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A GENERALIZATION OF THE HANNER'S INEQUALITY AND THE TYPE 2 (COTYPE 2) CONSTANT OF A BANACH SPACE

By

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1. Introduction

We shall extend the Hanner's 2-element inequality in L^p to the n-element inequality and introduce the notions of Hanner cotype p (Hanner type p). We determine the cotype 2 (type 2) constant of the Banach space of Hanner cotype p (Hanner type p). But our results are restricted for only real valued L^p and the general complex valued cases are left open.

Let $1 \leqq p < \infty$, (S, Σ, μ) be a measure space with $\mu(S) = 1$ and $L^p = L^p(S, \Sigma, \mu)$. The norm of L^p is given by $\|x\| = (\int |x(t)|^p d\mu(t))^{1/p}$.

Hanner [3] proved the following inequalities. For $x_1, x_2 \in L^p$, it holds that for $1 < p \leqq 2$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \geqq |\,\|x_1\| + \|x_2\|\,|^p + |\,\|x_1\| - \|x_2\|\,|^p$$

and for $2 \leqq p < \infty$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \leqq |\,\|x_1\| + \|x_2\|\,|^p + |\,\|x_1\| - \|x_2\|\,|^p.$$

Remark that, by the triangular inequality, $\|x_1 + x_2\| \leqq \|x_1\| + \|x_2\|$ and $\|x_1 - x_2\| \geqq |\,\|x_1\| - \|x_2\|\,|$. If we neglect the second term of the right hand side, then a special case of the Clarkson's inequality [1] follows. We can rewrite the Hanner's inequality as follows. Let $\varepsilon_1, \varepsilon_2$ be the independent Rademacher random variables with the distribution $\varepsilon_i = \pm 1$ with probability $1/2$. Then the Hanner's inequality is given by

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^{2} \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^{2} \varepsilon_i \|x_i\|\,|^p \qquad \text{for } 1 < p \leqq 2, \text{ and}$$

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^{2} \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^{2} \varepsilon_i \|x_i\|\,|^p \qquad \text{for } 2 \leqq p < \infty,$$

where E means the expectation with respect to the Rademacher distribution.

We shall extend the Hanner's inequality naturally as follows. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots, x_n \in L^p$. We show that if all x_i are real valued then it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^{n} \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^{n} \varepsilon_i \|x_i\|\,|^p \qquad \text{for } 1 \leqq p \leqq 2, \text{ and}$$

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^{n} \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^{n} \varepsilon_i \|x_i\|\,|^p \qquad \text{for } 2 \leqq p < \infty.$$

The general complex valued cases are left open.

Let E be a Banach space with norm $\| \ \|$, $0 < s < \infty$ and let $\{\varepsilon_i\}$ be the independent Rademacher sequence. Then E is called of cotype 2 if there exists a constant $C_{2,s} > 0$ such that

$$(\sum_{i=1}^{n} \|x_i\|^2)^{1/2} \leqq C_{2,s}(\mathrm{E}\|\sum_{i=1}^{n} \varepsilon_i x_i\|^s)^{1/s}$$

for every n and every $x_1, x_2, \cdots, x_n \in E$. Denote by $C_{2,s}(E)$ the smallest constant in the inequality. $C_{2,s}(E)$ is called the cotype 2 constant of E. The Banach space E is called of type 2 if there exists a constant $T_{s,2} > 0$ such that

$$(\mathrm{E}\|\sum_{i=1}^{n} \varepsilon_i x_i\|^s)^{1/s} \leqq T_{s,2}(\sum_{i=1}^{n} \|x_i\|^2)^{1/2}$$

for every n and every $x_1, x_2, \cdots, x_n \in E$. Denote by $T_{s,2}(E)$ the smallest constant in the inequality. $T_{s,2}(E)$ is called the type 2 constant of E. It is well known that L^p is of cotype 2 for $1 \leqq p \leqq 2$ and of type 2 for $2 \leqq p < \infty$, see Hoffmann-Jørgensen [4], Lindenstrauss and Tzafriri [5].

Let E be a Banach space with norm $\| \ \|$. We say that E is of Hanner cotype p $(1 \leqq p \leqq 2)$ if it holds that

$$\mathrm{E}\|\sum_{i=1}^{n} \varepsilon_i x_i\|^p \geqq \mathrm{E}|\sum_{i=1}^{n} \varepsilon_i \|x_i\||^p$$

for every n and every $x_1, x_2, \cdots, x_n \in E$, where $\{\varepsilon_i\}$ are independent Rademacher random variables. We say that E is of Hanner type p $(2 \leqq p < \infty)$ if it holds that

$$\mathrm{E}\|\sum_{i=1}^{n} \varepsilon_i x_i\|^p \leqq \mathrm{E}|\sum_{i=1}^{n} \varepsilon_i \|x_i\||^p$$

for every n and every $x_i, x_2, \cdots, x_n \in E$.

We shall show that each Banach space of Hanner cotype p $(1 \leqq p \leqq 2)$ is of cotype 2 and determines the cotype 2 constant $C_{2,p}(E)$ explicitly. We shall show also that each Banach space of Hanner type p $(2 \leqq p < \infty)$ is of type 2 and determines the type 2 constant $T_{p,2}(E)$ explicitly. These constants are in fact identical to the best constants in the Khinchin's inequality:

$$C_{2,p}(\mathbb{R})^{-1}(\sum_{i=1}^{n} a_i^2)^{1/2} \leqq (\mathrm{E}|\sum_{i=1}^{n} a_i\varepsilon_i|^p)^{1/p}$$
$$\leq T_{p,2}(\mathbb{R})(\sum_{i=1}^{n} a_i^2)^{1/2}.$$

that is, identical to the cotype 2 and the type 2 constants of the real numbers $\mathbb{R}$. The correct values of $C_{2,p}(\mathbb{R})$ and $T_{p,2}(\mathbb{R})$ are determined by Haagerup [2] and Szarek [6].

2. Generalization of Hanner's Inequality

LEMMA 1. Let p be $1 \leqq p < \infty$, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $u_1, u_2, \cdots, u_n$ be real numbers. then it holds that $\mathrm{E}|\sum_{i=1}^{n} \varepsilon_i u_i|^p = \mathrm{E}|\sum_{i=1}^{n} \varepsilon_i |u_i||^p$.

PROOF. Since $\{\varepsilon_i\}$ are symmetric and independent, $\{u_i\varepsilon_i\}$ and $\{|u_i|\varepsilon_i\}$ have the same distribution, hence the assertion follows.

LEMMA 2 (Hanner [3]). Let $\alpha \geqq 0$ and $u \geqq 0$. Let $f(u)$ be

$$f(u) = |u^{1/p} + \alpha|^p + |u^{1/p} - \alpha|^p.$$

If $1 \leqq p \leqq 2$, then $f(u)$ is a convex function, and if $2 \leqq p < \infty$, then $f(u)$ is a concave function.

PROOF. If $p = 1$, then $f(u)$ is convex since

$$f(u) = \begin{cases} 2\alpha & \text{for } 0 \leqq u \leqq \alpha \\ 2u & \text{for } u \geqq \alpha. \end{cases}$$

In the case where $p > 1$, the second defivative $f''(u)$ is given by

$$f''(u) = \alpha(p-1)/p \cdot u^{1/p-2}(|u^{1/p} - \alpha|^{p-2} - |u^{1/p} + \alpha|^{p-2}),$$

which implies the assertions.

LEMMA 3. Let $u_1, u_2, \cdots, u_n \geqq 0$ and let $F(u_1, u_2, \cdots, u_n)$ be

$$F(u_1, u_2, \cdots, u_n) = \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i u_i^{1/p}|^p.$$

Then regarding F as a function of each u_i, F is convex for $1 \leqq p \leqq 2$ and F is concave for $2 \leqq p < \infty$.

PROOF. We can write

$$F(u_i) = 1/2 \cdot \mathrm{E}\,[|u_i^{1/p} + (\textstyle\sum_{j \neq i} u_j^{1/p}\varepsilon_j)|^p + |u_i^{1/p} - (\sum_{j \neq i} u_j^{1/p}\varepsilon_j)|^p].$$

By Lemma 2, the integrant of the right hand side is a convex (resp. concave) function of u_i for $1 \leqq p \leqq 2$ (resp. for $2 \leqq p < \infty$), hence so is F.

THEOREM 1. Let n be a natural number, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $x_1, x_2, \cdots, x_n$ be real valued functions in L^p.

(1) If $1 \leqq p \leqq 2$, then it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i \|x_i\||^p.$$

(2) If $2 \leqq p < \infty$, then it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i \|x_i\||^p.$$

PROOF. (1) Suppose that $1 \leqq p \leqq 2$. By Lemma 1, we have

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p = \mathrm{E}\left(\displaystyle\int_S |\textstyle\sum_{i=1}^n \varepsilon_i(\omega)x_i(t)|^p\, d\mu(t)\right)$$

$$= \int_S \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i(\omega) x_i(t)|^p \, d\mu(t)$$

$$= \int_S \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i(\omega)\, |x_i(t)|\, |^p \, d\mu(t)$$

$$= \mathrm{E}\, \|\textstyle\sum_{i=1}^n \varepsilon_i |x_i|\, \|^p,$$

where $|x_i|(t) = |x_i(t)|$. So we can assume in advance that each x_i is non-negative, $x_i(t) \geqq 0$. By Lemma 3 and by the Jensen's inequality, we have

$$\int_S F(x_1(t)^p, x_2(t)^p, \cdots, x_n(t)^p)\, d\mu(t)$$
$$\geqq F\left(\int_S x_1(t)^p d\mu(t), \int_S x_1(t)^p d\mu(t), \cdots, \int_S x_1(t)^p d\mu(t)\right),$$

where F is the function given in Lemma 3 (we have also used the assumption $\mu(S) = 1$). This is the inequality desired.

(2) The case where $2 \leqq p < \infty$ is obtained by the manner same to the case (1). In this case, F is concave and we obtain the converse inequality

$$\int_S F(x_1(t)^p, x_2(t)^p, \cdots, x_n(t)^p) d\mu(t)$$
$$\leqq F\left(\int_S x_1(t)^p d\mu(t), \int_S x_1(t)^p d\mu\, (t), \cdots, \int_S x_1(t)^p d\mu(t)\right),$$

by the Jensen's inequality. This completes the proof.

REMARK. In the case where $p = 1$, Hanner's 2-element inequality

$$\|x_1 + x_2\| + \|x_1 - x_2\| \geqq \|x_1\| + \|x_2\| + |\, \|x_1\| - \|x_2\|\, |$$

does not imply any geometric information of the space L^1. In fact, this inequality holds for every Banach space. In fact, if we suppose that $\|x_1\| \geqq \|x_2\|$ without loss of generality, then this inequality is a consequence of the triangular inequality. On the contrary, our n-element inequality

$$\mathrm{E}\, \|\textstyle\sum_{i=1}^n \varepsilon_i x_i\| \geqq \mathrm{E}\, |\textstyle\sum_{i=1}^n \varepsilon_i \|x_i\|\, |$$

does not hold for any Banach space. If this n-element inequality is valid in a Banach space E, then E is of cotype 2 since $\mathbb{R}$ is of cotype 2 as follows:

$$\mathrm{E}\, \|\textstyle\sum_{i=1}^n \varepsilon_i x_i\| \geqq \mathrm{E}\, |\textstyle\sum_{i=1}^n \varepsilon_i \|x_i\|\, |$$
$$\geqq C_{2,1}(\mathbb{R})^{-1} (\textstyle\sum_{i=1}^n \|x_i\|^2)^{1/2}.$$

3. Hanner Type and Hanner Cotype of Banach space

THEOREM 2. (1) Let E be a Banach space of Hanner cotype p $(1 \leqq p \leqq 2)$. Then E is of cotype 2 and the cotype 2 constant $C_{2,p}(E)$ coincides with $C_{2,p}(\mathbb{R})$, where $C_{2,p}(\mathbb{R})$ is the best constant in the Khinchin's inequality.

(2) Let E be a Banach space of Hanner type p $(2 \leqq p < \infty)$. Then E is of type 2 and the type 2 constant $T_{p,2}(E)$ coincides with $T_{p,2}(\mathbb{R})$, where $T_{p,2}(\mathbb{R})$ is the best constant in the Khinchin's inequality.

PROOF. (1) By Theorem 1 and by the Khinchin's inequality, we have

$$\begin{aligned}(\mathrm{E}\,\|\textstyle\sum_{i=1}^{n}\varepsilon_i x_i\|^p)^{1/p} &\geqq (\mathrm{E}\,|\textstyle\sum_{i=1}^{n}\varepsilon_i\,\|x_i\|\,|^p)^{1/p}\\ &\geqq C_{2,p}(\mathbb{R})^{-1}(\textstyle\sum_{i=1}^{n}\|x_i\|^2)^{1/2}.\end{aligned}$$

This implies, by the minimality of $C_{2,p}(E)$, that $C_{2,p}(E) \leqq C_{2,p}(\mathbb{R})$. Conversely, if we imbedd $\mathbb{R}$ isometrically into E, we have $C_{2,p}(E) \leqq C_{2,p}(\mathbb{R})$ by the minimality of $C_{2,p}(\mathbb{R})$.

(2) By Theorem 1 and by the Khinchin's inequality, we have

$$\begin{aligned}(\mathrm{E}\,\|\textstyle\sum_{i=1}^{n}\varepsilon_i x_i\|^p)^{1/p} &\leqq (\mathrm{E}\,|\textstyle\sum_{i=1}^{n}\varepsilon_i\,\|x_i\|\,|^p)^{1/p}\\ &\leqq T_{p,2}(\mathbb{R})(\textstyle\sum_{i=1}^{n}\|x_i\|^2)^{1/2}.\end{aligned}$$

By the minimality of $T_{p,2}(E)$, it follows that $T_{p,2}(E) \leqq T_{p,2}(\mathbb{R})$. If we imbedd $\mathbb{R}$ isometrically into E, we have $T_{p,2}(\mathbb{R}) \leqq T_{p,2}(E)$ by the minimality of $T_{p,2}(\mathbb{R})$. This completes the proof.

COROLLARY. Let $L^p(\mathbb{R})$ be the space of all real functions in L^p.

(1) If $1 \leqq p \leqq 2$, then $C_{2,p}(L^p(\mathbb{R})) = C_{2,p}(\mathbb{R})$.

(2) If $2 \leqq p < \infty$, then $T_{p,2}(L^p(\mathbb{R})) = T_{p,2}(\mathbb{R})$.

4. Concluding Remarks

Our extensions of Hanner's inequality (Theorem 1) are valid only for real functions in L^p. The original result of Hanner is valid for complex valued case. So the extension of Theorem 1 to the complex valued case is left open. To show the complex valued case it is sufficient (and also necessary) to prove the next inequalities in $\mathbb{C}$. Let $z_1, z_2, \cdots, z_n$ be complex numbers in $\mathbb{C}$. Then

(1) if $1 \leqq p \leqq 2$, then $\mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i z_i|^p \geqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i |z_i|\,|^p$,

(2) if $2 \leqq p < \infty$, then $\mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i z_i|^p \leqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i |z_i|\,|^p$.

For example, in the case where $n = 3$

$$\begin{aligned}&|z_1+z_2+z_3|^p + |z_1+z_2-z_3|^p + |z_1-z_2+z_3|^p + |z_1-z_2-z_3|^p\\ &\geqq (\leqq)\,|\,|z_1|+|z_2|+|z_3|\,|^p + |\,|z_1|+|z_2|-|z_3|\,|^p\\ &\quad + |\,|z_1|-|z_2|+|z_3|\,|^p + |\,|z_1|-|z_2|-|z_3|\,|^p.\end{aligned}$$

According to the computer serch, these inequalities seem true.

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