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ASYMPTOTIC CONTRACTIONS OF INTEGRAL TYPE

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Abstract

We discuss whether asymptotic contractions of integral type are still asymptotic contractions.

1. Introduction

In 2002, Branciari [2] proved the following fixed point theorem, which is one of generalizations of the Banach contraction principle [1].

THEOREM 1 (Branciari [2]). *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exist $r \in [0, 1)$ and a locally integrable function f from $[0, \infty)$ into itself such that*

$$\int_0^s f(t)dt > 0 \qquad \text{and} \qquad \int_0^{d(Tx,Ty)} f(t)dt \le r \int_0^{d(x,y)} f(t)dt$$

for all $s > 0$ and $x, y \in X$. Then T has a unique fixed point.

Suzuki [11] showed that Theorem 1 is a corollary of the famous Meir-Keeler fixed point theorem [4]. Moreover, he proved the following theorem.

THEOREM 2 ([11]). *Let (X, d) be a metric space and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying the following:*
(A1) *$\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$.*
(A2) *θ is nondecreasing and right continuous.*
(A3) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\theta(d(x, y)) < \varepsilon + \delta \qquad \text{implies} \qquad \theta(d(Tx, Ty)) < \varepsilon$$

for all $x, y \in X$.
Then T is a Meir-Keeler contraction, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) < \varepsilon$ for all $x, y \in X$.

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On the other hand, motivated by [3, 9, 10], Suzuki [12] introduced the notion of asymptotic contractions of the final type and proved the following fixed point theorem.

THEOREM 3 ([12]). *Let (X, d) be a complete metric space and let T be an asymptotic contraction of the final type (ACF, for short) on X, i.e.,*

(B3) $\lim_{\delta\to+0} \sup\left\{\limsup_{n\to\infty} d(T^n x, T^n y) : d(x, y) < \delta\right\} = 0.$

(B4) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\varepsilon < d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbf{N}$ such that $d(T^\nu x, T^\nu y) \le \varepsilon$.*

(B5) *For $x, y \in X$ with $x \neq y$, there exists $\nu \in \mathbf{N}$ such that $d(T^\nu x, T^\nu y) < d(x, y)$.*

(B6) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbf{N}$ such that*

$$\varepsilon < d(T^i x, T^j x) < \varepsilon + \delta \quad \text{implies} \quad d(T^\nu \circ T^i x, T^\nu \circ T^j x) \le \varepsilon$$

for all $i, j \in \mathbf{N}$.

Assume that T^ℓ is continuous for some $\ell \in \mathbf{N}$. Then T has a unique fixed point.

In this paper, we prove that ACF of integral type are still ACF.

2. Preliminaries

In this section, we give some preliminaries. Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers and by $\mathbf{R}$ the set of all real numbers. The following theorem is one of the most important results concerning ACF.

THEOREM 4 ([12]). *Let T be a mapping on a metric space (X, d). Then the following are equivalent:*

(i) *T is an ACF.*

(ii) $\lim_n d(T^n x, T^n y) = 0$ *holds and $\{T^n x\}$ is a Cauchy sequence for all $x, y \in X$.*

In 2001, Suzuki introduced the notion of τ-distances.

DEFINITION ([5]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$.

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_n p(w, x_n)$ for all $w \in X$.

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and propositions are stated in [5–8] and references therein. The following lemmas are proved in [5].

LEMMA 1 ([5]). *Let X be a metric space with a τ-distance p. Then $p(z,x)=0$ and $p(z,y)=0$ imply $x=y$.*

LEMMA 2 ([5]). *Let (X,d) be a metric space with a τ-distance p. If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n,x_m) : m>n\}=0$, then $\{x_n\}$ is a Cauchy sequence. Moreover if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n,y_n)=0$, then $\lim_n d(x_n,y_n)=0$.*

3. Results

In this section, we give our results. We begin with the following theorem.

THEOREM 5. *Let X be a metric space with a τ-distance p and let T be a mapping on X. Assume that the following hold:*

(C3) $\lim_{\delta\to+0} \sup\left\{ \limsup_{n\to\infty} p(T^n x, T^n y) : p(x,y)<\delta \right\} = 0.$

(C4) *For each $\varepsilon>0$, there exists $\delta>0$ such that for $x,y\in X$ with $\varepsilon<p(x,y)<\varepsilon+\delta$, there exists $\nu\in\mathbf{N}$ such that $p(T^\nu x, T^\nu y)\le\varepsilon$.*

(C5) *For $x,y\in X$ with $p(x,y)>0$, there exists $\nu\in\mathbf{N}$ such that $p(T^\nu x,T^\nu y)<p(x,y)$.*

(C6) *For $x\in X$, there exist sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ such that $0<\alpha_n$, $0<\beta_n$ and $\lambda_n\in\mathbf{N}$ for $n\in\mathbf{N}$, $\lim_n \alpha_n=0$ and*

$$\alpha_n < p(T^i x, T^j x) < \alpha_n+\beta_n \qquad \textit{implies} \qquad p(T^{\lambda_n}\circ T^i x, T^{\lambda_n}\circ T^j x)\le \alpha_n$$

for all $i,j,n\in\mathbf{N}$.

Then T is an ACF.

PROOF. We first show

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0 \qquad \text{for all } x,y\in X. \tag{1}$$

Fix $x,y\in X$. We consider the following two cases:

- $p(T^j x, T^j y)=0$ for some $j\in\mathbf{N}$.
- $p(T^j x, T^j y)>0$ for all $j\in\mathbf{N}$.

In the first case, we have

$$\begin{aligned}\limsup_{n\to\infty} p(T^n x, T^n y) &= \limsup_{n\to\infty} p(T^n\circ T^j x, T^n\circ T^j y)\\ &\le \sup\left\{ \limsup_{n\to\infty} p(T^n u, T^n v) : p(u,v)<\delta \right\}\end{aligned}$$

for all $\delta > 0$. By (C3), we have $\lim_n p(T^n x, T^n y) = 0$. In the second case, we put $\gamma = \liminf_n p(T^n x, T^n y)$. Then from (C5), we have

$$\gamma < p(T^j x, T^j y) \qquad \text{for all } j \in \mathbf{N}. \tag{2}$$

Arguing by contradiction, we assume $\gamma > 0$. Then from (C4), there exists $\delta_1 > 0$ satisfying the following:

- For $u, v \in X$ with $\gamma < p(u,v) < \gamma + \delta_1$, there exists $\nu \in \mathbf{N}$ such that $p(T^\nu u, T^\nu v) \leq \gamma$.

From the definition of γ, we can take $j \in \mathbf{N}$ satisfying $\gamma < p(T^j x, T^j y) < \gamma + \delta_1$. Hence, there exists $\nu \in \mathbf{N}$ such that

$$p(T^{\nu+j} x, T^{\nu+j} y) = p(T^\nu \circ T^j x, T^\nu \circ T^j y) \leq \gamma.$$

This contradicts (2). Therefore we obtain $\gamma = 0$. Fix $\varepsilon > 0$. Then from (C3), there exists $\delta_2 > 0$ such that

$$\sup\left\{ \limsup_{n\to\infty} p(T^n u, T^n v) : p(u,v) < \delta_2 \right\} < \varepsilon.$$

Taking $j \in \mathbf{N}$ with $p(T^j x, T^j y) < \delta_2$, we have

$$\limsup_{n\to\infty} p(T^n x, T^n y) = \limsup_{n\to\infty} p(T^n \circ T^j x, T^n \circ T^j y) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (1). We next show

$$\lim_{n\to\infty} \sup_{m>n} p(T^n x, T^m x) = 0 \qquad \text{for all } x \in X. \tag{3}$$

Fix $x \in X$. We let $\varepsilon > 0$ be arbitrarily fixed and choose $K \in \mathbf{N}$ and $\delta > 0$ such that $\alpha_K < \varepsilon$ and $\delta < \min\{\beta_K, \alpha_K\}$. Then from (C6),

$$\alpha_K < p(T^i x, T^j x) < \alpha_K + \delta \qquad \text{implies} \qquad p(T^{\lambda_K + i} x, T^{\lambda_K + j} x) \leq \alpha_K$$

for all $i, j \in \mathbf{N}$. By (1), we can choose $N \in \mathbf{N}$ such that $p(T^n x, T^{n+1} x) < \delta/\lambda_K$ for every $n \geq N$. Fix $L \in \mathbf{N}$ with $L \geq N$. We shall show

$$p(T^L x, T^{L+n} x) < \alpha_K + \delta < 2\varepsilon \tag{4}$$

for all $n \in \mathbf{N}$ by induction. For every $n \in \{1, 2, \dots, \lambda_K\}$, we have

$$p(T^L x, T^{L+n} x) \leq \sum_{j=0}^{n-1} p(T^{L+j} x, T^{L+j+1} x) < n\delta/\lambda_K \leq \delta < \alpha_K + \delta.$$

For $m \in \mathbf{N}$ with $m > \lambda_K$, we assume (4) holds for every $n \in \mathbf{N}$ with $n < m$. In particular, $p(T^L x, T^{L+m-\lambda_K} x) < \alpha_K + \delta$. In the case where $p(T^L x, T^{L+m-\lambda_K} x) \leq \alpha_K$, we have

$$p(T^L x, T^{L+m} x) \le p(T^L x, T^{L+m-\lambda_K} x) + \sum_{j=0}^{\lambda_K - 1} p(T^{L+m-\lambda_K+j} x, T^{L+m-\lambda_K+j+1} x)$$

$$< \alpha_K + \lambda_K \delta / \lambda_K = \alpha_K + \delta.$$

In the other case, where $\alpha_K < p(T^L x, T^{L+m-\lambda_K} x) < \alpha_K + \delta$, we have

$$\begin{aligned} p(T^L x, T^{L+m} x) &\le p(T^L x, T^{L+\lambda_K} x) + p(T^{L+\lambda_K} x, T^{L+m} x) \\ &= p(T^L x, T^{L+\lambda_K} x) + p(T^{\lambda_K} \circ T^L x, T^{\lambda_K} \circ T^{L+m-\lambda_K} x) \\ &< \delta + \alpha_K. \end{aligned}$$

Therefore (4) holds when $n = m$. Thus, by induction, we obtain (4) for all $n \in \mathbf{N}$. Since $\varepsilon > 0$ is arbitrary, we obtain (3). By (1), (3) and Lemma 2, we have (ii) of Theorem 4. By Theorem 4, we obtain the desired result. □

Next, using Theorem 5, we prove the following.

THEOREM 6. *Let X be a metric space with a τ-distance p and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying* (A1) *and the following:*

(D2) *θ is nondecreasing.*

(D3) $\lim_{\delta \to +0} \sup\left\{ \limsup_{n \to \infty} \theta(p(T^n x, T^n y)) : \theta(p(x, y)) < \delta \right\} = 0.$

(D4) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\varepsilon < \theta(p(x, y)) < \varepsilon + \delta$, there exists $\nu \in \mathbf{N}$ such that $\theta(p(T^\nu x, T^\nu y)) \le \varepsilon$.*

(D5) *For $x, y \in X$ with $\theta(p(x, y)) > 0$, there exists $\nu \in \mathbf{N}$ such that $\theta(p(T^\nu x, T^\nu y)) < \theta(p(x, y))$.*

(D6) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbf{N}$ such that*

$$\varepsilon \le \theta(p(T^i x, T^j x)) < \varepsilon + \delta \qquad \textit{implies} \qquad \theta(p(T^\nu \circ T^i x, T^\nu \circ T^j x)) < \varepsilon$$

for all $i, j \in \mathbf{N}$.

Then T is an ACF.

Before proving Theorem 6, we give one lemma.

LEMMA 3. *If $\lim_{t \to +0} \theta(t) = 0$, then* (D3) *is equivalent to* (C3).

PROOF. We first show that (D3) implies (C3). Fix $\varepsilon > 0$. Then since $\theta(\varepsilon/2) > 0$, from (D3), there exists $\beta_1 > 0$ such that

$$\sup\left\{ \limsup_{n \to \infty} \theta(p(T^n x, T^n y)) : \theta(p(x, y)) < \beta_1 \right\} < \theta(\varepsilon/2).$$

This implies

$$\sup\left\{\limsup_{n\to\infty} p(T^n x, T^n y) : \theta(p(x,y)) < \beta_1\right\} < \varepsilon.$$

By hypothesis, we can choose $\delta_1 > 0$ such that $\theta(\delta_1) < \beta_1$. Since $p(x,y) < \delta_1$ implies $\theta(p(x,y)) < \beta_1$, we have

$$\sup\left\{\limsup_{n\to\infty} p(T^n x, T^n y) : p(x,y) < \delta_1\right\} < \varepsilon.$$

Thus (C3) holds. We next show that (C3) implies (D3). Fix $\varepsilon > 0$. Then by hypothesis, there exists $\alpha > 0$ with $\theta(\alpha) < \varepsilon$. From (C3), there exists $\beta_2 > 0$ such that

$$\sup\left\{\limsup_{n\to\infty} p(T^n x, T^n y) : p(x,y) < \beta_2\right\} < \alpha.$$

Hence

$$\sup\left\{\limsup_{n\to\infty} \theta(p(T^n x, T^n y)) : p(x,y) < \beta_2\right\} \le \theta(\alpha) < \varepsilon.$$

We put $\delta_2 = \theta(\beta_2)$. Since $\theta(p(x,y)) < \delta_2$ implies $p(x,y) < \beta_2$, we have

$$\sup\left\{\limsup_{n\to\infty} \theta(p(T^n x, T^n y)) : \theta(p(x,y)) < \delta_2\right\} < \varepsilon.$$

Thus (D3) holds. □

PROOF OF THEOREM 6. Put $\tau = \lim_{t\to+0} \theta(t)$. We consider the following two cases:
- $\tau > 0$.
- $\tau = 0$.

In the case $\tau > 0$, we shall show the following:
- For $x, y \in X$, there exists $N \in \mathbf{N}$ such that $T^n x = T^n y$ and $p(T^n x, T^n x) = 0$ for all $n \ge N$.

We note that from this formula, we can easily prove (1) and (3). Thus, T is an ACF by Theorem 4. In order to prove this formula, we fix $x, y \in X$. Arguing by contradiction, we assume $p(T^n x, T^n y) > 0$ for all $n \in \mathbf{N}$. We put $\gamma = \liminf_n \theta(p(T^n x, T^n y))$. Then from (D5), we have

$$\gamma < \theta(p(T^j x, T^j y)) \qquad \text{for all } j \in \mathbf{N}. \tag{5}$$

Since $0 < \tau \le \gamma$, from (D4), there exists $\delta_1 > 0$ satisfying the following:
- For $u, v \in X$ with $\gamma < \theta(p(u,v)) < \gamma + \delta_1$, there exists $\nu \in \mathbf{N}$ such that $\theta(p(T^\nu u, T^\nu v)) \le \gamma$.

From the definition of γ, we can take $j \in \mathbf{N}$ satisfying $\gamma < \theta(p(T^j x, T^j y)) < \gamma + \delta_1$. Hence, there exists $\nu \in \mathbf{N}$ such that

$$\theta(p(T^{\nu+j}x, T^{\nu+j}y)) \leq \gamma.$$

This contradicts (5). Therefore $p(T^m x, T^m y) = 0$ for some $m \in \mathbf{N}$. On the other hand, since $\tau/2 > 0$, from (C3), there exists $\delta_2 \in (0, \tau)$ such that

$$\sup\left\{\limsup_{n\to\infty} \theta(p(T^n u, T^n u)) : \theta(p(u, v)) < \delta_2\right\} < \frac{\tau}{2},$$

which is equivalent to the following:

- If $p(u, v) = 0$, then there exists $N \in \mathbf{N}$ such that $p(T^n u, T^n v) = 0$ for all $n \geq N$.

Since $p(T^m x, T^m y) = 0$, there exists $N_1 \in \mathbf{N}$ such that $p(T^n x, T^n y) = 0$ for all $n \geq N_1$. In the same way, we can prove that there exists $N_2 \in \mathbf{N}$ such that $p(T^n y, T^n x) = 0$ for all $n \geq N_2$. We put $N = \max\{N_1, N_2\}$. Then $p(T^n x, T^n y) = 0$ and $p(T^n y, T^n x) = 0$ for all $n \geq N$ and hence

$$p(T^n x, T^n x) \leq p(T^n x, T^n y) + p(T^n y, T^n x) = 0.$$

So by Lemma 1, we obtain $T^n x = T^n y$ and $p(T^n x, T^n x) = 0$ for all $n \geq N$. Thus T is an ACF. In the case $\tau = 0$, we shall prove (C3)–(C6). By Lemma 3, we know that (D3) implies (C3). Let us prove (C4). Fix $\varepsilon > 0$ and put $\eta = \lim_{t\to\varepsilon+0} \theta(t)$. In the case where $\eta < \theta(\varepsilon + \gamma)$ holds for every $\gamma > 0$, from (D4), there exists $\beta_1 > 0$ satisfying the following:

- For $u, v \in X$ with $\eta < \theta(p(u, v)) < \eta + \beta_1$, there exists $\nu \in \mathbf{N}$ such that $\theta(p(T^\nu u, T^\nu v)) \leq \eta$.

We can choose $\delta_3 > 0$ satisfying $\theta(\varepsilon + \delta_3) < \eta + \beta_1$. Fix $x, y \in X$ with $\varepsilon < p(x, y) < \varepsilon + \delta_3$. Then we have

$$\eta < \theta(p(x, y)) \leq \theta(\varepsilon + \delta_3) < \eta + \beta_1.$$

Hence there exists $\nu \in \mathbf{N}$ with $\theta(p(T^\nu x, T^\nu y)) \leq \eta$. This implies $p(T^\nu x, T^\nu y) \leq \varepsilon$. In the other case, where there exists $\delta_4 > 0$ such that $\eta = \theta(\varepsilon + \delta_4)$, we also fix $x, y \in X$ with $\varepsilon < p(x, y) < \varepsilon + \delta_4$. Then from (D5), there exists $\nu \in \mathbf{N}$ such that $\theta(p(T^\nu x, T^\nu y)) < \theta(p(x, y))$. Thus $\theta(p(T^\nu x, T^\nu y)) < \eta$ holds. This implies $p(T^\nu x, T^\nu y)) \leq \varepsilon$. It is obvious that (D5) implies (C5). In order to show (C6), we prove the following, which is stronger than (C6):

(C6)′ For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbf{N}$ such that

$$\varepsilon < p(T^i x, T^j x) < \varepsilon + \delta \qquad \text{implies} \qquad p(T^\nu \circ T^i x, T^\nu \circ T^j x) \leq \varepsilon$$

for all $i, j \in \mathbf{N}$.

Fix $x \in X$ and $\varepsilon > 0$ and put $\eta = \lim_{t\to\varepsilon+0} \theta(t)$. Then from (D6), there exists $\beta_2 > 0$ and $K \in \mathbf{N}$ satisfying the following:

- $\eta \le \theta(p(T^k x, T^\ell x)) < \eta + \beta_2$ implies $\theta(p(T^K \circ T^k x, T^K \circ T^\ell x)) < \eta$ for all $k, \ell \in \mathbf{N}$.

We can choose $\delta_4 > 0$ satisfying $\theta(\varepsilon + \delta_4) < \eta + \beta_2$. Fix $i, j \in \mathbf{N}$ with $\varepsilon < p(T^i x, T^j x) < \varepsilon + \delta_4$. Then we have

$$\eta \le \theta(p(T^i x, T^j x)) \le \theta(\varepsilon + \delta_4) < \eta + \beta_2.$$

and hence

$$\theta(p(T^K \circ T^i x, T^K \circ T^j x)) < \eta.$$

This implies $p(T^K \circ T^i x, T^K \circ T^j x) \le \varepsilon$. We have shown (C3)–(C6). By Theorem 5, we obtain that T is an ACF. □

As a direct consequence of Theorem 6, we obtain the following.

COROLLARY 1. *Let X be a metric space with a τ-distance p and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying* (A1), (D2), (D3) *and the following:*

(E45) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\theta(p(x, y)) < \varepsilon + \delta$, there exists $v \in \mathbf{N}$ such that $\theta(p(T^v x, T^v y)) < \varepsilon$.*

(E6) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbf{N}$ such that*

$$\theta(p(T^i x, T^j x)) < \varepsilon + \delta \qquad \textit{implies} \qquad \theta(p(T^v \circ T^i x, T^v \circ T^j x)) < \varepsilon$$

for all $i, j \in \mathbf{N}$.

Then T is an ACF.

Using Theorem 5 again, we shall prove the following.

THEOREM 7. *Let X be a metric space with a τ-distance p and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying* (A1), (D3)–(D5) *and the following:*

(F2) *θ is nondecreasing and continuous*

(F6) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbf{N}$ such that*

$$\varepsilon < \theta(p(T^i x, T^j x)) < \varepsilon + \delta \qquad \textit{implies} \qquad \theta(p(T^v \circ T^i x, T^v \circ T^j x)) \le \varepsilon$$

for all $i, j \in \mathbf{N}$.

Then T is an ACF.

PROOF. It is sufficient to prove the following because we have shown the remainder in the proof of Theorem 6:

- (C6) holds in the case $\lim_{t \to +0} \theta(t) = 0$.

We choose a strictly decreasing sequence $\{\gamma_n\}$ in $(0, \infty)$ satisfying $\gamma_1 < \theta(1)$ and $\lim_n \gamma_n = 0$. We can put

$$\alpha_n = \max\{t : \theta(t) = \gamma_n\}$$

for $n \in \mathbf{N}$. Then $\{\alpha_n\}$ is a strictly decreasing sequence in $(0, 1)$ satisfying the following:

- $\{\theta(\alpha_n)\}$ is a strictly decreasing sequence.
- $\lim_n \alpha_n = 0$.
- $\theta(\alpha_n) < \theta(t)$ for all $n \in \mathbf{N}$ and $t > 0$ with $\alpha_n < t$.

Fix $x \in X$. From (F6), there exist sequences $\{\delta_n\}$ and $\{\lambda_n\}$ such that $0 < \delta_n$ and $\lambda_n \in \mathbf{N}$ for $n \in \mathbf{N}$, and

$$\theta(\alpha_n) < \theta(p(T^k x, T^\ell x)) < \theta(\alpha_n) + \delta_n \qquad \text{implies} \qquad \theta(p(T^{\lambda_n} \circ T^k x, T^{\lambda_n} \circ T^\ell x)) \le \theta(\alpha_n)$$

for all $k, \ell, n \in \mathbf{N}$. Since θ is continuous, we can choose a sequence $\{\beta_n\}$ in $(0, \infty)$ such that

$$\theta(\alpha_n + \beta_n) < \theta(\alpha_n) + \delta_n$$

for all $n \in \mathbf{N}$. Fix $i, j, n \in \mathbf{N}$ with

$$\alpha_n < p(T^i x, T^j x) < \alpha_n + \beta_n.$$

Then we have

$$\theta(\alpha_n) < \theta(p(T^i x, T^j x)) \le \theta(\alpha_n + \beta_n) < \theta(\alpha_n) + \delta_n$$

and hence $\theta(p(T^{\lambda_n} \circ T^i x, T^{\lambda_n} \circ T^j x)) \le \theta(\alpha_n)$. This implies $p(T^{\lambda_n} \circ T^i x, T^{\lambda_n} \circ T^j x) \le \alpha_n$. We have shown (C6). □

As a direct consequence of Theorem 7, we can obtain the following.

COROLLARY 2. *Let (X, d) be a metric space and let T be a mapping on X. Let f be a locally integrable function from $[0, \infty)$ into itself satisfying $\int_0^s f(t)dt > 0$ for all $s > 0$. Assume* (B3) *and the following hold:*

(G4) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with*

$$\varepsilon < \int_0^{d(x,y)} f(t)dt < \varepsilon + \delta,$$

there exists $v \in \mathbf{N}$ such that

$$\int_0^{d(T^v x, T^v y)} f(t)dt \le \varepsilon.$$

(G5) *For $x, y \in X$ with $x \ne y$, there exists $v \in \mathbf{N}$ such that*

$$\int_0^{d(T^v x, T^v y)} f(t)dt < \int_0^{d(x,y)} f(t)dt.$$

(G6) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \mathbf{N}$ such that*

$$\varepsilon < \int_0^{d(T^i x, T^j x)} f(t)dt < \varepsilon + \delta \qquad \textit{implies} \qquad \int_0^{d(T^\nu \circ T^i x, T^\nu \circ T^j x)} f(t)dt \le \varepsilon$$

for all $i, j \in \mathbf{N}$.
Then T *is an ACF.*

We finally give an example which shows that we cannot omit the continuity of θ in Theorem 7.

EXAMPLE 1. Define a complete subset X of the Euclidean space $\mathbf{R}$ by $X = \{x_n : n \in \mathbf{N}\}$, where

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$

for $n \in \mathbf{N}$. Define a mapping T on X by $Tx_n = x_{n+1}$ for $n \in \mathbf{N}$. Define a function θ from $[0, \infty)$ into itself by

$$\theta(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1/([1/t] + 1) & \text{if } t > 0 \end{cases}$$

for $t \in [0, \infty)$, where $[1/t]$ is the maximum integer not exceeding $1/t$. Then all the assumptions of Theorem 7 except the continuity of θ are satisfied, that is, (A1), (D2)–(D5) and (F6) hold. However, T is not an ACF.

PROOF. We note

$$\theta(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } 1 < t, \\ 1/2 & \text{if } 1/2 < t \le 1, \\ 1/3 & \text{if } 1/3 < t \le 1/2, \\ 1/4 & \text{if } 1/4 < t \le 1/3, \\ \vdots & \vdots \end{cases}$$

for $t \in [0, \infty)$. Thus, (A1) and (D2) clearly hold. For all $x, y \in X$, $\lim_n d(T^n x, T^n y) = 0$ holds, which implies (D3)–(D5). Since

$$\lim_{n \to \infty} T^n x_0 = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sum_{j=1}^{n+1} \frac{1}{j} = \infty,$$

the sequence $\{T^n x_0\}$ in X is not a Cauchy sequence. So by Theorem 4, T is not an ACF. Let us prove (F6). Fix $x \in X$ and $\varepsilon > 0$. In the case where $\varepsilon \ge 1$, we put $\delta = 1$. Then there is no $(i, j) \in \mathbf{N}^2$ satisfying $\varepsilon < \theta(d(T^i x, T^j x)) < \varepsilon + \delta$. In the other case, where $0 < \varepsilon < 1$, there exists $\ell \in \mathbf{N}$ such that $1/(\ell + 1) \le \varepsilon < 1/\ell$. We put

$\delta := 1/\ell - \varepsilon > 0$. Then there is no $(i,j) \in \mathbf{N}^2$ satisfying $\varepsilon < \theta(d(T^i x, T^j x)) < \varepsilon + \delta$ because

$$\{\theta(t) : t \in [0,\infty)\} = \{0\} \cup \{1/n : n \in \mathbf{N}\}.$$

Therefore (F6) holds. $\square$

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