Properties of Light Nuclei with Harmonic Oscillator Wave Functions
II. Application of the jj Conpling Model and its Configuration Mixing to Ne^{21} and Na^{23}

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In this paper, we attempt to calculate the matrix elements of the central, tensor and mutual spin-orbit interactions in the $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations by Talmi's method. Based on the odd-group model, these results are applied to Ne^{21} and Na^{23}. In jj coupling, by using nucleon-nucleon interactions with the Yukawa potential, hitherto proposed by various authors to explain two-body and three-body data, we find it impossible to explain the occurrence of the ground state with $J=3/2$ for the $(3d_{5/2})^3$ and $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ configurations. Then we take into account the mixing of above three configurations and assume a two-body charge symmetric interaction which contains three parameters g, x and y describing the spin dependence of the central force and the relative strengths of the tensor and mutual spin-orbit forces, respectively. For both central and tensor forces the Yukawa radial dependence is used, while the spin-orbit term is of the kind proposed by Case & Pais. The three parameters g,x,y are determined by fitting the ground-state angular momentum $J=3/2$ and magnetic moment $\mu=2.217$ n.m. of Na^{23}. Then we have a value of $+\ 0.066\times10^{-24}\text{cm}^2$ for the quadrupole moment of Na^{23}.

I. Introduction

We investigate the properties of $3d$- and $2s$–shell nuclei on the basis of individual particle model with harmonic oscillator wave functions. Mayer[1] has proposed by her strong spin–orbit coupling "shell model" that the level order is $3d_{5/2}$, $2s_{\frac{1}{2}}$, $3d_{3/2}$. In this paper, we consider the $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations of like particles in jj coupling and finally these interconfigurational mixing. In order to calculate the matrix elements of the central and tensor interactions and the mutual spin orbit interaction introduced by Case and Pais,[2] the Talmi method[3] is extensively used. If we

1) M. G. Mayer, Phys. Rev. 75, 1969 (1949), 78, 16 (1950), 78, 22 (1950)
2) K. M. Case & A. Pais, Phys. Rev. 80, 203 (1950)
3) I. Talmi, Helv. Phys. Acta 25, 185 (1952)

assume that the even nucleons of even-group are coupled to zero spin, we can apply these results to Ne^{21} and Na^{23}.

The $(3d_{5/2})^3$ configuration with various central potentials has been investigated by Talmi[3) 4)] and Kurath[5)]. They have shown that the occurrence of the spin 3/2 in the ground state of these nuclei is unlikely to be due to the effect of Majorana forces if we assume that the potential is a "deep hole" potential. Then we discuss the effect of non-central interactions introduced to explain the two-and three-body problems, but we cannot explain its occurrence by using just the same interactions as those given by many authors.

For Na^{23}, Mayer has concluded in view of its spin and magnetic moment that there are 3 protons in the $3d_{5/2}$ level and that the $2s_{1/2}$ level is empty. Indeed, a calculation of the magnetic moment with jj coupling gives for this configuration a value of 2.87 n.m., in fairly good agreement with the measured value of 2.217 n.m., while the quadrupole moment in this configuration comes out to be zero. On the other hand, Sengupta[6)] has recently shown that a calculation of the magnetic and quadrupole moments gives for the $(3d_{5/2})^2(2s_{1/2})^1$ configuration results which are in good agreement with the experimental values. Therefore, we perform the same calculation in the $(3d_{5/2})^2(2s_{1/2})^1$ configuration as in the $(3d_{5/2})^3$ configuration, and also we cannot explain the occurrence of the ground state with $J = 3/2$.

From the fact that the ground state of F^{19} has a spin 1/2 and a magnetic moment in very good agreement with the calculated one for the $(2s_{1/2})^1$ configuration, we may assume that the $3d_{5/2}$ and $2s_{1/2}$ levels have very closely the same energy. With the above situation and such crude wave functions as are used, we suppose that interconfigurational mixing must play an important part. Therefore, we consider the mixing of $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{1/2})^1$ and $(3d_{5/2})^1(2s_{1/2})^2$ configurations, and assume a two-body charge-symmetric interaction which contains three parameters g,x,y describing the spin dependence of the central force and the relative strengths of the tensor and mutual spin-orbit forces, respectively. These three parameters are determined by fitting the ground-state spin and magnetic moment of Na^{23} and sign of its quadrupole moment. In consequence, we have for the quadrupole moment a value of $+ 0.066\times10^{-24}$ cm^2 in good agreement with the measured value, and we have shown that for the case of $y = 0$ there exist the constants of such an interaction which

4) I. Talmi, Phys. Rev. 82, 101 (1951)
5) D. Kurath, Phys. Rev. 80, 98 (1950), 91, 1430 (1953)
6) S. Sengupta, Phys. Rev. 96, 235 (1954)

are consistent with the deuteron data.

2. Calculation of Energy Matrices

By writing down the complete set of states in the $(m_{j_1}\ m_{j_2}\ m_{j_3})$-scheme classified by $\mathrm{M_J}$ for each of the $(3d_{5/2})^3$ and $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ configurations, we see that for the $(3d_{5/2})^3$ there are 3 independent states, namely those with $J = 9/2$, $5/2$ and $3/2$, and that for the $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ there are 5 independent states with $J = 9/2$, $7/2$, $5/2$, $3/2$ and $1/2$. Since in each single configuration considered here only one state corresponds to each of the total angular momenta, the matrices of both central and non-central forces can be calculated with the aid of the theorem of trace invariance in the single configuration. In order to calculate the matrix elements for interconfigurational mixing, starting from the $(n\ j\ m_j)$-scheme with the antisymmetrized eigenfunctions the J eigenfunctions in each configuration are found by the method of Gray & Wills[7] using angular momentum operators, and then matrix elements of the two-body interaction operators are calculated by the method of Condon & Shortey[8] using these J-eigenfunctions. Therefore the calculation of the matrices of two-body interaction operators is reduced to the calculation of the matrix elements in the $(n\ l\ m_l\ \mathrm{m}_s)$-scheme :

$$\sum_{\sigma_1\sigma_2}\int\int u_a^*(\vec{r}_1,\sigma_1)\, u_b^*(\vec{r}_2,\sigma_2)\, V(12)\, u_c(\vec{r}_1,\sigma_1)\, u_d(\vec{r}_2,\sigma_2)\, d^3r_1 d^3r_2 \qquad (1)$$

where $u = r^{-1}\ R_{n_l}(r)\ \Theta_{lm_l}\ (\theta)\ \Phi_{m_l}\ (\varphi)\chi_{\frac{1}{2}}^{m_s}(\sigma)$, the subscripts on the u's referring to the set of quantum numbers n, l, m_l, m_s. Even for usual central forces the evaluation of these matrix elements by the Slater method is so laborious and complicated that it is impractical and still more so for complicaed interactions, such as tensor and mutual spin-orbit interactions. However, in this paper we employ the harmonic oscillator wave functions as single nucleon wave functions. Therefore, as Talmi[3] has shown, when we transform two nucleon coordinates $\vec{\mathrm{r}}_1$ and $\vec{\mathrm{r}}_2$ to the relative coordinate $\vec{\mathrm{r}}$ and the coordinate of the center of gravity $\vec{\mathrm{R}}$, we can express the wave function $\phi_{n_1l_1}^{m_1}(\vec{r}_1)\phi_{n_2l_2}^{m_2}(\vec{r}_2)$ of two nucleons with definite quantum numders $n_1\ l_1\ m_1$ and $n_2\ l_2\ m_2$ as a finite sum of products $\psi_{NL}^{M}(\vec{R})\psi_{n\Lambda}^{m}(\vec{r})$ of eigen-functions of harmonic oscillators with the total mass $M = 2m$ and the reduced mass $\mu = \mathrm{m}/2$, respectively, where n_1, n_2, n and N are the number of nodes which characterize these wave functions, and l_1, l_2, L, Λ and m_1, m_2, M, m are the angular momenta and these z-components, respectively. Concerning what values of N, L,

7) N. M Gray & L.A. Wills, Phys. Rev. 38, 248 (1931)

8) E. U. Condon & G. H. Shortley. Theory of Atomic Spectra (Cambridge University Press, Cambridge 1935).

M and n, Λ, m should appear in such an expansion, we have four restrictions : the conservation law of the z-component of the orbital angular momentum, of the energy, and of the parity and the symmetry requirement. Such expansions which will be used in the following are given in Appendix I of the paper I (Bull. Kyushu Inst. Tech., Math. Natur. Sci. No.1, 23, 1955).

Thus, when we carry out the summation over the spin coodinates and the integration with respect to the coordinates of the center of gravity and the angular part of the relative coordinates, the evaluation of matrix elements (1) is reduced to the calculation of integrals of the forms :

$$I_{nl}=\int_{0}^{\infty}R_{nl}^{2}(r)\,V(r)\,dr \tag{2a}$$

and

$$I_{nl,\,n'l'}=\int_{0}^{\infty}R_{nl}(r)R_{n'l'}(r)\,V(r)\,dr. \tag{2b}$$

Here

$$R_{nl}(r)=N_{nl}e^{-\frac{\nu}{2}r^{2}}r^{l+1}v_{nl}(r)$$

where $\nu=\frac{\omega\mu}{\hbar}$, N_{nl} is a normalization factor, and v_{nl} is an associated Laguerre polynomial. These integrals with n, $n'\neq 0$ can be easily expressed as sums of integrals I_{0l} which we shall write simply as I_l in the following.

The wave functions of a single nucleon with given $n, l, j=l+\frac{1}{2}$ and m_j is given by

$$u(nlj=l+\tfrac{1}{2}m_j\,|\,1)=\sqrt{\frac{j+m_j}{2j}}u(nlm_j-\tfrac{1}{2}\,1)\chi_{\frac{1}{2}}^{\frac{1}{2}}(\sigma_1)+\sqrt{\frac{j-m_j}{2j}}u(nlm_j+\tfrac{1}{2}\,|\,1)\chi_{\frac{1}{2}}^{-\frac{1}{2}}(\sigma_1)$$

or briefly

$$u(nlj=l+\tfrac{1}{2}m_j\,|\,1)=f\,u(nlm_j-\tfrac{1}{2}\,|\,1)\chi_{\frac{1}{2}}^{\frac{1}{2}}(\sigma_1)+g\;u(nlm_j+\tfrac{1}{2}\,|\,1)\chi_{\frac{1}{2}}^{-\frac{1}{2}}(\sigma_1). \tag{3}$$

We define the direct integral J and the exchange integral K of any two-body interaction $V(12)$ in the $(j_1\,j_2\,m_{j_1}\,m_{j_2})$-scheme by ;

$$\left.\begin{aligned}
&J(n_1\,l_1\,j_1\,m_{j_1},\;n_2\,l_2\,j_2\,m_{j_2};\;n_3\,l_3\,j_3\,m_{j_3},\;n_4\,l_4\,j_4\,m_{j_4})\\
&=\sum_{\sigma_1\sigma_2}\int\!\!\int u^*(n_1l_1j_1m_1\,|\,1)u^*(n_2l_2j_2m_{j_2};2)V(12)u(n_3l_3j_3m_{j_3}\,|\,1)u(n_4l_4j_4m_{j_4}\,2)\,d^3r_1\,d^3r_2\,,\\
&K(n_1\,l_1\,j_1\,m_{j_1},\;n_2\,l_2\,j_2\,m_{j_2};\;n_3\,l_3\,j_3m_{j_3},\;n_4\,l_4\,j_4\,m_{j_4})\\
&=\sum_{\sigma_1\sigma_2}\int\!\!\int u^*(n_1l_1j_1m_{j_1}\,|\,1)u^*(n_2l_2j_2m_{j_2}\,|\,2)V(12)u(n_3l_3j_3m_{j_3}\,|\,2)u(n_4l_4j_4m_{j_4}\,1)\,d^3r_1\,d^3r_2.
\end{aligned}\right\} \tag{4}$$

When we introduce (3) into (4), we have a sum of matrix elements of type (1) in the

(nlm_lm_s)-scheme, the coefficients of which are products of f_i, g_i (i = 1, 2, 3, 4) and as mentioned above, these matrix elements can be easily expressed in terms of the Talmi integrals.

2.1 The Matrices of Central Interactions

The general two body central interaction operator may be written

$$V(12) = J(r)\{w + h\,P_H + b\,P_B + m\,P_M\}$$

Where P_H, P_B and P_M are Heisenberg, Bartlett and Majorana operators, respectively. In the case of the odd-group model in which only interactions between like nucleons are taken into account, there exists the relation $P_H = P_M P_B = -1$. It is therefore enough to calculate the cases of Wigner and Majorana interactions. However, since both Wigner and Majorana forces are spin-independent, when the summation over the spin coordinates in (4) is carried out, for both interactions there remains the same sum of the integrals of the form

$$J(m_1\,m_2\,;\,m_3\,,\,m_4) = \int\int J(r)\,u^*_{m_1}(\vec{r}_1)\,u^*_{m_2}(\vec{r}_2)\,u_{m_3}(\vec{r}_1)\,u_{m_4}(\vec{r}_2)\,d^3r_1\,d^3r_2$$

with products of f_i, g_i as coefficients, except for a change in the order of the last two quantum numbers in $J(m_1\,m_2\,;\,m_3\,m_4)$. Consequently, the matrix elements of the Majorana interaction are obtained from the Wigner by changing sign of the Talmi integrals I_l which arise from functions antisymmetric in the space coordinates of the two nucleons (those of odd Λ, like I_1, I_3, I_{11}, etc.). We list below the matrix elements which occur in the $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations and these interconfigurational mixing.

Table I. The non-vanishing elemnts of the central interaction

Row and column	The elements
	(a) $J=9/2$
	Diagonal elements
$(3d_{5/2})^3$	$(w-h)\frac{1}{25}\left\{\frac{117}{16}(I_0+I_4)+\frac{129}{4}(I_1+I_3)-\frac{33}{8}I_2\right\}$
	$+(m-b)\frac{1}{25}\left\{\frac{117}{16}(I_0+I_4)-\frac{165}{4}(I_1+I_3)+\frac{255}{8}I_2\right\}$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$(w-h)\left[\frac{1}{5}\left\{\frac{6}{16}(I_0+I_4)+\frac{3}{4}(I_1+I_3)+\frac{2}{8}I_2\right\}+2A-\frac{9}{25}B\right]$
	$+(m-b)\left[\frac{1}{5}\left\{\frac{6}{16}(I_0+I_4)-\frac{8}{4}(I_1+I_3)+\frac{2}{8}I_2\right\}-\frac{9}{5}A+\frac{2}{5}B\right]$

Non-diagonal element

$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1 \qquad (w+m-h-b)\frac{\sqrt{30}}{50}\left\{\frac{3}{8}I_0-\frac{1}{4}I_1+\frac{1}{2}I_2-\frac{7}{4}I_3+\frac{9}{8}I_4\right\}$

(*b*) $J=7/2$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$

$$(w-h)\left[\frac{1}{5}\left\{\frac{6}{16}(I_0+I_4)+\frac{8}{4}(I_1+I_3)+\frac{2}{8}I_2\right\}+2A\right]$$
$$+(m-b)\left[\frac{1}{5}\left\{\frac{6}{16}(I_0+I_4)-\frac{8}{4}(I_1+I_3)+\frac{2}{8}I_2\right\}+\frac{2}{5}B\right]$$

(*c*) $J=5/2$

Diagonal elements

$(3d_{5/2})^3$

$$(w-h)\frac{1}{5}\left\{\frac{63}{16}(I_0+I_4)+\frac{7}{4}(I_1+I_3)+\frac{29}{8}I_2\right\}$$
$$+(m-b)\frac{1}{5}\left\{\frac{63}{16}(I_0+I_4)-\frac{63}{4}(I_1+I_3)+\frac{157}{8}I_2\right\}$$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$

$$(w-h)\left[\frac{1}{5}\left\{\frac{9}{10}(I_0+I_4)+\frac{27}{10}(I_1+I_3)-\frac{22}{10}I_2\right\}+2A-\frac{7}{25}B\right]$$
$$+(m-b)\left[\frac{1}{5}\left\{\frac{9}{10}(I_0+I_4)-\frac{55}{10}(I_1+I_3)+\frac{45}{5}I_2\right\}-\frac{7}{5}A+\frac{2}{5}B\right]$$

$(3d_{5/2})^1(2s_{\frac{1}{2}})^2$

$$(w+m-h-b)\frac{1}{48}\left\{\frac{123}{4}I_0-79I_1+\frac{1155}{6}I_2-175I_3+\frac{315}{4}I_4\right\}$$
$$+(w+h)(2A-\frac{1}{5}B)+(m-b)(-A+\frac{2}{5}B)$$

Non-diagonal elements

$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1 \qquad -(w+m-h-b)\frac{7}{15}\sqrt{\frac{3}{14}}\left\{\frac{3}{8}I_0-\frac{1}{4}I_1+\frac{1}{2}I_2-\frac{7}{4}I_3+\frac{9}{8}I_4\right\}$

$(3d_{5/2})^3-(3d_{5/2})^1(2s_{\frac{1}{2}})^2 \qquad (w+m-h-b)\frac{\sqrt{2}}{8}\left\{\frac{5}{4}I_0-\frac{11}{3}I_1+\frac{53}{6}I_2-\frac{35}{3}I_3+\frac{21}{4}I_4\right\}$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1-(3d_{5/2})^1(2s_{\frac{1}{2}})^2 \qquad (w+m-h-b)\frac{7}{5\sqrt{21}}\left\{\frac{3}{8}I_0-\frac{1}{4}I_1+\frac{1}{2}I_2-\frac{7}{4}I_3+\frac{9}{8}I_4\right\}$

(*d*) $J=3/2$

Diagonal elements

$(3d_{5/2})^3$

$$(w-h)\frac{1}{5}\left\{\frac{36}{16}(I_0+I_4)+\frac{30}{4}(I_1+I_3)-\frac{36}{8}I_2\right\}$$
$$+(m-b)\frac{1}{5}\left\{\frac{36}{16}(I_0+I_4)-\frac{54}{4}(I_1+I_3)+\frac{156}{8}I_2\right\}$$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$

$$(w-h)\left[\frac{1}{5}\left\{\frac{9}{10}(I_0+I_4)+\frac{27}{10}(I_1+I_3)-\frac{22}{10}I_2\right\}+2A-\frac{2}{25}B\right]$$
$$+(m-b)\left[\frac{1}{5}\left\{\frac{9}{10}(I_0+I_4)-\frac{55}{10}(I_1+I_3)+\frac{45}{5}I_2\right\}-\frac{2}{5}A+\frac{2}{5}B\right]$$

Non-diagonal element

$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})$,	$(w+m-h-b)\frac{44}{105\sqrt{2}}\left\{\frac{3}{8}I_0-\frac{1}{4}I_1+\frac{1}{2}I_2-\frac{7}{4}I_3+\frac{9}{8}I_4\right\}$

(e) $J=1/2$

$(3d_{5/2\frac{1}{2}})^2(2s_{\frac{1}{2}})^1$	$(w-h)\left[\frac{1}{5}\left\{\frac{63}{16}(I_0+I_4)-\frac{11}{4}(I_1+I_3)-\frac{47}{8}I_2\right\}+2A-\frac{1}{5}B\right]$ $+(m-b)\left[\frac{1}{5}\left\{\frac{63}{16}(I_0+I_4)-\frac{29}{4}(I+I_3)+\frac{141}{8}I_2\right\}-A+\frac{2}{5}B\right]$

where

$$A=F^0(d,2s)=\frac{5}{32}I_0+\frac{5}{12}I_1+\frac{17}{48}I_2-\frac{7}{12}I_3+\frac{21}{32}I_4$$

$$B=G^2(d,2s)=\frac{25}{32}I_0-\frac{55}{24}I_1+\frac{255}{48}I_2-\frac{175}{24}I_3+\frac{105}{32}I_4.$$

2.2. The Matrices of the Tensor Interaction

The usual form

$$V_T(r)=J(r)\left\{\frac{3(\vec{\sigma}_1.\vec{r})(\vec{\sigma}_2.\vec{r})}{r^2}-(\vec{\sigma}_1.\vec{\sigma}_2)\right\}$$

of the tensor interaction operator can be written as :

$$V_T(r)=12J(r)\left\{\sqrt{\frac{8}{45}}\left[s_z^1s_z^2-\frac{1}{4}(s_+^1s_-^2+s_-^1s_+^2)\right]Y_2^0(\theta,\varphi)\right.$$
$$-\frac{1}{\sqrt{15}}(s_z^1s_-^2+s_-^1s_z^2)Y_2^1(\theta,\varphi)+\frac{1}{\sqrt{15}}(s_z^1s_+^2+s_+^1s_z^2)Y_2^{-1}(\theta,\varphi)$$
$$\left.+\frac{1}{\sqrt{15}}s_-^1s_-^2Y_2^2(\theta,\varphi)+\frac{1}{\sqrt{15}}s_+^1s_+^2Y_2^{-2}(\theta,\varphi)\right\} \tag{5}$$

where $s_+=s_x+is_y$, $s_-=s_x-is_y$. We substitute (5) for $V(12)$ in (4) and carry out the summation over the spin coordinates. Then there remains a sum of integrals on the space coordinates with products of f_i, g_i as coefficients. The angular integrations can be easily done by use of the Gaunt formula[9] and the radial integrals can be immediately written down in terms of I_l. The exchange integral $K(m_{j_1}, m_{j_2}; m_{j_3}, m_{j_4})$ can be obtained from the direct integral $J(m_{j_1}, m_{j_2}; m_{j_3}, m_{j_4})$ by changing sign of I_l's which arise from functions antisymmetric in the space coordinates of the two nucleons, because only that part of the wave function which is in the triplet state of the two nucleons contributes to the matrix elements of the tensor interaction. Therefore, the matrix element

9) Gaunt, Trans, Roy. Soc. London A, 228, 151 (1929)

$J-K$ in the (m_j, m_j')-scheme contains only the integrals I_l with odd l. Owing to the same reason, the results for a interaction which is multiplied by the Majorana operator are derived from those for an ordinary interaction by changing sign. We list below the matrix elements which occur in the configurations. considered here and interconfigurational mixing.

Table II. The non-vanishing elements of the tensor interaction

Row and column	The elments
	(a) $J=9/2$
	Diagonal elements
$(3d_{5/2})^3$	$\frac{3}{5}I_1-\frac{18}{35}I_2+\frac{3}{25}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$-\frac{1}{5}I_1+\frac{12}{35}I_2-\frac{3}{5}I_3$
	Non-diagonal element
$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{6\sqrt{30}}{25}(-\frac{1}{40}I_1+\frac{5}{28}I_2-\frac{1}{8}I_3)$
	(b) $J=7/2$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{7}{10}I_1-\frac{3}{7}I_2+\frac{3}{10}I_3$
	(c) $J=5/2$
	Diagonal elements
$(3d_{5/2})^3$	$\frac{7}{5}I_1-2I_2+\frac{7}{5}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$-\frac{22}{35}I_2+\frac{18}{25}I_3$
	Non-diagonal elements
$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$-\frac{14}{5}\sqrt{\frac{6}{7}}\left\{-\frac{1}{40}I_1+\frac{5}{28}I_2-\frac{1}{8}I_3\right\}$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1-(3d_{5/2})^1(2s_{\frac{1}{2}})^2$	$-\frac{\sqrt{21}}{25}\left\{\frac{1}{2}I_1-\frac{25}{7}I_2+\frac{5}{2}I_3\right\}$
	(b) $J=3/2$
	Diagonal elements
$(3d_{5/2})^3$	$\frac{3}{5}I_1-\frac{12}{7}I_2+\frac{9}{5}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{1}{2}I_1-\frac{37}{35}I_2+\frac{61}{50}I_3$

	Non-diagonal element
$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{88\sqrt{2}}{35}\left\{-\frac{1}{40}I_1+\frac{5}{28}I_2-\frac{1}{8}I_3\right\}$
	(e) $J=1/2$
$(3d_{5/2})_2(2s_{\frac{1}{2}})^1$	$\frac{7}{5}I_1-2I_2+\frac{7}{5}I_3$

2.3 The Matrices of the Mutual Spin-orbit Interaction

The operator

$$V_{SO}=J(r)\,(\vec{s}^{(1)}+\vec{s}^{(2)})\vec{L}_{12},$$

where $$\hbar\vec{L}_{12}=\hbar\vec{\Lambda}=(\vec{r}_2-\vec{r}_1)\times(\vec{p}_2-\vec{p}_1),$$

can be written in the form :

$$V_{SO}=J(r)\Big[\tfrac{1}{2}(s_+^{(1)}+s_+^{(2)})\,\Lambda_-+\tfrac{1}{2}(s_-^{(1)}+s_-^{(2)})\,\Lambda_++(s_z^{(1)}+s_z^{(2)})\,\Lambda_z\Big], \tag{6}$$

where $\Lambda_+=\Lambda_x+i\Lambda_y$, $\Lambda_-=\Lambda_x-i\Lambda_y$. The matrix elements (4) for (6) can be easily calculated with the help of the equations

$$\Lambda_\pm\Psi_\Lambda^m=\Big[(\Lambda\mp m)(\Lambda\pm m+1)\Big]^{\frac{1}{2}}\Psi_\Lambda^{m\pm1},\quad \Lambda_z\Psi_\Lambda^m=m\,\Psi_\Lambda^m .$$

The K integrals in the (m_j, m_j')-scheme differ from the J only by the sign of the I_l arising from functions antisymmetric in the space coordinates of the two nucleons. The reason for it is the same as in the case of the tensor interaction. The results are given in Table III.

Table III. The non-vanishing elements of the mutual spin-orbit interaction

Row and column	The elements
	(a) $J=9/2$
	Diagonal elements
$(3d_{5/2})^3$	$\frac{12}{25}I_1-\frac{9}{25}I_2+\frac{69}{25}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{25}{40}I_1-\frac{27}{25}I_2+\frac{149}{40}I_3$
	Non-diagonal element
$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{1}{5}\sqrt{\frac{6}{5}}(\frac{1}{8}I_1-\frac{3}{4}I_2+\frac{5}{8}I_3)$

(b) $J=7/2$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{1}{10}I_1+\frac{11}{10}I_3$

(c) $J=5/2$

Diagonal elements

$(3d_{5/2})^3$	$\frac{1}{5}I_2+\frac{7}{5}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{473}{600}I_1-\frac{161}{100}I_2+\frac{1381}{600}I_3$
$(3d_{5/2})^1(2s_{\frac{1}{2}})^2$	$\frac{7}{24}I_1-\frac{3}{4}I_2+\frac{35}{24}I_3$

Non-diagonal elements

$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$-\frac{2}{5}\sqrt{\frac{7}{6}}(\frac{1}{8}I_1-\frac{3}{4}I_2+\frac{5}{8}I_3)$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1-(3d_{5/2})^1(2s_{\frac{1}{2}})^2$	$\frac{\sqrt{21}}{2}(\frac{1}{30}I_1-\frac{1}{5}I_2+\frac{1}{6}I_3)$

(d) $J=3/2$

Diagonal elemnts

$(3d_{5/2})^3$	$\frac{9}{10}I_1-\frac{6}{5}I_2+\frac{3}{2}I_3$
$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{298}{600}I_1-\frac{86}{100}I_2+\frac{506}{600}I_3$

Non-diagonal element

$(3d_{5/2})^3-(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$\frac{44\sqrt{2}}{105}(\frac{1}{8}I_1-\frac{3}{4}I_2+\frac{5}{8}I_3)$

(e) $J=1/2$

$(3d_{5/2})^2(2s_{\frac{1}{2}})^1$	$-\frac{49}{120}I_1+\frac{1}{4}I_2+\frac{91}{120}I_3$

3. Numerical calculations with some two-body fitting interactions

In this chapter, based on the odd-group model results obtained in chap. 2 are applied to Na^{21} and Na^{23}, and then we assume two-body nuclear interactions with the Yukawa potential, hitherto proposed by various authors to explain two-body and sometimes three-body data :

$$V(12)=\frac{V_c}{3}(\vec{\tau}_1 . \vec{\tau}_2)\{1-g/2+(g/2)(\vec{\sigma}_1 . \vec{\sigma}_2)\}\left(\frac{e^{-r/r_o}}{r/r_o}\right), \tag{7}$$

$V_c=67.8$Mev, $r_0=1.18\times10^{-13}$cm, $g=0.157$;

$$V(12)=-V_c\left\{\frac{(1+P_M)}{2}\left(\frac{e^{-r/r_o}}{r/r_o}\right)\pm\gamma\, S_{12}\left(\frac{e^{-r/r_t}}{(r/r_t)^2}\right)\right\}, \tag{8}$$

$V_c=49.35$Mev, $\gamma\, V_c=18$Mev, $r_o=1.14\times10^{-13}$cm, $r_t=1.6\times10^{-13}$cm;

$S_{12}=3(\vec{\sigma}_1 . \vec{r})(\vec{\sigma}_2 . \vec{r})/r^2-(\vec{\sigma}_1 . \vec{\sigma}_2)$

$$V(12)=-V_c\left[\frac{1+P_M}{4}\left\{(1+\eta)+(1-\eta)P_B\right\}\left(\frac{e^{-r/r_o}}{r/r_o}\right)+\gamma(0.37+0.63P_M)\, S_{12}\left(\frac{e^{-r/r_t}}{r/r_t}\right)\right], \tag{9}$$

$V_c=25.5$Mev, $\eta=1.4$, $\gamma=1.9$, $r_o=r_t=1.35\times10^{-13}$cm;

$$V_{CP}(12)=V_{CP}\frac{1}{x}\frac{d}{dx}\left(\frac{e^{-x}}{x}\right)\vec{L}\cdot(\vec{s}^{(1)}+\vec{s}^{(2)}), \tag{10}$$

where $x=r/r_s$, $r_s=1.18\times10^{-13}$cm, $\hbar\vec{L}=(\vec{r}_2-\vec{r}_1)\times(\vec{p}_2-\vec{p}_1)$, $V_{CP}=24$Mev.

The explicit expressions for the Talmi integrals I_l for the Yukawa potential have been given by Talmi, while the corresponding expressions for the singular Yukawe potential in (8) and the Case & pais potential in (10) are given in Appendix I. There $\mu=\frac{1}{2\nu^{\frac{1}{2}}r_o}$, and we can fix a value of ν(*i.e.* of μ) by using the formula for the nuclear radius ;

$$R^2=\langle r^2\rangle=N_l^2\int_o^\infty e^{-2\nu r^2}r^{2l+4}dr=\frac{2l+3}{4\nu},$$

$$\text{or}\qquad =N_{1l}^2\int_o^\infty e^{-2\nu r^2}\left(1-\frac{4\nu}{2l+3}r^2\right)r^{2l+4}dr$$

$$=\frac{1}{8\nu}\left\{(2l+3)^2-(2l+5)(2l-1)\right\},$$

and $R=1.4\times A^{\frac{1}{3}}\times10^{-13}$ cm,

For the $3d$- and $2s$-shells, we obtain the same result:

$$\mu=\frac{R}{r_o\sqrt{7}},$$

where $R=4.0\times10^{-13}$ cm for $A=23$.

At first, we calculate the energy levels with the nuclear interaction (7) containing only central forces, discussed by Chew- and Goldberger[10]. In the $(3d_{5/2})^3$ configuration the ground state has $J=9/2$ and the first excited state has $J=3/2$ and the state with $J=5/2$ is next to it. In the $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ configuration the state with $J=9/2$ is lowest

10) G. F. Chew and M. L. Goldberger, Phys. Rev. 73, 1409 (1948)

and above it in order lie the states with $J=7/2$, $5/2$, $3/2$ and $1/2$. As mentioned in Chap. 1, we consider interconfigurational mixing of the $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations. In this case, since it has been shown by the (d,p) stripping reaction that the first excited state with $J=1/2$ $(2s_{\frac{1}{2}})$of F^{17} is higher by 0.536 Mev than the ground state with $J=5/2$ $(3d_{5/2})$, we assume that in the zeroth order the $(2s_{\frac{1}{2}})$-level of a single nucleon is higher by this value than the $3d_{5/2}$-level. By calculation with the off-diagonal elements given in Table I, it turns out that the level order is 9/2, 3/2, 5/2, 7/2 and 1/2.

Next, we calculate the energy levels with the nuclear interaction (8) proposed by Christian & Noyes[11] in analyzing high energy proton-proton scattering. The central force acts only on the singlet states of two nucleons owing to its Serber exchange character, while the tensor force acts only on the triplet states as pointed out in Chap. 2. If we take into account only the singlet interaction, the level order is 5/2, 3/2, 9/2 in the $(3d_{5/2})^3$ configuration and 1/2, 3/2, 7/2, 5/2, 9/2 in the $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$. Then we calculate the contribution of the tensor interaction with the singular Yukawa radial dependence by using Table II and Appendix I (b). The results are :
for the $(3d_{5/2})^3$ configuration,

$E^T_{9/2}=\mp 0.2148$, $E^T_{5/2}=\mp 0.4531$, $E^T_{3/2}=\mp 0.1427$ (Mev) ; for the $(3d_{5/2})^2(2s_{\frac{1}{2}})_1$ configuration

$E^T_{9/2}=\pm 0.0785$, $E^T_{7/2}=\mp 0.2823$, $E^T_{5/2}=\pm 0.0447$, $E^T_{3/2}=\mp 0.1541$, $E^T_{\frac{1}{2}}=\mp 0.4531$(Mev).
where the upper(or lower) sign corresponds to the upper (or lower) sign of the tensor term of (8). These contributions have no magnitude enough to change the order of levels. With the lower sign of the tensor term the level spacing between the first excited state with $J=3/2$ and the ground state diminishes for both $(3d_{5/2})^3$ and $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ configurations. Thus with the lower sign we consider inter-configurational mixing in the same way as in the case of (7). The off-diagonal elements of the tensor force are so small that they have almost no influence on interconfigurational mixing.The state with $J=1/2$ is lowest and the first excited state have $J=3/2$, above it lie the states with $J=5/2$, 9/2 and 7/2.

Finally, we investigate the energy levels with the nuclear interaction (9) discussed by Christian and Hart[12] in analyzing high energy proton-neutron scattering, and the

11) R. S Ghristian and H. P. Noyes, Phys. Rev. 79, 85 (1951)
12) R. S. Chtistan aud E, W, Hart, Phys. Rev. 77,441 (1950)

contribution of the mutual spin-orbit introaction (10) introduced by Case & Pais in order to preserve charge symmetry of nuclear forces in analyzing high energy nucleon-nucleon scattering. The level order with the singlet interaction energy (i. e. the central) is the same as in the case of (8). A change of the central range gives rise to little change in their splittings. The contributions of the tensor force have no magnitude enough to change the order of levels. The results are :

for $(3d_{5/2})^3$

$E^T_{9/2}=0.1488,\ E^T_{5/2}=0.3123,\ E^T_{3/2}=0.0963$ (Mev) ;

for $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$

$E^T_{9/2}=-0.0645,\ E^T_{7/2}=0.2091,\ E^T_{5/2}=-0.0387,\ E^T_{3/2}=0.1133,\ E^T_{\frac{1}{2}}=0.3123$ (Mev).

We calculate the contributions of the mutual spin-orbit interaction by using Table III and Appendix I (a).

For $(3d_{5/2})^3$,

$E^{CP}_{9/2}=-0.2868,\ E^{CP}_{5/2}=-0.0646,\ E^{CP}_{3/2}=-0.3681$ (Mev),

and for $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$

$E^{CP}_{9/2}=-0.2905,\ E^{CP}_{7/2}=-0.0840,\ E^{CP}_{5/2}=-0.2988,\ E^{CP}_{3/2}=-0.1841,\ E^{CP}_{\frac{1}{2}}=0.150$ (Mev).

By adding the Case & Pais spin-orbit force (10) to the interaction (9), also, the levels in each configuration does not change in order. In interconfigurational mixing the off-diagonal elements of non-central forces are very smaller than those of the central force, and then the level order is the same as in the case of the interaction (8).

By the way, we consider the nuclear interaction

$$V(12)=\tfrac{1}{2}(1+P_M)\ V_c\left[\left(\frac{e^{-r/r_c}}{r/r_c}\right)+\gamma S_{12}\left(\frac{e^{-r/r_t}}{r/r_t}\right)\right],$$

where $V_c=-46.1$ Mev, $\gamma=0.54$, $r_c=1.18\times10^{-13}$cm, $r_t=1.69\times10^{-13}$cm.

This interaction has been initially proposed by Pease and Feshbach[13] on the H^3 problem, and improved by Feynman[14] to explain high energy neutron-proton scattering. With this interaction, owing to Pauli principle, the contributions to the energy levels arise from only the central part. The result is the same as in above two cases.

Thus, in both jj coupling and its interconfigurational mixing, based on the odd-group

13) P. L. Pease and H Feshbach, Phys. Rev. 81, 142 (1951), 88, 945(1952)

14) R. P. Feynman, Lectures on high energy phenomena and meson theories at C.I.T (1952)

model. we cannot explain the occurrence of the ground state with $J = 3/2$ by using some nuclear interactions with Yukawa potential, hitherto proposed by various authors to explain two-body and sometimes three-body data.

4. Fitting the Na^{23} data

Because of the reasons, mentioned in the end of Chap.1, we shall attempt here, with interconfigurational mixing of the $(3d_{5/2})^3$, $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations, to relate the known groud state data of Na^{23} to the interaction constants of a mixed interaction.

We shall assume a two-body charge-symmetric interaction of the form

$$V(12) = \frac{V_c(\vec{\tau}_1 \cdot \vec{\tau}_2)}{3}\Big[\Big\{1-g/2+(g/2)(\vec{\sigma}_1 \cdot \vec{\sigma}_2)\Big\}\Big(\frac{e^{-r/a}}{r/a}\Big)$$

$$+x\Big\{\frac{3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r})}{r^2} - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)\Big\}\Big(\frac{e^{-r/a}}{r/a}\Big)+y\Big\{(\vec{s}^{(1)}+\vec{s}^{(2)})\cdot\vec{L}\Big\}\frac{b^2}{r}\frac{d}{dr}\Big(\frac{e^{-r/b}}{r/b}\Big)\Big] \quad (11)$$

where $\hbar L = (\vec{r}_2-\vec{r}_1)\times(\vec{p}_2-\vec{p}_1)$, $a = 1.35\times 10^{-13}$ cm, $b = 1.18\times 10^{-13}$cm.
From fitting the deuteron data[15] we may suppose that V_c have a value between about 20 and 30 Mev. Apart from this overall constant V_c, (11) contains three parameters g, x, y which describe the spin dependence of the central force, and the relative strengths of the tensor and mutual spin-orbit forces, respectively. The purpose of the calculations of this chapter is to find values for g, x, y which are consistent with the ground state data of Na^{23}.

Since we perform the calculations with interconfigurational mixing, we need an appropriate assumption about how far the $2s_{\frac{1}{2}}$ level of a single nucleon is from its $3d_{5/2}$ level in the zeroth approximation. From the ground state data of F^{19} we may suppose that the $3d_{5/2}$ and $2s_{\frac{1}{2}}$ levels have very closely the same energy. On the other hand, it has been shown by the (d,p) stripping reaction[16] that the first excited state with $J = 1/2$ of F^{17} is higher by 0.536 Mev than its ground state with $J = 5/2$. Hence as a value $\Delta E(3d_{5/2} - 2s_{\frac{1}{2}})$ by which the $2s_{\frac{1}{2}}$ level is higher than the $3d_{5/2}$ level, we take two values : 0.2 and 0.5 Mev. If we further assume that V_c have an approximate value between 20 and 30 Mev, the former value of $\Delta E(3d_{5/2} - 2s_{\frac{1}{2}})$ (which we shall denote

15) H. Feshbach and J. Schwinger, Phys. Rev. 82, 194 (1951); W. J. Robinson, Phys. Rev. 93, 1296 (1954).

16) F. Ajzenberg and T. Lauritzen, Rev. Mod. Phys. 24, 321 (1952)

by Case I) corresponds to about 0.008 V_c, the latter (Case II) to 0.02 V_c. In order to calculate the relative level positions, we evaluate the energy matrices of the two-body interaction (11) and add ΔE and $2\Delta E$ to the diagonal elements which correspond to the $(3d_{5/2})^2(2s_{\frac{1}{2}})^1$ and $(3d_{5/2})^1(2s_{\frac{1}{2}})^2$ configurations, respectively, so that V_c will not enter.

The ground state of Na^{23} is known to have $J=3/2$, a magnetic moment of $\mu=2.217$ n.m., and a quadrupole moment of $Q=0.1\times10^{-24}\text{cm}^2$. The interaction (11) allows mixing of the two states $(3d_{5/2})^3{}_{3/2}$, $\{(3d_{5/2})^2(2s_{\frac{1}{2}})^1\}_{3/2}$ with $J=3/2$. Thus the ground state wave function Ψ_G can be written as

$$\Psi_G=\alpha\Psi\left[(3d_{5/2})^3{}_{3/2}\right]+\beta\Psi\left[\left\{(3d_{5/2})^2(2s_{\frac{1}{2}})^1\right\}_{3/2}\right],\ \text{with}\ \alpha>0.\ \alpha^2+\beta^2=1 \tag{12}$$

The magnetic moment is given by the expectation value of the operator

$$\mu=\sum_i(m_l^i\,g_l^i+m_s^i\,g_s^i)\ n.m., \tag{13}$$

where m_l^i and m_s^i are the z-components of the orbital and spin angular momentum operators of the nucleons, respectively, and g_l^i and g_s^i are the gyromagnetic ratios of orbit and spin, respectively.

$$g_l^P=1,\ g_l^N=0,\ g_s^P=5.587,\ g_s^N=-3.827.$$

Applying the operator (13) to the wave function (12), we obtain for the magnetic moment of Na^{23}

$$\mu=2.87\alpha^2+1.775\beta^2=1.775+1.095\alpha^2.$$

This expression gives, on inserting the known value of μ,

$$\alpha=0.63534,\quad \beta=\pm\,0.77224. \tag{14}$$

In order to remove the arbitrariness of sign of β, we can use sign of the quadrupole momen Q. The quadrupole moment is given by the expectation value of the operator.

$$Q=\sum_i g_l^i(3z_i^2-r_i^2)\,. \tag{15}$$

Using the wave function (12), the quadrupole moment of Na^{23} is found to be

$$Q=\frac{\langle r^2\rangle}{175}\left\{40\beta^2-44\sqrt{5}\,\alpha\beta\right\}, \tag{16}$$

where $\langle r^2\rangle=R^2=1.6\times10^{-26}\text{cm}^2$, and by using the relation $\alpha^2+\beta^2=1$, we plot Q against α in Fig. 1. At $\alpha\simeq0.685$, Q is negative or positive according to whether β is positive or negative. However, since the measured value is $+0.1\times10^{-24}\text{cm}^2$, β must be negative. Thus,

$$\alpha = 0.63534, \quad \beta = -0.77224. \tag{14'}$$

By introducing (14') into (16), we obtain, for the quadrupole moment of Na^{23}, a value of $Q = +0.066 \times 10^{-24}\mathrm{cm}^2$, which is in good agreement with the measured value.

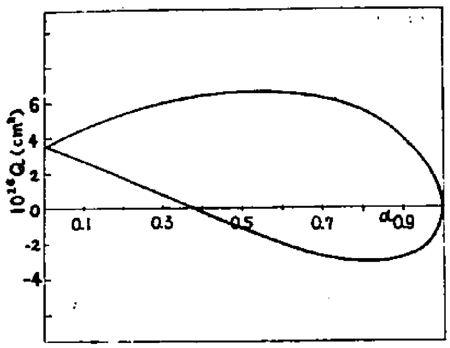

Fig 1 QUADRUPOLE MOMENT

From the tables I, II, III and Appendix II we can write down the matrix of (11) for $J = 3/2$ in interconfigurational mixing as a function of g, x, y. The condition that this matrix should have an eigen-vector $\{\alpha, \beta\}$ then results in two equations in the four unknowns g, x, y and λ, the corresponding eigen-value. Hence for a fixed g, we can find the parameter y as a function of x, and plot against x the levels not only for $J = 3/2$ matrix but for each J value. At first, for Case I this is done in each of $g = 0.5$, 0.67, 0.83 and 1 in Fig. 2. The only regions of x in which the analysis has sense are those in which the eigen-value λ, to which the ground state properties of Na^{23} have been fitted, lies lowest.

Fig. 2. shows that in every case considered here λ lies lowest in a region of $x > x_g$, decided by a value of g, and that x_g is positive and increases with increasing g, while

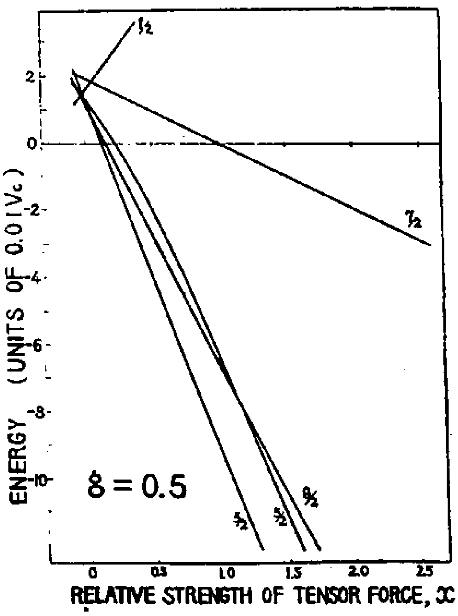

Fig.2.a LEVEL PLOTTING AGAINST x

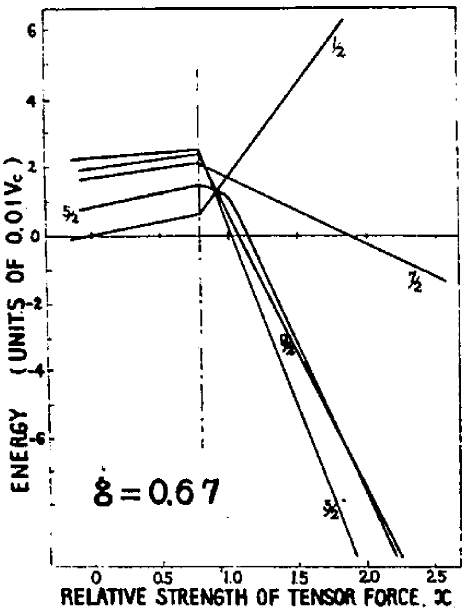

Fig.2.b LEVEL PLOTTING AGAINST x

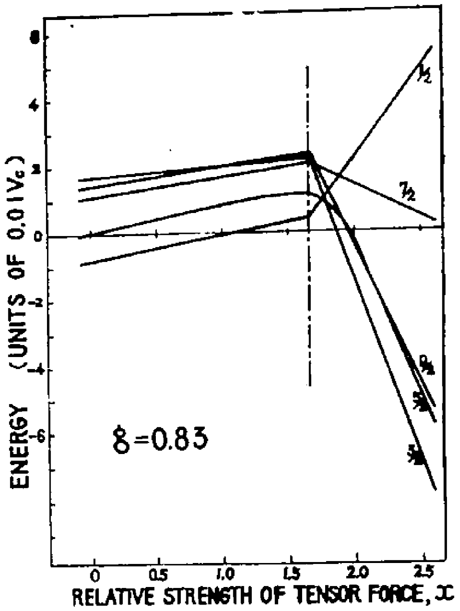

Fig.2.c LEVEL PLOTTING AGAINST x

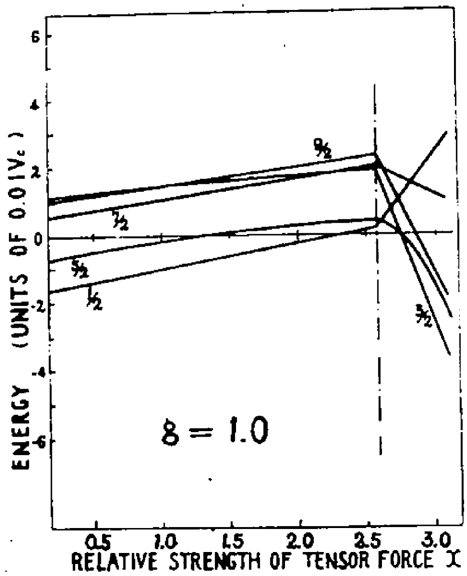

Fig.2.a LEVEL PLOTTING AGAINST x

in this region y is positive on the whole, though sometimes negative. On the other hand, if the spin-orbit term of the kind suggested by Case & Pais is presented in the nuclear interaction, a sign of y must be negative. The reason for it is as follows. If the spin-orbit term were attractive in the $^3D_1$ state of the deuteron, its strong singularity would greatly counteract the centrifugal repulsion. This would allow a large $^3D_1$ admixture in the deuteron ground state, in contradiction with the information obtained from the magnetic moment and quadrupole moment measurements. However, if the spin-orbit term were repulsive in this state, it would add to the already large centrifugal repulsion and hence have little effect on the deuteron ground state. Therefore, it seems to be most reasonable to assume the spin-orbit term repulsive in the $^3D_1$ state. In order that the spin-orbit term is repulsive in the $^3D_1$ state of the deuteron, y must be negative, because $(\vec{\tau}_1 \cdot \vec{\tau}_2) = -3$ and $\vec{S} \cdot \vec{L} < 0$ in the $^3D_1$ state of the N-P system and $(1/x)d/dx(e^{-r}/x)$ is attractive.

As a result of calculation, we can easily find x_g, a critical value of x, for each value of g, and a value of y at $x = x_g$.

	g	x_g	y_g
Case I. a	0.5	0.042	1.172
Case I. b	0.67	0.914	−0.045
Case I. c	0.83	1.783	−0.025
Case I. d	1.0	2.748	1.069 .

y increases linearly with increasing x. Hence, for cases (I. a) and (I. d) y is always positive and considerably large in the region of x in which λ lies lowest, while for cases (I. b) and (I. c) there exists a range in which y is negative or zero in this region. Therefore, for y to be negative or zero, we must take cases (I. b) and (I. c). In these cases, the regions of x in which λ lies lowest and moreover y is negative are :

Case I. b	$0.916 > x > 0.914,$	$0 > y > -0.045$;
Case I. c	$1.784 > x > 1.783,$	$0 > y > -0.025$.

Finally, in order to investigate how a change of $\triangle E$ $(3d_{5/2} - 2s_{\frac{1}{2}})$ has an effect on the behaviour of the levels and the three force constants g, x, y, assuming $\triangle E = 0.02\,V_c$ (Case II) instead of $0.008\,V_c$ (Case I), we perform the calculation in the same way as in case I. If we take an approximate value of $V_c = 25$ Mev, this corresponds to that the $2s_{\frac{1}{2}}$ –level of a single nucleon is higher by 0.5 Mev than the $3d_{5/2}$ level. The level plotting against x is done in each of $g = 0.67$ and 0.83 in Fig. 3. The result is almost the same as in case I and insensitive to the value of $\triangle E$ which is to be practically small. The ranges of x in which λ lies lowest and moreover y is negative are :

Case II. a	$g = 0.67,$	$0.916 > x > 0.90,$	$0 > y > -0.347$;
Case II. b	$g = 0.83,$	$1.781 > x > 1.733,$	$0 > y > -1.011$.

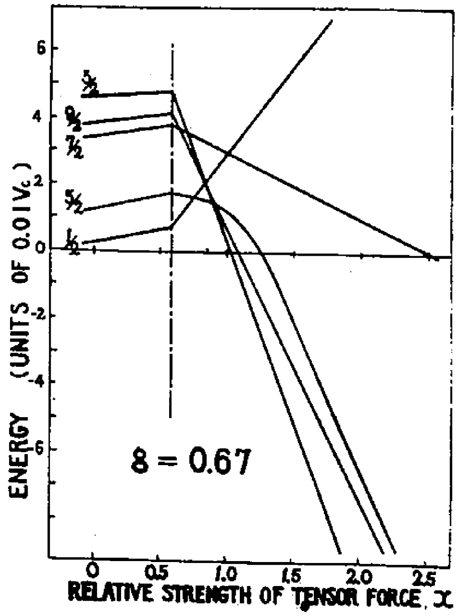

Fig. 3.a LEVEL PLOTTING AGAINST x

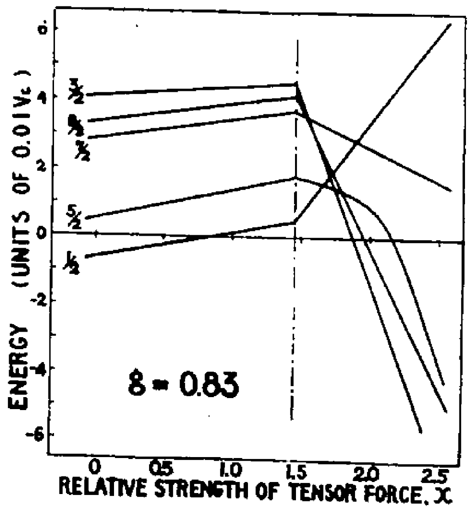

Fig. 3.b LEVEL PLOTTING AGAINST x

In order to compare these interactions, fitting the Na^{21} data, with those interactions containing only central and tensor forces which have been in detail discussed on the deuteron problem, we have particularly investigated the interaction with $y=0$. According to cases (I. c) and (II. b), we obtain

$$g = 0.83, \qquad x = 1.78, \qquad y = 0.$$

Then, if we further take a value of $V_c = 25.5$ Mev so as to obtain correct binding energy of the deuteron $E = 2.23$ Mev, we obtain for its quadrupole moment Q and percentage of D state P_D,

$$Q = 2.60 \times 10^{-27}\text{cm}^2, \qquad P_D = 4.2\,.$$

Hence we see that so far as concerns the triplet interaction, our interaction is fairly consistent with the deuteron ground state data. On the other hand, from low energy neutron-proton scattering it is known to be $\frac{V}{3}(1-2g) \sim 35.5$ Mev for $r_0 = 1.35 \times 10^{-13}$cm, while ~ 5.6 Mev in our case. This depth of the singlet potential is too small to explain the low energy neutron-proton scattering. On the contrary, if we take $g = 2.6$ to explain the low energy n-p scattering, it turns out easily from Figs. 2 and 3 that x_g is too large to be adjusted to the deuteron data. The negative y is so small in magnitude that we cannot attribute the doublet splitting required by the jj coupling shell model to it. In this paper do not take into account configurations of nucleons of even group in unfilled shells, but it is hoped to include these in future work.

Appendix I. Talmi integrals

(a). $I(\mu)$ for the Case-Pais Potential

$$I_1 = \frac{V_0}{3}\left[\frac{2}{\sqrt{\pi}}\,\frac{2\mu^2-1}{4\mu^3} - (1-\phi(\mu))e^{\mu^2}\right]$$

$$I_2 = \frac{V_0}{15}\left[\frac{2}{\sqrt{\pi}}\,\frac{4\mu^4+8\mu^2-2}{4\mu^3} - (2\mu^2+5)(1-\phi(\mu))e^{\mu^2}\right]$$

$$I_3 = \frac{V_0}{105}\left[\frac{4}{\sqrt{\pi}}\,\frac{4\mu^6+26\mu^4+24\mu^2-4}{4\mu^3} - (35+28\mu^2+4\mu^4)(1-\phi(\mu))e^{\mu^2}\right]$$

$$I_4 = \frac{V_0}{945}\left[\frac{8}{\sqrt{\pi}}\,\frac{4\mu^8+52\mu^6+165\mu^4+96\mu^2-12}{4\mu^3} - (315+378\mu^2+84\mu^4+8\mu^6)(1-\phi(\mu))e^{\mu^2}\right]$$

(b). $I(\mu)$ for the singular Yukawa potential

$$I_0=2V_0\lambda^2(1-\phi(\mu))e^{\mu^2}$$

$$I_1=-\frac{4}{3}V_0\lambda^2\left[\frac{1}{\sqrt{\pi}}\mu-(\tfrac{1}{2}+\mu^2)(1-\phi(\mu))e^{\mu^2}\right]$$

$$I_2=-\frac{8}{15}V_0\lambda^2\left[\frac{1}{\sqrt{\pi}}(\mu^3+\frac{5}{2}\mu)-(\frac{3}{4}+3\mu^2+\mu^4)(1-\phi(\mu))e^{\mu^2}\right]$$

$$I_3=-\frac{16}{105}V_0\lambda^2\left[\frac{1}{\sqrt{\pi}}(\mu^5+7\mu^3+\frac{33}{4}\mu)-(\frac{15}{8}+\frac{45}{4}\mu^2+\frac{15}{2}\mu^4+\mu^6)(1-\phi(\mu))e^{\mu^2}\right]$$

$$I_4=-\frac{32}{945}V_0\lambda^2\left[\frac{1}{\sqrt{\pi}}(\mu^7+\frac{27}{2}\mu^5+\frac{185}{4}\mu^3+\frac{279}{8}\mu)-(\frac{105}{16}+\frac{105}{2}\mu^2+\frac{105}{2}\mu^4+14\mu^6+\mu^8)\right.$$
$$\left.(1-\phi(\mu))e^{\mu^2}\right]$$

where

$$\phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)\ dt,\qquad \lambda=\sqrt{\nu}\ r_0$$

Appendix. II.

(a). Wave functions for $(3d_{5/2})^3$ in jj coupling

$\Psi_{9/2,9/2}=(5/2,3/2,1/2)$

$\Psi_{5/2,5/2}=1/\sqrt{2}\{(5/2,1/2,-1/2)-(5/2,3/2,-3/2)\}$

$\Psi_{3/2,3/2}=\sqrt{8/21}(3/2,1/2,-1/2)+\sqrt{5/21}(5/2,1/2,-3/2)+\sqrt{8/21}(5/2,3/2,-5/2)$.

(b). Wave functions for $(3d_{5/2})^2(2s_{1/2})^1$ in jj coupling

$\Psi_{9/2,9/2}=(5/2,3/2,1/2_s)$

$\Psi_{7/2,7/2}=\sqrt{1/9}(5/2,1/2,1/2_s)-\sqrt{8/9}(5/2,3/2,-1/2_s)$

$\Psi_{5/2,5/2}=3/\sqrt{14}(3/2,1/2,1/2_s)-\sqrt{5/14}(5/2,-1/2,1/2_s)$

$\Psi_{3/2,3/2}=\sqrt{2/35}(3/2,-1/2,1/2_s)-3\sqrt{2}/\sqrt{35}(3/2,1/2,-1/2_s)-\sqrt{1/7}(5/2,-3/2,1/2_s)$
$+\sqrt{2/7}(5/2,-1/2,-1/2_s)$

$\Psi_{1/2,1/2}=1/\sqrt{3}\{(1/2,-1/2,1/2_s)-(3/2,-3/2,1/2_s)+(5/2-5/2,1/2_s)\}$.