + \begin{pmatrix} n_2 \\ n_3 \end{pmatrix}\right], \end{cases}$$

where $n_2 = B_1 + B_3 + 2B_4$, $n_3 = C_1 + C_3 + 2C_4$ and the vector $\boldsymbol{y}(t) = {}^t(y^1\ y^2\ y^3)$ is given by

$$(23) \qquad \boldsymbol{y}(t) = P^{-1}\boldsymbol{x}(t) - \begin{pmatrix} -\dfrac{\kappa_1}{\lambda_1} \\ B_6 \\ C_6 \end{pmatrix}.$$

The constants B_2 and C_2 are written in the matrix form

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} = \varDelta \begin{pmatrix} \Xi_2 \\ \Xi_3 \end{pmatrix},$$

where the matrix

$$\varDelta = \frac{1}{(\rho-2\lambda)^2+4\theta^2}\begin{pmatrix} (\rho-2\lambda)(b_2-c_3)+2\theta(b_3+c_2) & -(\rho-2\lambda)(b_3+c_2)+2\theta(b_2-c_3) \\ (\rho-2\lambda)(b_3+c_2)-2\theta(b_2-c_3) & (\rho-2\lambda)(b_2-c_3)+2\theta(b_3+c_2) \end{pmatrix}$$

has the determinant

$$|\varDelta| = \frac{[(\rho-2\lambda)(b_2-c_3)+2\theta(b_3+c_2)]^2+[(\rho-2\lambda)(b_3+c_2)-2\theta(b_2-c_3)]^2}{[(\rho-2\lambda)^2+4\theta^2]^2},$$

which vanishes if and only if

$$\begin{cases} (\rho-2\lambda)(b_2-c_3)+2\theta(b_3+c_2)=0, \\ (\rho-2\lambda)(b_3+c_2)-2\theta(b_2-c_3)=0, \end{cases} \quad \text{i.e.,} \quad \begin{pmatrix} \rho-2\lambda & 2\theta \\ -2\theta & \rho-2\lambda \end{pmatrix}\begin{pmatrix} b_2-c_3 \\ b_3+c_2 \end{pmatrix} = 0.$$

Therefore, since $\theta \neq 0$, i.e.,

$$\begin{vmatrix} \rho-2\lambda & 2\theta \\ -2\theta & \rho-2\lambda \end{vmatrix} = (\rho-2\lambda)^2+4\theta^2 \neq 0,$$

the determinant $|\varDelta|$ vanishes if and only if $b_2-c_3=0$ and $b_3+c_2=0$. So, assuming $b_2-c_3 \neq 0$ or $b_3+c_2 \neq$, choose the constants Ξ_2 and Ξ_3 such that ${}^t(\Xi_2\ \Xi_3) \neq \mathbf{0}$. Then ${}^t(B_2\ C_2) \neq \mathbf{0}$ so that

$$\begin{vmatrix} C_2 & B_2 \\ -B_2 & C_2 \end{vmatrix} = B_2^2+C_2^2 \neq 0. \tag{24}$$

Therefore, the vector ${}^t(\cos 2\theta t\ \sin 2\theta t)$ in (22) is written as

$$\begin{pmatrix} \cos 2\theta t \\ \sin 2\theta t \end{pmatrix} = \frac{2}{B_2^2+C_2^2}e^{-(\rho-\lambda)t}\begin{pmatrix} C_2 & -B_2 \\ B_2 & C_2 \end{pmatrix}\begin{pmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{pmatrix}\begin{pmatrix} y^2 \\ y^3 \end{pmatrix} - \begin{pmatrix} \ell_2 \\ \ell_3 \end{pmatrix},$$

where ℓ_i $(i=2,3)$ are the following constants, respectively:

$$\ell_2 = \frac{(B_1+B_3+2B_4)C_2-(C_1+C_3+2C_4)B_2}{B_2^2+C_2^2},$$

$$\ell_3 = \frac{(B_1+B_3+2B_4)B_2+(C_1+C_3+2C_4)C_2}{B_2^2+C_2^2},$$

which can be put as $\ell_2=\ell_3=0$ by a suitable choice of B_4 and C_4 (this fact is guaranteed by (24)). Thus, the identity $\cos^2 2\theta t+\sin^2 2\theta t=1$ yields the equation of the spiral

$$(y^2)^2 + (y^3)^2 = \frac{B_2^2 + C_2^2}{4} e^{2(\rho-\lambda)t}. \tag{25}$$

In conclusion, we have the following result.

THEOREM 3. *Let a_μ, b_μ and c_μ ($\mu = 1, 2, 3$) in the matrix in* (10) *satisfy $b_2 - c_3 \neq 0$ or $b_3 + c_2 \neq 0$. Then, for* (iv) *in the case of infinite horizon $T = \infty$, the feasible optimal path $\boldsymbol{x}(t)$ of the form* $(21)'$ *with $\delta_1 = 0$ is transformed by* (23), *under a suitable choice of the constants B_4 and C_4, to $\boldsymbol{y}(t) = {}^t(y^1 \ y^2 \ y^3)$ in which y^2 and y^3 satisfy the equation of the spiral* (25).

References

[1] E. Bessel-Hagen, Über die Erhaltungssätze der Electrodynamik, Math. Ann., **84** (1972), 258–276.

[2] M. Braun, Differential equations and their applications, Springer-Verlag, New York, 1983.

[3] F. Fujiwara, F. Mimura and T. Nôno, New derivation of conservation laws and its application to more general neoclassical optimal growth models, Tensor, N.S., **60** (1998), 63–73.

[4] F. Fujiwara, F. Mimura and T. Nôno, New derivation of conservation laws in one and two sector growth models, to be published in Tensor, N.S., **62** (2000).

[5] F. Fujiwara, F. Mimura and T. Nôno, Conservation laws and optimal paths in external two-sector growth model, Economic theory, dynamics and markets (Essays in honor of R. Sato,), ed., T. Negishi, R. V. Ramachandran and K. Mino, Kluwer Academic Publishers, Boston, (2001), 153–164.

[6] F. Fujiwara, F. Mimura and T. Nôno, New derivation of conservation laws in three-sector growth models, Reports of the Japan Institute of Industrial Science, **10** (2000), 1–10.

[7] F. Mimura and T. Nôno, A method for deriving new conservation laws, Bull. Kyushu Inst. Tech. Math. Natur. Sci., **42** (1995), 1–17.

[8] F. Mimura and T. Nôno, A method for the derivation of new conservation laws in economic growth models, Bull. Kyushu Inst. Tech. Math. Natur. Sci., **44** (1997), 7–22.

[9] F. Mimura, F. Fujiwara and T. Nôno, New derivation of conservation laws for optimal economic growths, Tensor, N.S., **58** (1997), 195–207.

[10] F. Mimura, F. Fujiwara and T. Nôno, New derivation of conservation laws for optimal control problem and its application to economic growth models, Global competition and integration, ed., R. Sato, R. V. Ramachandran and K. Mino, Kluwer Academic Publishers, Boston, 1999, 240–266.

[11] E. Noether, Invariante Variations Probleme, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. II, **1918** (1918), 235–257.

[12] F. Ramsey, A mathematical theory of saving, Econom. J., **38** (1928), 543–559.

[13] A. Takayama, Mathematical economics, Dryden Press, Illinois, 1974; Cambridge University Press, New York, 1985.

**Department of Mathematics*
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550, Japan
and
***Department of Economics*
Fukuyama University