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NONCOMMUTATIVE POISSON ALGEBRA STRUCTURES ON POSET ALGEBRAS AND MORPHISMS OF LEIBNIZ PAIRS

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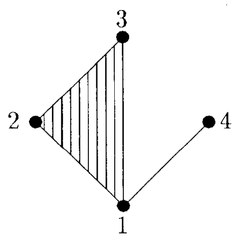
A non-commutative Poisson algebras A is an algebra over the field **C** of the complex numbers (we shall take the field **C** for simplicity) having an associative algebra product, being denoted by xy the associative algebra product of x, y in A and the Lie algebra bracket $\{-, -\}$ connected with the Leibniz law: $\{xy, z\} = \{x, z\}y + x\{y, z\}$ for $x, y, z \in A$. For any associative algebra A one can always construct a noncommutative Poisson algebra structure in which we take a Lie product $\{-, -\}$ to be the scalar multiple of the ordinary associative commutator, i.e, by setting $\{x, y\} := \lambda[x, y] = \lambda(xy - yx)$, $\lambda \in \mathbf{C}$. Let us denote such a noncommutative Poisson algebra by

$$A^{\lambda},$$

which is called *standard* or said to have a standard structure. Note that A^1 is of the ordinal associative commutator. We shall use the notion of "standard" in this way rather than that of one which we allow a direct sum of noncommutative Poisson algebras $A_1^{\lambda_1}, \ldots, A_r^{\lambda_r}$ to be called in [3, 4]. Every finite-dimensional noncommutative Poisson algebra A of a semisimple associative algebra structure must be decomposed into a direct sum of the standard ones $A_1^{\lambda_1}, \ldots, A_r^{\lambda_r}$ of simple associative algebra structures ([3, Theorem 7]). Every noncommutative Poisson algebra structure on the algebra $M_{\infty}(\mathbf{C})$ of all **C**-endomorphisms of a countable-dimensional **C**-vector space must be also standard ([4, Proposition]). The associative algebra structure of the noncommutative Poisson algebra structures on the Kac-Moody algebras L of affine type must be almost trivial, that is, $L'L' = 0$ for the derived Lie ideal L' of L ([4, Theorem]). These results were found while facing to our problem of what deformations of Lie algebra structures or associative algebra structures on noncommutative Poisson algebras keep the Leibniz law satisfied, so that one obtains noncommutative Poisson algebras deformed a given one.

In this short article, we first describe the noncommutative Poisson algebra structures on some associative subalgebras, so called "poset algebras" in the $n \times n$ full matrix algebra $M_n(\mathbf{C})$ ([2, §15]). For example, let $I = \{1, 2, 3, 4\}$ a poset on which a partial ordering is given by $1 \prec 2 \prec 3$, $1 \prec 4$. The corresponding poset algebra A in $M_4(\mathbf{C})$ is the associative subalgebra of $M_4(\mathbf{C})$ consisting the matrices pictured as on the left below (The entries in position marked by $*$ may be chosen arbitrarily from $\mathbf{C}$), and a geometric realization of I (the nerve $\Sigma(I)$ of I) is given on the right below.

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$



We can construct a noncommutative Poisson algebra structure on A which is not necessarily standard. $A_1^\lambda = \langle e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{13} \rangle$, $A_2^\mu = \langle e_{11}, e_{44}, e_{14} \rangle$ be the subalgebras spanned by the contents in the triangular brackets and their Lie products $\{-, -\}$ are defined by $\{-, -\} = \lambda[-, -]$ on A_1^λ and $\{-, -\} = \mu[-, -]$ on A_2^μ, where e_{ij} is the matrix with 1 in the (i, j)th place and 0 elsewhere, and the bracket $[-, -]$ is the ordinary associative commutator. Then $A = A_1^\lambda + A_2^\mu$ and $A_1^\lambda \cap A_2^\mu = \mathbf{C}e_{11}$. We will see that every poset algebra in $M_n(\mathbf{C})$ having a noncommutative Poisson algebra structure is expressed in the form $A = A_1^{\lambda_1} + \cdots + A_r^{\lambda_r}$ such that for $i \neq j$, $A_i^{\lambda_i} \cap A_j^{\lambda_j} = 0$ or $\mathbf{C}e_{kk}$ for some k (Theorem 1).

A morphism of noncommutative Poisson algebras is simultaneously an associative algebra homomorphism and a Lie algebra homomorphism. If f is such a morphism of A^λ into A^1 then the formula $f(\lambda[x, y]) = [f(x), f(y)]$ implies $(\lambda - 1)[f(x), f(y)] = 0$ for any $x, y \in A$. Hence A^λ is isomorphic to A^1 as noncommutative Poisson algebras only when $\lambda = 1$. There is a nice concept of a morphism to understand a relation between A^λ and A^1, namely a morphism of Leibniz pairs. A Leibniz pair (A, L) consists of an associative algebra A and a Lie algebra L over the common field $\mathbf{C}$ of the complex numbers (A Leibniz pair is actually defined over a ring in [1]), connected by a Lie algebra morphism, so called a *structural* morphism $\mu: L \to \mathrm{Der}_{\mathbf{C}} A$, where denoting by $\mathrm{Der}_{\mathbf{C}} A$ the Lie algebra of $\mathbf{C}$-linear derivations of the associative algebra A into itself. When L is identical with A as a $\mathbf{C}$-module, one obtains a noncommutative Poisson algebra A on which the Lie product $\{-, -\}$ is defined by $\{x, y\} := \mu(x)(y)$ for $x, y \in A$. Since we do not deform the associative algebra structure here, let us denote by

$$A_{\{-,-\}}$$

a noncommutative Poisson algebra on which a Lie product is taken to be $\{-, -\}$. Note that $A^1 = A_{[-,-]}$ in our terminology. Then for any poset algebra $A = A_{\{-,-\}}$ in $M_n(\mathbf{C})$, there exists a morphism $(1_A, \psi)$ of the Leibniz pairs $(A, A_{\{-,-\}})$ into $(A, A_{[-,-]})$, as we will see in the section 2.

1 Noncommutative Poisson algebra structures on poset algebras

When a subalgebra A of $M_n(\mathbf{C})$ contains all the diagonal matrices, hence, in particular, all e_{ii}, $i = 1,\dots,n$, A is spanned by those e_{ij} which it contains. For if an a in A has the form $a = \Sigma\lambda_{ij}e_{ij}$ then $e_{ii}ae_{jj} = \lambda_{ij}e_{ij}$, so if $\lambda_{ij} \neq 0$ then $e_{ij} \in A$. Such algebras A are called *poset algebras* ([2]). Now define a poset $I = I(A)$ by setting $i \prec j$ if $e_{ij} \in A$, and let $\bar{I}$ be the poset (without loops) determined by reducing I modulo the equivalence relation defined by the loops, i.e., by identifying to a single element any i and j for which both $i \prec j$ and $j \prec i$ (hence identifying any $i_1, i_2, \dots, i_r$ whenever $i_1 \prec i_2 \prec \cdots \prec i_r$.) Let $\Sigma = \Sigma(A)$ be the nerve of $\bar{I}(A)$. This is a finite simplicial complex.

Let $A_{\{-,-\}}$ be a poset algebra in $M_n(\mathbf{C})$ having a Lie algebra structure denoted its product by $\{-, -\}$ satisfying the Leibniz law: $\{xy, z\} = \{x, z\}y + x\{y, z\}$ for $x, y, z \in A$. Then A must be standard when $\Sigma(A)$ is connected and $\Sigma(A)$ has the property that for any pair of 1-faces there exists a polygon which has these two 1-faces as its edges ([3, Theorem 2]). In particular, the algebra $M_n(\mathbf{C})$ and its subalgebra $T_n(\mathbf{C})$ of all upper triangular matrices are allowed to have only the standard structures, because the corresponding simplical complex to these algebras is just a $(n-1)$-simplex. The analysis of the noncommutative Poisson algebra structures on the poset algebra A in $M_n(\mathbf{C})$ is based on the formulas $\{e_{ii}, e_{jj}\} = 0$, $\{e_{ii}, e_{jk}\} = \lambda_i(e_{jk})$ where $\lambda_i(e_{jk}) = 0$ if $i \neq j$ and $i \neq k$, and

$$\{e_{ij}, e_{kl}\} = -\lambda_k(e_{ij})e_{ij}e_{kl} + \lambda_i(e_{kl})e_{kl}e_{ij},$$

when $e_{ij}, e_{kl} \in A$ with $i \neq j$, $k \neq l$, where of course $\lambda_i(e_{jk}) \in \mathbf{C}$. Let $i \prec j_1 \prec \cdots \prec j_p$ be a p-simplex in $\Sigma = \Sigma(A)$. Since $\{e_{ii}, e_{ij_k}\} = \{e_{ii}, e_{ij_1}e_{j_1j_2}\cdots e_{j_{k-1}j_k}\} = \{e_{ii}, e_{ij_1}\}e_{j_1j_2}\cdots e_{j_{k-1}j_k} = \lambda_i(e_{ij_1})e_{ij_k}$ for $1 \leq k \leq p$, one gets $\lambda_i(e_{ij_k}) = \lambda_i(e_{ij_1})$ for $1 \leq k \leq p$. Hence we can assign a scalar $\lambda(i \prec j_1 \prec \cdots \prec j_p)$ to each p-simplex in Σ. If two simplecies X, Y in Σ have common 1-face then obviously $\lambda(X) = \lambda(Y)$. Let $X = i_0 \prec i_1 \prec \cdots \prec i_p$ be a p-simplex in Σ and $A(X)$ a subalgebra of A spanned by $\{e_{i_ji_k} \mid 0 \leq j \leq k \leq p\}$. Then the equality $\{-, -\} = \lambda(X)[-, -]$ holds on $A(X)$, so that one has a noncommutative Poisson subalgebra $A(X)^{\lambda(X)}$ of A. Now let us introduce some terminology to describe such matters. Assume that $\Sigma = \Sigma(A)$ is connected. By *standard component* Σ_i of Σ we shall mean a simplical subcomplex maximal respect to the property that each simplex of dimension ≥ 2 in Σ_i has a common 1-face with some other simplex in Σ_i. Then Σ can be described as a sum, say a *standard sum*, $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ of the standard components Σ_i such that $\Sigma_i \cap \Sigma_j$ consists of one vertex or has no element for $i \neq j$. If a simplical complex Σ_i consists of the simplecies $X_{i_1}, \dots, X_{i_s}$, then we have a corresponding subalgebra $A(\Sigma_i) := A(\Sigma_{i_1}) + \cdots + A(\Sigma_{i_s})$ on which the equality $\{-, -\} = \lambda(\Sigma_i)[-, -]$ holds where $\lambda(\Sigma_i) := \lambda(X_{i_1}) = \cdots = \lambda(X_{i_s})$, one can then write $A(\Sigma_i) = A(\Sigma_i)^{\lambda(\Sigma_i)}$ in our terminology. Therefore we have

THEOREM 1. *Let* $A = A_{\{-,-\}}$ *be a poset algebra in* $M_n(\mathbf{C})$, $\Sigma = \Sigma(A)$ *the simplical complex associated to* A. *Suppose that* Σ *is connected and let* $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ *be a standard sum of* Σ. *Then there exists a* $\lambda_i \in \mathbf{C}$ *and a subalgebra* A_i *for each* i *such that* A *is expressed as a sum*

$$A = A_1^{\lambda_1} + \cdots + A_r^{\lambda_r}$$

of standard noncommutative Poisson subalgebras A_i, *where* A_i *and* λ_i *can be taken to be* $A(\Sigma_i)$ *and* $\lambda(\Sigma_i)$ *respectively corresponding to a standard component* Σ_i *of* Σ. *Here the* A_i*'s satisfy that if* $i \neq j$ *then* $A_i \cap A_j$ *is* 0 *or equal to a* 1*-dimensional subspace* $\mathbf{C}e_{kk}$ *for some* k.

2 Morphisms of Leibniz pairs $(A, A_{\{-,-\}})$ and $(A, A_{[-,-]})$

As in the introduction let $A_{\{-,-\}}$ be a noncommutative Poisson algebra A whose Lie product is given by $\{-,-\}$. Then we have the structural morphism $\mu: A_{\{-,-\}} \to \mathrm{Der}_{\mathbf{C}} A$ defined by $\mu(x)(b) := \{x, b\}$ for $x, b \in A$. For the typical standard noncommutative Poisson algebra $A_{[-,-]} = A^1$ one also has the structural morphism $\mu_0: A_{[-,-]} \to \mathrm{Der}_{\mathbf{C}} A$ defined similarly by $\mu_0(x)(b) := [x, b]$ for $x, b \in A$. A Leibniz pair morphism of $(A, A_{\{-,-\}})$ into $(A, A_{[-,-]})$ is a pair (ϕ, ψ) consisting an associative algebra homomorphism $\phi: A \to A$ and a Lie algebra homomorphism $\psi: A_{\{-,-\}} \to A_{[-,-]}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu(x)} & A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ A & \xrightarrow{\mu_0(\psi(x))} & A \end{array}$$

commutes for any $x \in A$.

Now let A be a poset algebra in $M_n(\mathbf{C})$ and $\Sigma = \Sigma(A)$ the simplical complex associated to A. Suppose that Σ is connected. As in the theorem 1 let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ be a standard sum of Σ and $A_i = A(\Sigma_i)$ a subalgebra corresponding to each standard component Σ_i of Σ and $\lambda_i := \lambda(\Sigma_i)$. Then one has a noncommutative Poisson algebra $A_i = A_i^{\lambda_i}$. Let us denote simply by I_i a poset $I(A)$. We shall construct a Leibniz pair morphism $(1_A, \psi): (A, A_{\{-,-\}}) \to (A, A_{[-,-]})$, so that the condition $\{x, y\} = [\psi(x), y]$ holds for $x, y \in A$. Our Lie algebra homomorphism ψ is defined as follow. For $k, l \in I_i$ and $k \prec l$ we put

$$\psi(e_{kl}) := \lambda_i e_{kl}.$$

If $\{j_1, \ldots, j_s\} = \{j \mid p \in I_j\}$ then a matrix e_{pp} is sent to

$$\psi(e_{pp}) := e_{pp} + \sum_{t=1}^{s} (1 - \lambda_{j_t}) \sum_{i \in I_{j_t} \setminus \{p\}} e_{ii}.$$

It is easy to verify that ψ is a Lie algebra homomorphism and satisfy the required condition that $\{x, y\} = [\psi(x), y]$ for $x, y \in A$. For exmple, if $A = A_1^{\lambda} + A_2^{\mu}$ is the poset algebra in $M_4(\mathbf{C})$ given in the introduction then ψ is of the form

$$\psi(e_{22}) = \lambda e_{22},\ \psi(e_{33}) = \lambda e_{33},\ \psi(e_{12}) = \lambda e_{12},\ \psi(e_{23}) = \lambda e_{23},$$
$$\psi(e_{13}) = \lambda e_{13},\ \psi(e_{44}) = \mu e_{44},\ \psi(e_{14}) = \mu e_{14},\ \text{and}$$
$$\psi(e_{11}) = e_{11} + (1 - \lambda)(e_{22} + e_{33}) + (1 - \mu)e_{44}.$$

THEOREM 2. *Let A be a poset algebra in $M_n(\mathbf{C})$, $A_{\{-,-\}}$ and $A_{[-,-]}$ the noncommutative Poisson algebras of A whose Lie algebra products are given by $\{-, -\}$ and the ordinal associative commutator $[-, -]$ respectively. Then there exists a morphism $(1_A, \psi)$ of a Leibniz pair $(A, A_{\{-,-\}})$ into a Leibniz pair $(A, A_{[-,-]})$. When $A = A_1^{\lambda_1} + \cdots + A_r^{\lambda_r}$ as in the theorem* 1 *and $\lambda_1 \cdots \lambda_r \neq 0$, ψ can be taken to be a Lie algebra isomorphism, hence, these two pairs of above are isomorphic as Leibniz pairs.*

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