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POISSON LIE ALGEBRA STRUCTURES ASSOCIATED WITH THREE-DIMENSIONAL LIE ALGEBRAS

By

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1. Introduction

On a m-dimensional differentiable manifold with a second rank skew-symmetric differentiable tensor field ω, F. A. Berezin [1, 2] introduced the Poisson Lie algebras (Poisson-Berezin algebras, briefly PB-algebras) provided with the infinite dimensional Lie algebra structure; and illustrated an interesting example of PB-algebras associated with the structure constants of arbitray m-dimensional Lie algebras $\mathfrak{g}$. Moreover, A. A. Kirillov [3] investigated the local Lie algebra with one-dimensional fibre, which gave birth to the generalized PB-algebras (Poisson-Kirillov algebras, briefly PK-algebras) defined by the pair of the structure constants of $\mathfrak{g}$ and a set of differentiable functions $F^i (i = 1, \cdots, m)$ on the manifold.

In this paper, the centers of PB-algebras are completely determined by setting $\mathfrak{g}$ as the three-dimensional Lie algebras. By using of the elements of the center, it is effectively determined the set of functions F^i which, together with the structure constants of $\mathfrak{g}$, give rise to the bracket of PK-algebras (cf. [5] for the case that F^i are the constants). Moreover, there is given a procedure of constructing the bracket of PK-algebras from an arbitrary set of functions F^i. And the procedure is illustrated by some examples.

For the convenience, differentiability is assumed to be of sufficiently high order and the summation convention is employed throughout.

2. PB- and PK-algebras

Let $\mathfrak{M}$ be a m-dimensional differentiable manifold with local coordinates $x = (x^1, \cdots, x^m)$ and $\mathfrak{R}(\mathfrak{M})$ the ring (with respect to the operations of addition and multiplication) of differentiable functions on $\mathfrak{M}$. Then, in terms of the components $\omega^{ij}(x)$ of the tensor field ω on $\mathfrak{M}$, the generalized Poisson bracket on $\mathfrak{R}(\mathfrak{M})$ is defined by [2]

$$(1) \qquad [f, g] = \omega^{ij} f_{,i}\, g_{,j} \quad \text{for} \quad f, g \in \mathfrak{R}(\mathfrak{M}).$$

Here note the convention of the commas: $f_{,i} = \partial f/\partial x^i$, $g_{,j} = \partial g/\partial x^j$. Under this bracket operation, $\mathfrak{N}(\mathfrak{M})$ makes a Poisson Lie algebra, i.e., (1) defines an infinite dimensional Lie algebra structure (PB-algebra structure) if and only if (here the parenthesis of indices denotes the summation over all cyclic permutations of the indices)

(2) $$\omega^{(i|s|}\omega^{jk)}_{,s} = \omega^{is}\omega^{jk}_{,s}+\omega^{js}\omega^{ki}_{,s}+\omega^{ks}\omega^{ij}_{,s} = 0.$$

Let C^{ij}_k be the structure constants of a m-dimensional Lie algebra $\mathfrak{g}$ and put

(3) $$\omega^{ij} = C^{ij}_k x^k.$$

Then (2) is satisfied identically, since it is equivalent to the Jacobi identity for the structure constants

(4) $$C^{(i|s|}_t C^{jk)}_s = 0;$$

and the bracket (1) has the form

(5) $$[f, g] = C^{ij}_k x^k f_{,i}\, g_{,j} \quad \text{for } f, g \in \mathfrak{N}(\mathfrak{M}).$$

With a set of differentiable functions F^i on $\mathfrak{M}$, the Poisson bracket (1) can be generalized as [3]

(6) $$[f, g] = \omega^{ij} f_{,i}\, g_{,j} + F^i(f g_{,i} - g f_{,i}) \quad \text{for } f, g \in \mathfrak{N}(\mathfrak{M}),$$

which defines an infinite dimensional Lie algebra structure (PK-algebra structure) if and only if

(7) $$F^{(i}\omega^{jk)} + \omega^{(i|s|}\omega^{jk)}_{,s} = 0,$$

(8) $$F^i_{,k}\omega^{jk} - F^j_{,k}\omega^{ik} + F^k\omega^{ij}_{,k} = 0.$$

Similarly as before, let ω^{ij} be given by (3). Then, in viewing that (2) is equivalent to the Jacobi identity (4), the equations (7) and (8) are reduced respectively to

(9) $$F^{(i}C^{jk)}_s x^s = 0,$$

(10) $$(F^i_{,k}C^{jk}_s - F^j_{,k}C^{ik}_s)x^s + F^k C^{ij}_k = 0;$$

and the bracket (6) has the form

(11) $$[f, g] = C^{ij}_k x^k f_{,i}\, g_{,j} + F^i(f g_{,i} - g f_{,i}) \quad \text{for } f, g \in \mathfrak{N}(\mathfrak{M}).$$

3. Centers of the PB-algebras

Let $\mathfrak{G}(\mathfrak{M})$ be the PB-algebra with the bracket (5) on $\mathfrak{N}(\mathfrak{M})$ and $\mathfrak{H}(\mathfrak{M})$ the center of $\mathfrak{G}(\mathfrak{M})$. An element $f \in \mathfrak{H}(\mathfrak{M})$ is a differentiable function on $\mathfrak{M}$ such that $C^{ij}_k x^k f_{,i}\, g_{,j} = 0$ for all $g \in \mathfrak{N}(\mathfrak{M})$, i.e., it is a solution of the first-order system of partial differential equations

$$C_k^{ij} x^k f_{,i} = 0 \quad (j = 1, \cdots, m), \tag{12}$$

which may be not always effectively solved to determine the center $\mathfrak{H}(\mathfrak{M})$.

However our discussion is continued by setting $\mathfrak{g}$ as the three-dimensional Lie algebras. We set also $\mathfrak{M} = \mathbb{R}^3$ for the covenience. The structure constants C_k^{ij} of $\mathfrak{g}$:

$$P = \begin{pmatrix} C_1^{23} & C_1^{31} & C_1^{12} \\ C_2^{23} & C_2^{31} & C_2^{12} \\ C_3^{23} & C_3^{31} & C_3^{12} \end{pmatrix} \tag{13}$$

take the following forms under a suitable isomorphism of $\mathfrak{g}$ (Levy-Nahas [4]):

$$P^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ P^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ P_\lambda^8 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda \neq -1$ in P_λ^8. Let $\mathfrak{G}^\alpha(\mathbb{R}^3)$ be the PB-algebra associated with the structure constants P^α and $\mathfrak{H}^\alpha(\mathbb{R}^3)$ the center of $\mathfrak{G}^\alpha(\mathbb{R}^3)$, respectively for $\alpha = 1, \cdots, 8$; and denote the coordinates of $\mathbb{R}^3$ by $(x^1, x^2, x^3) = (x, y, z)$ as well as $(f_{,1}, f_{,2}, f_{,3}) = (f_x, f_y, f_z)$.

THEOREM 1. *Elements f of the centers $\mathfrak{H}^\alpha(\mathbb{R}^3)$ of the PB-algebras $\mathfrak{G}^\alpha(\mathbb{R}^3)$ associated with P^α $(\alpha = 1, \cdots, 8)$ are of the following forms, respectively* (Φ *is an arbitrary differentiable function of the indicated variables*):

$\mathfrak{H}^1(\mathbb{R}^3)$: $f = \Phi(x, y, z)$;

$\mathfrak{H}^2(\mathbb{R}^3)$: $f = \Phi(x)$;

$\mathfrak{H}^3(\mathbb{R}^3)$: $f = \Phi(x^{-1}y)$ $(x \neq 0)$ *or* $f = \Phi(xy^{-1})$ $(y \neq 0)$;

$\mathfrak{H}^4(\mathbb{R}^3)$: $f = \Phi(x^2 - y^2)$;

$\mathfrak{H}^5(\mathbb{R}^3)$: $f = \Phi(x^2 + y^2)$;

$\mathfrak{H}^6(\mathbb{R}^3)$: $f = \Phi(x^2 - y^2 + z^2)$;

$\mathfrak{H}^7(\mathbb{R}^3)$: $f = \Phi(x^2 + y^2 + z^2)$;

$\mathfrak{H}_\lambda^8(\mathbb{R}^3)$: $f = \Phi\left(\log(x^2 + \lambda y^2) + \frac{2}{\sqrt{\lambda}} \arctan \frac{\sqrt{\lambda}\, y}{x}\right)$ $(x \neq 0)$ *or*

$$f=\Phi\Big(\log(x^2+\lambda y^2)-\frac{2}{\sqrt{\lambda}}\arctan\frac{x}{\sqrt{\lambda}y}\Big)\quad (y\neq 0)\qquad \textit{for } \lambda>0,$$
$$f=\Phi\Big((x+\sqrt{-\lambda}y)^{1+1/\sqrt{-\lambda}}(x-\sqrt{-\lambda}y)^{1-1/\sqrt{-\lambda}}\Big)\qquad \textit{for } \lambda<0,$$
$$f=\Phi\Big(xe^{x^{-1}y}\Big)\quad (x\neq 0)\qquad \textit{for } \lambda=0.$$

PROOF. Corresponding to the structure constants of P^α, the differential system (13) is written respectively as

P^1: $C_k^{ij}x^kf_{,i}\equiv 0$ $(j=1,2,3)$;

P^2: $C_k^{i1}x^kf_{,i}\equiv 0$, $C_k^{i2}x^kf_{,i}=-xf_z=0$, $C_k^{i3}x^kf_{,i}=xf_y=0$;

P^3: $C_k^{i1}x^kf_{,i}=xf_z=0$, $C_k^{i2}x^kf_{,i}=yf_z=0$, $C_k^{i3}x^kf_{,i}=-xf_x-yf_y=0$;

P^4: $C_k^{i1}x^kf_{,i}=-yf_z=0$, $C_k^{i2}x^kf_{,i}=-xf_z=0$, $C_k^{i3}x^kf_{,i}=xf_y+yf_x=0$;

P^5: $C_k^{i1}x^kf_{,i}=yf_z=0$, $C_k^{i2}x^kf_{,i}=-xf_z=0$, $C_k^{i3}x^kf_{,i}=xf_y-yf_x=0$;

P^6: $C_k^{i1}x^kf_{,i}=-yf_z-zf_y=0$, $C_k^{i2}x^kf_{,i}=zf_x-xf_z=0$, $C_k^{i3}x^kf_{,i}=xf_y+yf_x=0$;

P^7: $C_k^{i1}x^kf_{,i}=yf_z-zf_y=0$, $C_k^{i2}x^kf_{,i}=zf_x-xf_z=0$, $C_k^{i3}x^kf_{,i}=xf_y-yf_x=0$;

P_λ^8: $C_k^{i1}x^kf_{,i}=(x+\lambda y)f_z=0$, $C_k^{i2}x^kf_{,i}=-(x-y)f_z=0$,

$C_k^{i3}x^kf_{,i}=-(x+\lambda y)f_x+(x-y)f_y=0$.

Now, the solution can be obtained resectively in each case (α) for the structure constants of P^α as follows.

(i) For the no conditions, f is obviously an arbitrary function on $\mathbb{R}^3$.

(ii) The solution $f=\Phi(x)$ follows immediately from $f_x=f_y=0$ if $x\neq 0$, while $x\neq 0$ can be delated in the consequence by the continuity of f.

(iii) According to $x\neq 0$ or $y\neq 0$, the subsidiary equation (briefly s-equation) of the last one: $x^{-1}dx=y^{-1}dy$ have the solution $x^{-1}y=$ const. or $xy^{-1}=$ const., and therefore $f=\Phi(x^{-1}y)$ or $f=\Phi(xy^{-1})$ by the first or the second one, respectively.

(iv) The first (also the second) equation implies that f is a function of x and y (note the continuity of f as used in (ii)). Therefore $f=\Phi(x^2-y^2)$ follows from the solution $x^2-y^2=$ const. of the s-equation of the last one: $xdx-ydy=0$.

(v) Similarly to (iv), the solution $f=\Phi(x^2+y^2)$ is obtained through the s-equation: $xdx+ydy=0$.

(vi) By the transformation of variables

$$u=x^2-y^2+z^2,\ v=y,\ w=z,$$

with the Jacobian (functional determinant) $D(u,v,w)/D(x,y,z)=2x\neq 0$, the equations are rewritten respectively as

$$y f_w + z f_v = 0, \quad x f_w = 0, \quad x f_v = 0;$$

which have the solution $f = \Phi(u) = \Phi(x^2 - y^2 + z^2)$ (note also the continuity of f).

(vii) Similarly to (vi), the solution $f = \Phi(x^2 + y^2 + z^2)$ is obtained by the transformation of variables

$$u = x^2 + y^2 + z^2, \quad v = y, \quad w = z.$$

(viii) If $x \neq 0$, the s-equation of the last one: $(-x + y)\, dx = (x + \lambda y)\, dy$ is, by putting $x^{-1}y = u$, rewritten as

$$\frac{2dx}{x} + \left(\frac{2\lambda u}{\lambda u^2 + 1} + \frac{2}{\lambda u^2 + 1}\right) du = 0.$$

This equation has the solutions, for $\lambda > 0$, in viewing of $\lambda u^2 + 1 = \lambda(u^2 + (1/\sqrt{\lambda})^2)$:

$$\log x^2 + \log(\lambda u^2 + 1) + \frac{2}{\sqrt{\lambda}} \arctan \sqrt{\lambda}\, u = \text{const., i.e.,}$$

$$\log(x^2 + \lambda y^2) + \frac{2}{\sqrt{\lambda}} \arctan \frac{\sqrt{\lambda}\, y}{x} = \text{const.} \quad (x \neq 0);$$

for $\lambda < 0$, in viewing of $\lambda u^2 + 1 = \lambda(u^2 - (1/\sqrt{-\lambda})^2)$:

$$\log x^2 + \log|\lambda u^2 + 1| - \frac{1}{\sqrt{-\lambda}} \log\left|\frac{u - 1/\sqrt{-\lambda}}{u + 1/\sqrt{-\lambda}}\right| = \text{const., i.e.,}$$

$$(x + \sqrt{-\lambda}\, y)^{1 + 1/\sqrt{-\lambda}} (x - \sqrt{-\lambda}\, y)^{1 - 1/\sqrt{-\lambda}} = \text{const.};$$

and for $\lambda = 0$, since the equation is reduced to $x^{-1}dx + du = 0$:

$$\log|x| + u = \text{const.} \quad \text{i.e.,} \quad x e^{x^{-1}y} = \text{const.} \quad (x \neq 0).$$

If $y \neq 0$, also by putting $u = xy^{-1}$, the s-equation is rewritten as

$$\frac{2dy}{y} + \left(\frac{2u}{u^2 + \lambda} - \frac{2}{u^2 + \lambda}\right) du = 0;$$

which has the similar solutions as above for $\lambda > 0$, while the term $\arctan(\sqrt{\lambda}\, x^{-1}y)$ is replaced with $-\arctan(\sqrt{\lambda}\, xy^{-1})$ in the solution. This can be seen also by using the relation $\arctan t + \arctan t^{-1} = \pi/2\ (t > 0)$ or $-\pi/2\ (t < 0)$ in the above solution of the s-equation.

Since $\lambda \neq -1$, either $x + \lambda y \neq 0$ or $x - y \neq 0$ is satisfied if $(x, y) \neq (0, 0)$, so that $f_z = 0$ follows from the first or the second equation if $(x, y) \neq (0, 0)$. Thus the respective solutions $f = \Phi(x, y)$ are written by the above solutions of the s-equations.

REMARK. In (Nakanishi [5]), the center, i. e., $\mathfrak{H}^6(\mathbb{R}^3)$ in Theorem 1, played an important role for the study of infinitesimal automorphism of the PB-algebra associated with P^6, i.e., with $\mathfrak{sl}(2, \mathbb{R})$.

4. Poisson Lie brackets of PK-algebras

4. 1. We first assume that the differentiable functions F^i in the bracket (11) belong to the center $\mathfrak{H}(\mathfrak{M})$ of the PB-algebra $\mathfrak{G}(\mathfrak{M})$, i.e., (see (12))

$$C_k^{ij}F^s_{;i}x^k = -C_k^{ji}F^s_{;i}x^k = 0 \quad (j,\, s = 1, \cdots, m).$$

Then, in viewing that (10) is reduced to

$$F^i C_i^{jk} = 0, \tag{14}$$

the bracket (11) defines the PK-algebra structure if and only if (9) and (14) are satisfied.

The matrix P of (13) defined by the structure constants C_k^{ij} of the three-dimensional Lie algebra $\mathfrak{g}$ is now rearranged as $P = (\rho_{ij})$, in which the elements are related by $C_k^{ij} = \rho_{ks}\varepsilon^{ijs}$ where ε^{ijs} are the usual alternators. This relation makes (9) and (14) into more simple forms (cf. [5]). In fact (9) is equivalently rewritten as

$$F^{(i}C_s^{jk)}x^s = x^s\rho_{st}(\varepsilon^{jkt}F^i + \varepsilon^{kit}F^j + \varepsilon^{ijt}F^k) = 0, \tag{15}$$

which are satisfied identically if at least two indices in i, j, k are the same. So it can be assumed that the indices i, j, k are all distinguished. Then, since i, j, k, t take the values 1, 2 or 3, the alternators ε^{jkt}, ε^{kit} and ε^{ijt} do not vanish if and only if $t = i$, $t = j$ and $t = k$ respectively. Therefore the equation (15) is equivalent to

$$x^s(\rho_{si}F^i + \rho_{sj}F^j + \rho_{sk}F^k) = 0 \ \ \text{(not summing up for } i, j, k).$$

Since the indices i, j, k are all distinguished, the above equation is equal to

$$x^s\rho_{si}F^i = 0, \quad \text{i. e.,} \quad XP\,{}^tF = 0, \tag{16}$$

where $X = (x^1, x^2, x^3)$ and $F = (F^1, F^2, F^3)$ and ^{t} denotes the transposition. Moreover the equation (14) is equivalently rewritten as $F^i\rho_{is}\varepsilon^{jks} = 0$, in which $\varepsilon^{jks} \neq 0$ if and only if the indices j, k, s are all distinguished. Therefore

$$F^i\rho_{is} = 0, \quad \text{i. e.,} \quad FP = 0, \tag{17}$$

from which it follows that $F^1 = 0$ for $P = P^2$; $F^1 = F^2 = 0$ for $P = P^3, \cdots, P^5$ and P^8_λ; $F^1 = F^2 = F^3 = 0$ for $P = P^6, P^7$. And for such F^1, F^2 and F^3, (16) is satisfied identically. Thus the Poisson Lie brackets of PK-algebras are now obtained as follows (cf. [5]: the case of $F^i = $const.; note that the series of P_α in [5] is rearranged here as P^α).

THEOREM 2. *The Poisson Lie brackets* $[f, g]$ *for* $f, g \in \mathfrak{R}(\mathfrak{M})$ *of the PK-algebras associated with the structure constants of* P^α *and a set of differentiable functions* $(\Phi^\alpha, \Psi^\alpha, \Lambda^\alpha)$ *in the center* $\mathfrak{H}^\alpha(\mathbb{R}^3)$ *of the PB-algebras* $\mathfrak{G}^\alpha(\mathbb{R}^3)$ $(\alpha = 1, \cdots, 8)$ *have the following forms, respectively*:

P^1: $[f, g]=\Phi^1(fg_x-gf_x)+\Psi^1(fg_y-gf_y)+\Lambda^1(fg_z-gf_z)$,

P^2: $[f, g]=x(f_y g_z-f_z g_y)+\Phi^2(fg_y-gf_y)+\Psi^2(fg_z-gf_z)$,

P^3: $[f, g]=x(f_z g_x-f_x g_z)-y(f_y g_z-f_z g_y)+\Phi^3(fg_z-gf_z)$,

P^4: $[f, g]=x(f_y g_z-f_z g_y)-y(f_z g_x-f_x g_z)+\Phi^4(fg_z-gf_z)$,

P^5: $[f, g]=x(f_z g_y-f_y g_z)+y(f_z g_x-f_x g_z)+\Phi^5(fg_z-gf_z)$,

P^6: $[f, g]=x(f_y g_z-f_z g_y)-y(f_z g_x-f_x g_z)+z(f_x g_y-f_y g_x)$,

P^7: $[f, g]=x(f_y g_z-f_z g_y)+y(f_z g_x-f_x g_z)+z(f_x g_y-f_y g_x)$,

P_λ^8: $[f, g]=(x-y)(f_y g_z-f_z g_y)+(x+\lambda y)(f_z g_x-f_x g_z)+\Phi^8(fg_z-gf_z)$,

in which the brackets for P^6 and P^7 are those of the PB-algebras.

4.2. In general, by differentiating (9) by k and summing up for k, it follows that

$$F^k_{,k}C^{ij}_s x^s+F^iC^{jk}_k-F^jC^{ik}_k+(F^i_{,k}C^{jk}_s-F^j_{,k}C^{ik}_s)x^s+F^kC^{ij}_k=0;$$

and so, assuming (9), the equation (10) is equivalent to

$$F^k_{,k}C^{ij}_s x^s+F^iC^{jk}_k-F^jC^{ik}_k=0. \tag{18}$$

Particularly let the structure constants C^{ij}_k satisfy $C^{ij}_s x^s \not\equiv 0$ for some i, j and $C^{ik}_k = 0$ for all i (e.g., see P^α $(\alpha\neq 3, 4)$). Then (18) is reduced to $F^k_{,k}C^{ij}_s x^s=0$, so that $F^i_{,i}=0$ except for (x^i) such that $C^{ij}_s x^s=0$, and therefore $F^i_{,i}=0$ on the considering neighbourhood by the continuity of F^i.

In the following examples of the three-dimensional case ($\mathfrak{M}=\ ^3$), the functions F^i $(i=1, 2, 3)$ of $(x^1, x^2, x^3)=(x, y, z)$ are denoted by $(F^1, F^2, F^3)=(F, G, H)$ as well as $(F_{,1}, F_{,2}, F_{,3})=(F_x, F_y, F_z)$. Since (18) is skew-symmetric for the indices i, j and indentical for $i=j$, it may be considered for $(i, j)=(1, 2)$, $(2, 3)$, $(3, 1)$.

EXAMPLE 1: $P=P^3$. In viewing of $C^{1k}_k=C^{2k}_k=0$ and $C^{3k}_k=2$, (18) is identical for $(i, j)=(1, 2)$, and

$$(F_x+G_y+H_z)y-2G=0 \quad \text{for } (i, j)=(2, 3),$$

$$(F_x+G_y+H_z)x-2H=0 \quad \text{for } (i, j)=(3, 1).$$

Since (16) implies $yF=xG$, by putting $F=xP(x, y, z)$ and $G=yP(x, y, z)$ $(xy\neq 0$, which can be delated in the consequence by the continuity of F, G and $H)$, both of the above equations lead to

$$xP_x+yP_y+H_z=0.$$

So that H is integrated as

$$H=-x\int P_x dz-y\int P_y dz+\Psi(x, y),$$

and therefore the final forms of F, G and H are

$$F=x\Phi_z(x, y, z), \qquad G=y\Phi_z(x, y, z),$$
$$H=-x\Phi_x(x, y, z)-y\Phi_y(x, y, z)+\Psi(x, y),$$

where $\Phi(x, y, z)=\int Pdz$ and $\Psi=\Psi(x, y)$ are arbitrary differentiable functions.

EXAMPLE 2: $P=P^4$. Similarly as above, since $C_k^{1k}=C_k^{2k}=0$ and $C_k^{3k}=2$, (18) is identical for $(i, j)=(1, 2)$, and

$$(x-y)(F_x+G_y+H_z)+2G=0 \quad \text{for } (i, j)=(2, 3),$$
$$(x+\lambda y)(F_x+G_y+H_z)-2F=0 \quad \text{for } (i, j)=(3, 1).$$

And since (16) implies $(x-y)F+(x+\lambda y)G=0$, by putting $F=(x+\lambda y)P(x, y, z)$ and $G=-(x-y)P(x, y, z)$ $((x-y)(x+\lambda y)\neq 0$, which can also be delated in the consequence), both of the above equations lead to

$$(x+\lambda y)P_x-(x-y)P_y+H_z=0.$$

So that H is integrated as

$$H=-(x+\lambda y)\int P_x dz+(x-y)\int P_y dz+\Psi(x, y),$$

and therefore the final form of F, G and H are

$$F=(x+\lambda y)\Phi_z(x, y, z), \qquad G=(x-y)\Phi_z(x, y, z),$$
$$H=-(x+\lambda y)\Phi_x(x, y, z)+(x-y)\Phi_y(x, y, z)+\Psi(x, y),$$

where $\Phi(x, y, z)=\int Pdz$ and $\Psi=\Psi(x, y)$ are arbitrary differentiable functions.

EXAMPLE 3: $P=P^6$. This is the case that $C_s^{ij}x^s\neq 0$, i.e., $C_s^{23}x^s=x$, $C_s^{31}x^s=-y$, $C_s^{12}x^s=z$; and $C_k^{ik}=0$ for all $i=1, 2, 3$. So that $F^i_{;i}=0$, i.e.,

$$F_x+G_y+H_z=0.$$

Since (16) implies $xF-yG+zH=0$, by putting $(x^2+z^2\neq 0)$

$$F=\frac{xyP(x, y, z)}{x^2+z^2}, \quad H=\frac{yzP(x, y, z)}{x^2+z^2}+y\Psi(x, y, z),$$

and so $G=P+z\Psi$ $(y\neq 0$, which can also be delated in the consequence); the above equation leads to

$$xyP_x+(x^2+z^2)P_y+yzP_z+(x^2+z^2)(z\Psi_y+y\Psi_z)=0, \tag{19}$$

with the s-equations (note that the last term disappears if $z\Psi_y+y\Psi_z\equiv 0$)

$$\frac{dx}{xy}=\frac{dy}{x^2+z^2}=\frac{dz}{yz}=\frac{-dP}{(x^2+z^2)(z\Psi_y+y\Psi_z)}.$$

If $\Psi=\Psi(x, y^2-z^2)$ (this is the case of $z\Psi_y+y\Psi_z\equiv 0$), by the independent solutions $x^2-y^2+z^2=\text{const.}$ and $x^{-1}z=\text{const.}$ $(x\neq 0)$ which follow respectively from

$$\frac{xdx}{x^2y}=\frac{-ydy}{-y(x^2+z^2)}=\frac{zdz}{yz^2},\quad \frac{dx}{xy}=\frac{dz}{yz},$$

the solution of (19) is written as $P=\Phi(x^2-y^2+z^2, x^{-1}z)$ and therefore

$$F=\frac{xy\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2},$$

$$G=\Phi(x^2-y^2+z^2, x^{-1}z)+z\Psi(x, y^2-z^2),$$

$$H=\frac{yz\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2}+y\Psi(x, y^2-z^2).$$

As an example of Ψ such that $z\Psi_y+y\Psi_z\not\equiv 0$, we first take $\Psi=z$. Then, since one more solution $P+y^2/2=\text{const.}$ of the s-equation follows from

$$\frac{dy}{x^2+z^2}=\frac{-dP}{(x^2+z^2)y},$$

the solution of (19) is of the from $\Lambda(x^2-y^2+z^2, x^{-1}z, P+y^2/2)=0$; accordingly $P=\Phi(x^2-y^2+z^2, x^{-1}z)-y^2/2$ and therefore

$$F=\frac{xy\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2}-\frac{xy^3}{2(x^2+z^2)},$$

$$G=\Phi(x^2-y^2+z^2, x^{-1}z)-\frac{y^2}{2}+z^2,$$

$$H=\frac{yz\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2}-\frac{y^3z}{2(x^2+z^2)}+yz.$$

The other example of Ψ is $\Psi=\log(x^2+z^2)$. Since one more solution is $P+2z=\text{const.}$, P is of the from $P=\Phi(x^2-y^2+z^2, x^{-1}z)-2z$ and therefore

$$F=\frac{xy\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2}-\frac{2xyz}{x^2+z^2},$$

$$G=\Phi(x^2-y^2+z^2, x^{-1}z)-2z+z\log(x^2+z^2),$$

$$H=\frac{yz\Phi(x^2-y^2+z^2, x^{-1}z)}{x^2+z^2}-\frac{2yz^2}{x^2+z^2}+y\log(x^2+z^2).$$

Here notice that $x^{-1}z$ $(x \neq 0)$ may be replaced with $xz^{-1}(z \neq 0)$ in the above Φ.

REMARK. Particularly for $\Psi = 0$ and $\Phi = \Phi(x^2 - y^2 + z^2)$, the above solutions F, G and H in the case of $\Psi = \Psi(x)$ are reduced respectively to

$$F = \frac{xy\Phi(x^2 - y^2 + z^2)}{x^2 + z^2},$$

$$G = \Phi(x^2 - y^2 + z^2),$$

$$H = \frac{yz\Phi(x^2 - y^2 + z^2)}{x^2 + z^2},$$

which differ nothing, without the change of variables y and z, from the coefficients of the vector field (1.13) in [6] obtained for the study of infinitesimal automorphism.

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