Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 54, 2007, pp. 13–26

SOME COMMENTS ABOUT RECENT RESULTS ON ONE-PARAMETER NONEXPANSIVE SEMIGROUPS

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Abstract

We discuss common fixed points of one-parameter nonexpansive semigroups. In particular, we give some comments about recent results in Suzuki's papers [J. Math. Anal. Appl., 324 (2006), 1006–1019] and [Israel J. Math., 157 (2007), 239–257].

1. Introduction

Let C be a subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty in the case where E is a Hilbert space and C is bounded closed and convex; see [1–3, 7, 8] and others. A family of mappings $\{T(t) : t \ge 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*one-parameter nonexpansive semigroup*, for short) on C if the following are satisfied:

(ns 1) for each $t \ge 0$, $T(t)$ is a nonexpansive mapping on C;
(ns 2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \ge 0$;
(ns 3) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

We denote by $F(\mathscr{T})$ the set of common fixed points of $\{T(t) : t \ge 0\}$, that is,

$$F(\mathscr{T}) = \bigcap_{t \ge 0} F(T(t)).$$

We know that $F(\mathscr{T})$ is nonempty in the case where E is a Hilbert space and C is bounded closed and convex; see [3, 6]. In 1967, Browder [4] proved the following strong convergence theorem, and Reich [11] extended the theorem to uniformly smooth Banach spaces.

Date: Received November 30, 2006.
2000 *Mathematics Subject Classification.* Primary 47H20, Secondary 47H09, 47H10.
Key words and phrases. One-parameter nonexpansive semigroup, nonexpansive mapping, fixed point, Browder's type convergence theorem.
The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

Theorem 1 (Browder [4]). *Let C be a bounded closed convex subset of a Hilbert space E and let T be a nonexpansive mapping on C. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by*

$$u_n = (1 - \alpha_n)Tu_n + \alpha_n u$$

for $n \in \mathbf{N}$. Then $\{u_n\}$ converges strongly to the element of $F(T)$ nearest to u.

Motivated by Theorem 1, Suzuki [13, 23] proved the following theorem, which is a one-parameter version of Theorem 1.

Theorem 2 ([23]). *Let C be a weakly compact convex subset of a Banach space E. Assume that either of the following holds:*

- *E is uniformly convex with uniformly Gâteaux differentiable norm;*
- *E is uniformly smooth; or*
- *E is a smooth Banach space with the Opial property and the duality mapping J of E is weakly sequentially continuous at zero.*

Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbf{R}$ satisfying $0 < \alpha_n < 1$, $0 < \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbf{N}$, and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n)T(\tau + t_n)u_n + \alpha_n u$$

for $n \in \mathbf{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathscr{T})$.

In Theorem 2, we assume that $\{t_n\}$ converges to 0, that is, $\{\tau + t_n\}$ converges. The author thought that this condition is essential. However this is not true. In this paper, we shall show it.

Also, the author has studied characterizations of (common) fixed points for nonexpansive semigroups and nonexpansive mappings; see [12, 14–22, 24–26]. We shall give two characterizations, which concern results in [21].

2. Preliminaries

In this section, we give some preliminaries. Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers, and by $\mathbf{R}$ the set of real numbers.

Let E be a real Banach space. We denote by E^* the dual of E. E is called *uniformly convex* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|x + y\|}{2} < 1 - \delta$$

for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$. E is said to be *smooth* or said to have a *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in E$ with $\|y\| = 1$, the limit is attained uniformly in $x \in E$ with $\|x\| = 1$. E is said to be *uniformly smooth* or said to have a *uniformly Fréchet differentiable norm* if the limit is attained uniformly in $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have the *Opial property* [9] if for each weakly convergent sequence $\{x_n\}$ in E with weak limit x_0,

$$\liminf_{n \to \infty} \|x_n - x_0\| < \liminf_{n \to \infty} \|x_n - x\|$$

holds for $x \in E$ with $x \neq x_0$.

Let E be a smooth Banach space. The duality mapping J from E into E^* is defined by

$$\langle x, J(x) \rangle = \|x\|^2 = \|J(x)\|^2$$

for $x \in E$. J is said to be *weakly sequentially continuous at zero* if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

We recall that a closed convex subset C of a Banach space E is said to have the *fixed point property for nonexpansive mappings* (*FPP*, for short) if for every bounded closed convex subset K of C, every nonexpansive self-mapping on K has a fixed point. We remark that if E and C satisfy the assumption in Theorem 2, then C has FPP.

Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny [5] if

$$P(Px + t(x - Px)) = Px$$

for $x \in C$ with $Px + t(x - Px) \in C$ and $t \geq 0$. Reich [10] proved that if E is smooth, C is convex and K is a subset of C, then there is at most one sunny nonexpansive retraction from C onto K. See also [27].

The following theorem is one of the most celebrated fixed point theorems for families of nonexpansive mappings.

THEOREM 3 (Bruck [6]). *Suppose a weakly compact convex subset C of a Banach space has the fixed point property for nonexpansive mappings. Then for any commuting family S of nonexpansive mappings on C, the set of common fixed points of S is a nonempty nonexpansive retract of C. That is, there exists a nonexpansive mapping P from C onto*

$$\bigcap_{T \in S} F(T)$$

such that $P^2 = P$.

The following lemma is well known.

LEMMA 1. *Let $\{u_n\}$ be a sequence in a Banach space E and let z belong to E. Assume that every subsequence of $\{u_n\}$ has a subsequence converging to z. Then $\{u_n\}$ itself converges to z.*

3. Browder-type convergence theorem

In this section, using Theorem 2, we generalize Theorem 2. We also state some comments on Theorem 2.

THEOREM 4. *Let E, C and $\{T(t) : t \geq 0\}$ be as in Theorem 2. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbf{R}$ satisfying*

(i) $0 < \alpha_n < 1$, $0 \leq t_n$ *and* $s_n := \liminf_m |t_m - t_n| > 0$ *for* $n \in \mathbf{N}$*;*

(ii) $\{t_n\}$ *is bounded;*

(iii) $\lim_n \alpha_n / s_n = 0$.

Fix $u \in C$ and define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n) T(t_n) u_n + \alpha_n u$$

for $n \in \mathbf{N}$. Then $\{u_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathscr{T})$.

REMARK. From the proof below, we obtain $\lim_n s_n = 0$.

PROOF. Let $\{f(n)\}$ be an arbitrary subsequence of $\{n\}$. Then since $\{t_n\}$ is bounded, there exists a subsequence $\{g(n)\}$ of $\{n\}$ such that $\{t_{f \circ g(n)}\}$ converges to some nonnegative real number τ. We note $s_{f \circ g(n)} \leq |\tau - t_{f \circ g(n)}|$ for all $n \in \mathbf{N}$, which implies

$$\lim_{n \to \infty} \frac{\alpha_{f \circ g(n)}}{t_{f \circ g(n)} - \tau} = 0.$$

By Theorem 2, $\{u_{f \circ g(n)}\}$ converges strongly to Pu. Since $\{f(n)\}$ is arbitrary, we obtain that $\{u_n\}$ converges strongly to Pu. □

We give an example of sequences $\{\alpha_n\}$ and $\{t_n\}$.

EXAMPLE 1. Define functions f and g from $\mathbf{N}$ into $\mathbf{N} \cup \{0\}$ by

$$f(n) = \max\left\{k \in \mathbf{N} \cup \{0\} : \frac{k(k+1)}{2} < n\right\}$$

and

$$g(n) = n - \frac{f(n)(f(n)+1)}{2}$$

for $n \in \mathbf{N}$. Define sequences $\{\alpha_n\}$ and $\{t_n\}$ in $\mathbf{R}$ by

$$t_n = \frac{1}{2^{g(n)}} + \frac{1}{4^n} \quad \text{and} \quad \alpha_n = \frac{1}{4^{2n}}$$

for $n \in \mathbf{N}$. Then $\{\alpha_n\}$ and $\{t_n\}$ satisfy the assumptions in Theorem 4, and $1/2^\nu$ is a cluster point of $\{t_n\}$ for every $\nu \in \mathbf{N}$.

REMARK. The sequence $\{g(n)\}$ is

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, \dots.$$

Next, we prove the following lemma, which is one of key lemmas to prove Theorem 2.

LEMMA 2. *Let C be a closed convex subset of a Banach space E with the fixed point property for nonexpansive mappings. Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C. Let A be a nonempty weakly compact convex subset of C and let τ be a nonnegative real number such that the following hold:*

- *A is $T(\tau)$-invariant.*
- *If $y \in A$ satisfies $T(\tau)y = y$, then $T(t)y \in A$ for all $t \geq 0$.*

Then $\{T(t) : t \geq 0\}$ has a common fixed point in A.

PROOF. Put $D = A \cap F(T(\tau))$. Then since A has FPP, D is nonempty. By Bruck's theorem (Theorem 3), there exists a nonexpansive retraction P from A onto D. Fix $y \in D$ and $t \geq 0$. From the second assumption, we have $T(t)y \in A$. We also have

$$T(t)y = T(t) \circ T(\tau)y = T(t + \tau)y = T(\tau) \circ T(t)y$$

and hence $T(t)y \in F(T(\tau))$. Therefore $T(t)y \in D$. This means that D is $T(t)$-invariant for all $t \geq 0$. For each $t \geq 0$, we define a mapping $S(t)$ on A by $S(t) = T(t) \circ P$. We shall show that $\{S(t) : t \geq 0\}$ is a one-parameter nonexpansive semigroup on A. For $x, y \in A$, we have

$$\|S(t)x - S(t)y\| = \|T(t) \circ Px - T(t) \circ Py\| \leq \|Px - Py\| \leq \|x - y\|.$$

That is, $S(t)$ is a nonexpansive mapping on A. Since $S(t)A \subset D$, we have $P \circ S(t) = S(t)$ and hence

$$\begin{aligned} S(s) \circ S(t) &= T(s) \circ P \circ S(t) = T(s) \circ S(t) = T(s) \circ T(t) \circ P \\ &= T(s + t) \circ P = S(s + t) \end{aligned}$$

for all $s, t \geq 0$. The strong continuity of $(t \mapsto S(t)x)$ is obvious. Therefore we have shown that $\{S(t) : t \geq 0\}$ is a one-parameter nonexpansive semigroup on A. By using Bruck's theorem (Theorem 3) again, there exists a common fixed point $z \in A$ of $\{S(t) : t \geq 0\}$. We have $z = S(1)z \in D$ and hence $Pz = z$. So, for every $t \geq 0$, we have

$$z = S(t)z = T(t) \circ Pz = T(t)z.$$

That is, z is a common fixed point of $\{T(t) : t \geq 0\}$. □

Finally, we prove the following proposition, which shows that the condition $\lim_n \alpha_n/t_n = 0$ is essential in Theorem 2.

PROPOSITION 1. *Let E be the two dimensional real Hilbert space and put $C = E$. Define a one-parameter linear nonexpansive semigroup $\{T(t) : t \geq 0\}$ on C by*

$$T(t)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(t)x - \sin(t)y \\ \sin(t)x + \cos(t)y \end{bmatrix}$$

for all $(x, y) \in C$ and $t \geq 0$. Put $u = (1, 0)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbf{R}$ satisfying $0 < \alpha_n < 1$ and $0 < t_n$ for $n \in \mathbf{N}$, and $\lim_n \alpha_n = \lim_n t_n = 0$. Define a sequence $\{u_n\}$ in C by

$$u_n = (1 - \alpha_n)T(t_n)u_n + \alpha_n u$$

for $n \in \mathbf{N}$. Then the following are equivalent:
(i) *$\{u_n\}$ converges to $(0, 0)$;*
(ii) *$\lim_n \alpha_n/t_n = 0$ holds.*

PROOF. We note $F(\mathscr{T}) = \{(0, 0)\}$. By Theorem 2, (ii) implies (i). Let us prove that (i) implies (ii). We assume (i). For $\alpha \in (0, 1)$ and $t \in (0, \infty)$, we put $x(\alpha, t)$ and $y(\alpha, t)$ by

$$\begin{bmatrix} x(\alpha, t) \\ y(\alpha, t) \end{bmatrix} = \frac{\alpha}{4(1-\alpha)\sin^2(t/2) + \alpha^2} \begin{bmatrix} \alpha + 2(1-\alpha)\sin^2(t/2) \\ (1-\alpha)\sin(t) \end{bmatrix}.$$

It is easy to verify that

$$\begin{bmatrix} x(\alpha, t) \\ y(\alpha, t) \end{bmatrix} = (1-\alpha)T(t)\begin{bmatrix} x(\alpha, t) \\ y(\alpha, t) \end{bmatrix} + \alpha u.$$

That is, $u_n = (x(\alpha_n, t_n), y(\alpha_n, t_n))$ for $n \in \mathbf{N}$. Arguing by contradiction, we assume $\limsup_n \alpha_n/t_n > 0$. Thus, $\theta := \liminf_n t_n/\alpha_n < \infty$. We choose a subsequence $\{f(n)\}$ of $\{n\}$ such that $\lim_n t_{f(n)}/\alpha_{f(n)} = \theta$. We have

$$\begin{aligned}
0 &= \lim_{n\to\infty} x(\alpha_{f(n)}, t_{f(n)}) \\
&= \lim_{n\to\infty} \frac{\alpha_{f(n)}(\alpha_{f(n)} + 2(1-\alpha_{f(n)})\sin^2(t_{f(n)}/2))}{4(1-\alpha_{f(n)})\sin^2(t_{f(n)}/2) + \alpha_{f(n)}^2} \\
&= \lim_{n\to\infty} \frac{1 + 2(1-\alpha_{f(n)})\left(\dfrac{\sin(t_{f(n)}/2)}{t_{f(n)}/2}\right)^2 \dfrac{t_{f(n)}}{4}\dfrac{t_{f(n)}}{\alpha_{f(n)}}}{(1-\alpha_{f(n)})\left(\dfrac{\sin(t_{f(n)}/2)}{t_{f(n)}/2}\right)^2\left(\dfrac{t_{f(n)}}{\alpha_{f(n)}}\right)^2 + 1} \\
&= \frac{1}{\theta^2 + 1}.
\end{aligned}$$

This is a contradiction. $\square$

4. Characterizations

The following theorem is essentially proved in [21]. We give a simple proof of it.

THEOREM 5 ([21]). *Let $\{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on a subset C of a Banach space E. Define a nonexpansive mapping S from C into E by*

$$Sx = \lambda \int_0^\infty \exp(-\lambda s) T(s) x \, ds$$

for $x \in C$, where $\lambda > 0$. Then $F(S) = F(\mathscr{T})$ holds.

PROOF. It is obvious that $F(S) \supset F(\mathscr{T})$. Let us prove the converse inclusion. Let $z \in C$ be a fixed point of S. For $n \in \mathbf{N}$, we put

$$M_n = \max\{\|T(t)z - T(0)z\| : 0 \leq t \leq n\}$$

and $M = \sup_n M_n$. For every $n \in \mathbf{N}$, we choose $L_n \in \mathbf{R}$ satisfying

$$0 \leq L_n \leq n \qquad \text{and} \qquad M_n = \|T(L_n)z - T(0)z\|.$$

We consider the following three cases:

A. $M = 0$

B. $M > 0$ and $\limsup_n L_n < \infty$

C. $M > 0$ and $\limsup_n L_n = \infty$

(Case A) In this case, $T(0)z$ is a common fixed point of $\{T(t) : t \geq 0\}$. Hence

$$\begin{aligned} z &= Sz \\ &= \lambda \int_0^\infty \exp(-\lambda s) T(s + 0) z \, ds \\ &= \lambda \int_0^\infty \exp(-\lambda s) T(s) \circ T(0) z \, ds \\ &= \lambda \int_0^\infty \exp(-\lambda s) T(0) z \, ds \\ &= T(0)z. \end{aligned}$$

Thus, $z \in F(\mathscr{T})$.

(Case B) We put $L = \limsup_n L_n$. Then it is obvious that $\|T(L)z - T(0)z\| = M$. We choose $\delta > 0$ such that

$$\|T(t)z - T(0)z\| < M/2$$

for $t \in [0, \delta]$. We have

$$\begin{aligned} M &= \|T(L)z - T(0)z\| \\ &\leq \|T(L)z - z\| \end{aligned}$$

$$
\begin{aligned}
&= \|T(L)z - Sz\| \\
&\le \lambda \int_0^\infty \exp(-\lambda s)\|T(L)z - T(s)z\| ds \\
&= \lambda \int_0^L \exp(-\lambda s)\|T(L)z - T(s)z\| ds + \lambda \int_L^{L+\delta} \exp(-\lambda s)\|T(L)z - T(s)z\| ds \\
&\quad + \lambda \int_{L+\delta}^\infty \exp(-\lambda s)\|T(L)z - T(s)z\| ds \\
&\le \lambda \int_0^L \exp(-\lambda s)\|T(L-s)z - T(0)z\| ds \\
&\quad + \lambda \int_L^{L+\delta} \exp(-\lambda s)\|T(s-L)z - T(0)z\| ds \\
&\quad + \lambda \int_{L+\delta}^\infty \exp(-\lambda s)\|T(s-L)z - T(0)z\| ds \\
&\le \lambda \int_0^L \exp(-\lambda s) M\, ds + \lambda \int_L^{L+\delta} \exp(-\lambda s) M/2\, ds \\
&\quad + \lambda \int_{L+\delta}^\infty \exp(-\lambda s) M\, ds \\
&= M\left(1 - \exp(-\lambda L) + \frac{\exp(-\lambda L)}{2} - \frac{\exp(-\lambda(L+\delta))}{2} + \exp(-\lambda(L+\delta))\right) \\
&= M\left(1 - \frac{\exp(-\lambda L)}{2} + \frac{\exp(-\lambda(L+\delta))}{2}\right) \\
&< M,
\end{aligned}
$$

which implies a contradiction.

(Case C) We choose $\delta > 0$ such that

$$\|T(t)z - T(0)z\| < \min\{M/3, 1\}$$

for $t \in [0, \delta]$. For $k, n \in \mathbf{N}$ and $t \in [0, L_n]$, we have

$$
\begin{aligned}
&\|T((k+1)L_n + t)z - T(L_n)z\| \\
&\quad \le \|T(kL_n + t)z - T(0)z\| \\
&\quad \le \sum_{j=1}^{k} \|T(jL_n)z - T((j-1)L_n)z\| + \|T(kL_n + t)z - T(kL_n)z\|
\end{aligned}
$$

$$\leq \sum_{j=1}^{k} \|T(L_n)z - T(0)z\| + \|T(t)z - T(0)z\|$$

$$\leq (k+1)M_n.$$

Hence, for $n \in \mathbf{N}$ with $\delta < L_n$ and $\min\{M/3, 1\} < M_n/2$, we have

$$\begin{aligned}
M_n &= \|T(L_n)z - T(0)z\| \\
&\leq \|T(L_n)z - z\| \\
&= \|T(L_n)z - Sz\| \\
&\leq \lambda \int_0^\infty \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&= \lambda \int_0^{L_n-\delta} \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&\quad + \lambda \int_{L_n-\delta}^{L_n} \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&\quad + \lambda \int_{L_n}^{2L_n} \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&\quad + \lambda \sum_{k=1}^{\infty} \int_{(k+1)L_n}^{(k+2)L_n} \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&\leq \lambda \int_0^{L_n-\delta} \exp(-\lambda s)\|T(L_n - s)z - T(0)z\| ds \\
&\quad + \lambda \int_{L_n-\delta}^{L_n} \exp(-\lambda s)\|T(L_n - s)z - T(0)z\| ds \\
&\quad + \lambda \int_{L_n}^{2L_n} \exp(-\lambda s)\|T(s - L_n)z - T(0)z\| ds \\
&\quad + \lambda \sum_{k=1}^{\infty} \int_{(k+1)L_n}^{(k+2)L_n} \exp(-\lambda s)\|T(L_n)z - T(s)z\| ds \\
&\leq \lambda \int_0^{L_n-\delta} \exp(-\lambda s) M_n \, ds \\
&\quad + \lambda \int_{L_n-\delta}^{L_n} \exp(-\lambda s) M_n/2 \, ds
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda \int_{L_n}^{2L_n} \exp(-\lambda s) M_n \, ds \\
&+ \lambda \sum_{k=1}^{\infty} \int_{(k+1)L_n}^{(k+2)L_n} \exp(-\lambda s)(k+1) M_n \, ds \\
= {} & (1 - \exp(-\lambda(L_n - \delta))) M_n \\
&+ (\exp(-\lambda(L_n - \delta)) - \exp(-\lambda L_n)) M_n/2 \\
&+ (\exp(-\lambda L_n) - \exp(-\lambda 2L_n)) M_n \\
&+ \sum_{k=1}^{\infty} (\exp(-\lambda(k+1)L_n) - \exp(-\lambda(k+2)L_n))(k+1) M_n \\
= {} & M_n - (\exp(-\lambda(L_n - \delta)) + \exp(-\lambda L_n)) M_n/2 \\
&+ \sum_{k=1}^{\infty} \exp(-\lambda k L_n) M_n \\
= {} & M_n - (\exp(-\lambda(L_n - \delta)) + \exp(-\lambda L_n)) M_n/2 \\
&+ \frac{\exp(-\lambda L_n)}{1 - \exp(-\lambda L_n)} M_n,
\end{aligned}
$$

which implies

$$0 \le -\exp(\lambda\delta) - 1 + \frac{2}{1 - \exp(-\lambda L_n)}.$$

Since $\limsup_n L_n = \infty$, we obtain $0 \le -\exp(\lambda\delta) + 1$. This is a contradiction. □

Similarly, we can prove a characterization for single nonexpansive mappings.

THEOREM 6. *Let T be a nonexpansive mapping on a subset C of a Banach space E. Define a nonexpansive mapping S from C into E by*

$$Sx = (1 - r) \sum_{n=1}^{\infty} r^{n-1} T^n x$$

for $x \in C$, where $r \in (0, 1)$. Then $F(S) = F(T)$ holds.

PROOF. S is nonexpansive because

$$\|Sx - Sy\| \leq (1-r)\sum_{n=1}^{\infty} r^{n-1}\|T^n x - T^n y\|$$

$$\leq (1-r)\sum_{n=1}^{\infty} r^{n-1}\|x-y\| = \|x-y\|$$

for all $x, y \in C$. It is obvious that $F(S) \supset F(T)$. Let us prove the converse inclusion. Let $z \in C$ be a fixed point of S. Arguing by contradiction, we assume that z is not a fixed point of T. We put $T^0 z = z$. We also put

$$M_n = \max\{\|T^j z - z\| : 1 \leq j \leq n\}$$

for $n \in \mathbf{N}$ and $M = \sup_n M_n$. We note $M \geq M_n \geq M_1 > 0$. For every $n \in \mathbf{N}$, we choose $L_n \in \mathbf{N}$ satisfying $L_n \leq n$ and $M_n = \|T^{L_n} z - z\|$. We consider the following two cases:

A. $\limsup_n L_n < \infty$

B. $\limsup_n L_n = \infty$

(Case A) We put $L = \limsup_n L_n$. Then it is obvious that $\|T^L z - z\| = M$. For every $n \in \mathbf{N}$, we have

$$\begin{aligned}
M &= \|T^L z - z\| \\
&= \|T^L z - Sz\| \\
&\leq (1-r)\sum_{k=1}^{\infty} r^{k-1}\|T^L z - T^k z\| \\
&\leq (1-r)\sum_{k=1}^{L-1} r^{k-1}\|T^{L-k} z - z\| + (1-r)\sum_{k=L+1}^{\infty} r^{k-1}\|T^{k-L} z - z\| \\
&\leq (1-r)\sum_{k=1}^{L-1} r^{k-1} M + (1-r)\sum_{k=L+1}^{\infty} r^{k-1} M \\
&= (1 - r^{L-1})M + r^L M \\
&< M.
\end{aligned}$$

This is a contradiction.

(Case B) For $k, \ell, n \in \mathbf{N}$ with $1 \leq \ell \leq L_n$, we have

$$\begin{aligned}
\|T^{(k+1)L_n+\ell} z - T^{L_n} z\| &\leq \|T^{kL_n+\ell} z - z\| \\
&\leq \sum_{j=1}^{k} \|T^{jL_n} z - T^{(j-1)L_n} z\| + \|T^{kL_n+\ell} z - T^{kL_n} z\|
\end{aligned}$$

$$\leq \sum_{j=1}^{k} \|T^{L_n}z - z\| + \|T^{\ell}z - z\|$$

$$\leq (k+1)M_n.$$

Hence

$$\begin{aligned}
M_n &= \|T^{L_n}z - z\| \\
&= \|T^{L_n}z - Sz\| \\
&\leq (1-r)\sum_{k=1}^{\infty} r^{k-1}\|T^{L_n}z - T^k z\| \\
&\leq (1-r)\sum_{k=1}^{L_n-1} r^{k-1}\|T^{L_n-k}z - z\| + (1-r)\sum_{k=L_n+1}^{2L_n} r^{k-1}\|T^{k-L_n}z - z\| \\
&\quad + (1-r)\sum_{i=1}^{\infty}\sum_{j=1}^{L_n} r^{(i+1)L_n+j-1}\|T^{L_n}z - T^{(i+1)L_n+j}z\| \\
&\leq (1-r)\sum_{k=1}^{L_n-1} r^{k-1}M_n + (1-r)\sum_{k=L_n+1}^{2L_n} r^{k-1}M_n \\
&\quad + (1-r)\sum_{i=1}^{\infty}\sum_{j=1}^{L_n} r^{(i+1)L_n+j-1}(i+1)M_n \\
&= M_n\left(1 - r^{L_n-1} + r^{L_n} - r^{2L_n} + \sum_{i=1}^{\infty}(r^{(i+1)L_n} - r^{(i+2)L_n})(i+1)\right) \\
&= M_n\left(1 - r^{L_n-1} + \sum_{i=1}^{\infty} r^{iL_n}\right) \\
&= M_n\left(1 - r^{L_n-1} + \frac{r^{L_n}}{1-r^{L_n}}\right),
\end{aligned}$$

which implies

$$0 \leq -r^{-1} + \frac{1}{1-r^{L_n}}.$$

Since $\limsup_n L_n = \infty$, we obtain $0 \leq -r^{-1} + 1$. This is a contradiction. $\square$

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