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A DETERMINATION OF MOTIONS IN THE CENTRAL FORCE PROBLEM THROUGH CONSERVED QUANTITIES

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1. Introduction

As an illustration of the new operative method (Mimura *et al.* [4, 5]) which grew up from the application of a suitable version of Noether's theorem [6] to the composite variational principle (Caviglia [1, 2]), it was derived a couple of independent conserved quantities (first integrals) for the motions of the following particle in the central force problem (see Whittaker [7], p. 243):

A single particle moving in a plane under a central force directed towards a fixed center in a resisting medium, where the force is proportional to the particle's distance from the center and the medium imposes a retarding force equal to β times the velocity.

The origin is placed on the center of force and the position of particle at time t is defined by polar coordinates $(r(t), \varphi(t))$ to have the differential equations of the motion (e.g. Djukic [3]):

$$m(\ddot{r} - r\dot{\varphi}^2) + \beta\dot{r} + \sigma r = 0,$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) + \beta r\dot{\varphi} = 0,$$

where $m\,(m > 0)$ is the mass of particle and $\sigma\,(\sigma > 0)$ is the central force constant. So by putting

$$\mu = \frac{\beta}{2m}, \quad \omega^2 = \frac{\sigma}{m},$$

it follows that

$$\ddot{r} + 2\mu\dot{r} + \omega^2 r - r\dot{\varphi}^2 = 0, \tag{1}$$

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + 2\mu r\dot{\varphi} = 0, \tag{2}$$

which are the Euler-Lagrange equations with the Lagrangian

$$L = \tfrac{1}{2}e^{2\mu t}(\dot{r}^2 + (r\dot{\varphi})^2 - (\omega r)^2).$$

The couple of conserved quantities of the equations (1) and (2) are [4, §4; 5, §6]

$$\Omega_1 = e^{2\mu t}(\tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}(r\dot{\varphi})^2 + \tfrac{1}{2}(\omega r)^2 + \mu r\dot{r}), \tag{3}$$

(4) $$\Omega_2 = e^{2\mu t} r^2 \dot{\varphi},$$

in which Ω_1 was obtained by Djukic [3] under the symmetry for the gauge-variant Lagrangians, while the equation (2) can be put as

$$\frac{d(r^2\dot{\varphi})}{r^2\dot{\varphi}} = -2\mu dt,$$

whose solution $r^2\dot{\varphi} = \Omega_2 e^{-2\mu t}$ (Ω_2: const.) leads to the appearance of Ω_2 of (4). In this paper, we show that the conserved quantities (3) and (4) contribute to determine completely the motions of the particle in the central force problem.

2. A determination of motions through the conserved quantities

In the couple of the conserved quantities Ω_1 and Ω_2 of the equations of the motion, $\dot{\varphi}$ can be eliminated to see

$$(r\dot{r})^2 + 2\mu r^3\dot{r} + \omega^2 r^4 - 2\Omega_1 e^{-2\mu t} r^2 + (\Omega_2 e^{-2\mu t})^2 = 0,$$

which is transformed, by a change of variable $x = e^{\mu t} r$, into

(5) $$(x\dot{x})^2 + (\omega^2 - \mu^2)x^4 - 2\Omega_1 x^2 + \Omega_2^2 = 0;$$

while (4) is also into

(6) $$x^2\dot{\varphi} = \Omega_2.$$

The motions of the particle in the considering central force problem can be determined completely by the equations (5) and (6).

1. We first settle the case with $\Omega_2 \neq 0$ which implies by (6) that $\dot{\varphi} \neq 0$, i.e., the particle is moving out of the straight. Then, Ω_1 and Ω_2 lie in the root which comes from the equation (5):

(7) $$x\dot{x} = \pm\sqrt{-(\omega^2 - \mu^2)x^4 + 2\Omega_1 x^2 - \Omega_2^2},$$

satisfying the conditions: $\Omega_1 > 0$ if $\omega^2 - \mu^2 \geq 0$, and $\Omega_1^2 - (\omega^2 - \mu^2)\Omega_2^2 \geq 0$.

1.1. $\Omega_2 \neq 0$ and $\omega^2 - \mu^2 > 0$. By putting

$$a = \frac{\sqrt{\Omega_1^2 - (\omega^2 - \mu^2)\Omega_2^2}}{\omega^2 - \mu^2}, \qquad b = \frac{\Omega_1}{\omega^2 - \mu^2},$$

the equation (7) is written as

$$x\dot{x} = \pm\sqrt{\omega^2 - \mu^2}\sqrt{a^2 - (x^2 - b)^2}.$$

If $a = 0$, this equation has a solution $x^2 = b$; and if $a \neq 0$, by using a variable $y = x^2$, it leads to

$$\frac{dy}{\sqrt{a^2-(y-b)^2}} = \pm\, 2\sqrt{\omega^2-\mu^2}\, dt,$$

which is integrated:

$$\sin^{-1}\frac{y-b}{a} = \pm\, 2\sqrt{\omega^2-\mu^2}\, t + \alpha \qquad (\alpha\text{: const.}).$$

Consequently, together with $y = x^2 = b$ for $a = 0$, the solution y can be put as (replace $\pm\,\alpha$ with α)

$$y = \pm\, a \sin(2\sqrt{\omega^2-\mu^2}\, t + \alpha) + b,$$

in which the minus sign is nonessential, since the constant α can be replaced with $\alpha + \pi$. Here note that the constants a and $b (b > 0)$ satisfy $b^2 - a^2 = \Omega_2^2/(\omega^2-\mu^2) > 0$, so that $y \geq b \pm a > 0$. For the solution, by a change of variable $\tau = 2\sqrt{\omega^2-\mu^2}\, t + \alpha$, the equation (6) with $\Omega_2 = \pm\sqrt{(b^2-a^2)(\omega^2-\mu^2)}$ is transformed into

$$\frac{d\varphi}{d\tau} = \pm\frac{\sqrt{b^2-a^2}}{2(a\sin\tau + b)} \qquad (\Omega_2 \gtrless 0),$$

which is integrated:

$$\varphi = \pm\tan^{-1}\frac{b\tan(\frac{1}{2}\tau) + a}{\sqrt{b^2-a^2}} + k \qquad (k\text{: const.}).$$

In this way, the motion of the particle is determined:

$$r = e^{-\mu t}\sqrt{a\sin(2\sqrt{\omega^2-\mu^2}\, t + \alpha) + b},$$

$$\varphi = \pm\tan^{-1}\frac{b\tan(\sqrt{\omega^2-\mu^2}\, t + \frac{1}{2}\alpha) + a}{\sqrt{b^2-a^2}} + k \qquad (\Omega_2 \gtrless 0).$$

1.2. $\Omega_2 \neq 0$ and $\omega^2 - \mu^2 < 0$. By using the above b and

$$a = \frac{\sqrt{\Omega_1^2-(\omega^2-\mu^2)\Omega_2^2}}{\mu^2-\omega^2},$$

the equation (7) is written as

$$x\dot{x} = \pm\sqrt{\mu^2-\omega^2}\sqrt{(x^2-b)^2-a^2};$$

which, by the variable $y = x^2$, leads to

$$\frac{dy}{\sqrt{(y-b)^2-a^2}} = \pm 2\sqrt{\mu^2-\omega^2}\,dt.$$

So that, through the integration:

$$\cosh^{-1}\frac{y-b}{a} = \pm 2\sqrt{\mu^2-\omega^2}\,t+\alpha \qquad (\alpha\text{: const.}),$$

the solution y can be put as (replace $\pm\alpha$ with α)

$$y = a\cosh(2\sqrt{\mu^2-\omega^2}\,t+\alpha)+b,$$

where the constants $a(a>0)$ and b satisfy $a^2-b^2=\Omega_2^2/(\mu^2-\omega^2)>0$, so that $y \geq a+b>0$. Accordingly, by the variable $\tau = 2\sqrt{\mu^2-\omega^2}\,t+\alpha$, the equation (6) with $\Omega_2 = \pm\sqrt{(a^2-b^2)(\mu^2-\omega^2)}$ leads to

$$2\frac{d\varphi}{d\tau} = \pm\frac{\sqrt{a^2-b^2}}{a\cosh\tau+b} \qquad (\Omega_2 \gtrless 0).$$

Moreover, by a change of variable $\phi=\tanh(\frac{1}{2}\tau)$, this equation is transformed into

$$\frac{d\varphi}{d\phi} = \pm\frac{\sqrt{(a+b)/(a-b)}}{\phi^2+(a+b)/(a-b)},$$

which is integrated:

$$\varphi = \pm\tan^{-1}\frac{(a-b)\phi}{\sqrt{a^2-b^2}}+k \qquad (k\text{: const.}).$$

Therefore the motion of the particle is determined:

$$r = e^{-\mu t}\sqrt{a\cosh(2\sqrt{\mu^2-\omega^2}\,t+\alpha)+b},$$

$$\varphi = \pm\tan^{-1}\frac{(a-b)\tanh(\sqrt{\mu^2-\omega^2}\,t+\frac{1}{2}\alpha)}{\sqrt{a^2-b^2}}+k \qquad (\Omega_2 \gtrless 0).$$

1.3. $\Omega_2 \neq 0$ and $\omega^2-\mu^2=0$. By putting

$$a=\sqrt{2\Omega_1}, \quad b=\frac{\Omega_2}{a}=\frac{\Omega_2}{\sqrt{2\Omega_1}},$$

the equation (7) is written as

$$\frac{xdx}{\sqrt{x^2-b^2}} = \pm a\,dt.$$

Then, through the integration

$$\sqrt{x^2 - b^2} = \pm\, at + \alpha,$$

x^2 can be put as (replace $\pm\,\alpha$ with α)

$$x^2 = (at + \alpha)^2 + b^2.$$

Accordingly, by a change of variable $\tau = at + \alpha$, the equation (6) with $\Omega_2 = ab$ is transformed into

$$\frac{d\varphi}{d\tau} = \frac{b}{\tau^2 + b^2},$$

which is integrated:

$$\varphi = \tan^{-1}\frac{\tau}{b} + k \qquad (k\text{: const.}).$$

Therefore the motion of the particle is determined:

$$r = e^{-\mu t}\sqrt{(at + \alpha)^2 + b^2},$$

$$\varphi = \tan^{-1}\frac{at + \alpha}{b} + k.$$

2. In the following case with $\Omega_2 = 0$, we leave the particular solution $r = 0$ of (1) and (2) out of consideration, since it means that the particle stays at the origin (center of force). Then (6) implies $\dot{\varphi} = 0$, i.e., the particle is moving straight towards the origin with coordinate $r(t)$ on the line. In this case, (1) leads to the equation of linearly damped one-dimensional harmonic oscillator. And the equation (7) is reduced to

$$\dot{x} = \pm\sqrt{-(\omega^2 - \mu^2)x^2 + 2\Omega_1}. \tag{8}$$

2.1. $\Omega_2 = 0$ and $\omega^2 - \mu^2 > 0$. Since $\Omega_1 > 0$ in the root of (8), by putting

$$a = \sqrt{\frac{2\Omega_1}{\omega^2 - \mu^2}},$$

the equation (9) leads to

$$\frac{dx}{\sqrt{a^2 - x^2}} = \pm\sqrt{\omega^2 - \mu^2}\, dt,$$

whose solution

$$\sin^{-1}\frac{x}{a} = \pm\sqrt{\omega^2-\mu^2}\,t+\alpha \ (\alpha\colon \text{const.})$$

is arranged in $r = e^{-\mu t}x$ to obtain (the minus sign is omitted as remarked in 1.1)

$$r = ae^{-\mu t}\sin(\sqrt{\omega^2-\mu^2}\,t+\alpha).$$

2.2. $\Omega_2 = 0$ and $\omega^2-\mu^2<0$. By putting

$$a = \sqrt{\frac{2|\Omega_1|}{\mu^2-\omega^2}},$$

the equation (8) leads to

$$\frac{dx}{\sqrt{x^2\pm a^2}} = \pm\sqrt{\mu^2-\omega^2}\,dt,$$

in which $\pm a^2$ correspond respectively to $\Omega_1 \gtrless 0$, while $a=0$ if $\Omega_1=0$. The respective integrations

$$\sinh^{-1}\frac{x}{a} = \pm\sqrt{\mu^2-\omega^2}\,t+\alpha \quad (\alpha\colon \text{const.}),\quad \text{if } \Omega_1>0;$$

$$\cosh^{-1}\frac{x}{a} = \pm\sqrt{\mu^2-\omega^2}\,t+\alpha \quad (\alpha\colon \text{const.}),\quad \text{if } \Omega_1<0;$$

$$x = \alpha e^{\pm\sqrt{\mu^2-\omega^2}\,t} \quad (\alpha\colon \text{const.}),\quad \text{if } \Omega_1=0;$$

are arranged respectively in $r = e^{-\mu t}x$ to obtain

$$r = \pm\, ae^{-\mu t}\sinh(\sqrt{\mu^2-\omega^2}\,t+\alpha),\quad \text{if } \Omega_1>0;$$

$$r = ae^{-\mu t}\cosh(\sqrt{\mu^2-\omega^2}\,t+\alpha),\quad \text{if } \Omega_1<0;$$

$$r = \alpha e^{-\mu t}e^{\pm\sqrt{\mu^2-\omega^2}\,t},\quad \text{if } \Omega_1=0.$$

2.3. $\Omega_2=0$ and $\omega^2-\mu^2=0$. In this case, from $\dot{x} = \pm\sqrt{2\Omega_1}$ immediately follows the solution

$$r = \pm\, e^{-\mu t}(\sqrt{2\Omega_1}\,t+\alpha) \qquad (\alpha\colon \text{const}).$$

Thus the motions of the particle in the considering central force problem are determined completely. In conclusion, the results are summalized:

THEOREM. *Let a single particle of mass* $m\,(m>0)$ *with polar coodinates* $(r(t),\ \varphi(t))$ *is moving under a central force* $\sigma r\,(\sigma\colon$ *const.*, $\sigma>0)$ *directed towards the origin in a resisting medium which imposes a retarding force equal to* $\beta\,(\beta\colon$ *const.*, $\beta>0)$ *times the* *velocity.*

When initial position (r_0, φ_0) *and velocity* $(\dot{r}_0, \dot{\varphi}_0)$ *of the particle are given, the conserved quantities* Ω_1 *and* Ω_2 *can be evaluated by substituting the data for* (3) *and* (4). *And then the motions of the particle are determined completely as follows, where* $K = (4m\sigma - \beta^2)/4m^2$ *and* $K \geq 0$ *implies that* $\Omega_1 > 0$.

In the case with $\Omega_2 \neq 0$, *i.e., the particle is moving out of the straight, then two-dimensional motions are determined as*

$$\left.\begin{aligned} r &= e^{-\mu t}\sqrt{a \sin(2\sqrt{K}t + \alpha) + b}, \\ \varphi &= \pm \tan^{-1}\frac{b\tan(\sqrt{K}t + \frac{1}{2}\alpha) + a}{\sqrt{b^2 - a^2}} + k \qquad (\Omega_2 \gtrless 0), \end{aligned}\right\} \quad \text{if } K > 0;$$

$$\left.\begin{aligned} r &= e^{-\mu t}\sqrt{a \cosh(2\sqrt{-K}t + \alpha) + b}, \\ \varphi &= \pm \tan^{-1}\frac{(a - b)\tanh(\sqrt{-K}t + \frac{1}{2}\alpha)}{\sqrt{a^2 - b^2}} + k \qquad (\Omega_2 \gtrless 0), \end{aligned}\right\} \quad \text{if } K < 0;$$

$$\left.\begin{aligned} r &= e^{-\mu t}\sqrt{(at + \alpha)^2 + b^2}, \\ \varphi &= \tan^{-1}\frac{at + \alpha}{b} + k, \end{aligned}\right\} \quad \text{if } K = 0;$$

where a and b are the constants:

$$a = \frac{\sqrt{\Omega_1^2 - K\Omega_2^2}}{K}, \quad b = \frac{\Omega_1}{K}, \qquad \text{if } K > 0;$$

$$a = \frac{\sqrt{\Omega_1^2 - K\Omega_2^2}}{-K}, \quad b = \frac{\Omega_1}{K}, \qquad \text{if } K < 0;$$

$$a = \sqrt{2\Omega_1}, \qquad b = \frac{\Omega_2}{\sqrt{2\Omega_1}}, \qquad \text{if } K = 0.$$

respectively; while the constants k and α *are specified by the initial data.*

Particularly in the case with $\Omega_2 = 0$, *i.e., the particle is moving straight towards the origin with a coordinate* $r(t)$ *on the line, then one-dimensional motions are determined as*

$$r = ae^{-\mu t}\sin(\sqrt{K}t + \alpha), \qquad \text{if } K > 0;$$

$$\left.\begin{aligned} r &= \pm\, ae^{-\mu t}\sinh(\sqrt{-K}t + \alpha) && (\Omega_1 > 0), \\ r &= ae^{-\mu t}\cosh(\sqrt{-K}t + \alpha) && (\Omega_1 < 0), \\ r &= \alpha e^{-\mu t} e^{\pm\sqrt{-K}t} && (\Omega_1 = 0), \end{aligned}\right\} \quad \text{if } K < 0;$$

$$r = \pm e^{-\mu t}(at + \alpha), \qquad \text{if } K = 0;$$

where a is the constant:

$$a = \sqrt{\frac{2\Omega_1}{K}}, \qquad \text{if } K > 0;$$

$$a = \sqrt{\mp \frac{2\Omega_1}{K}} \quad (\Omega_1 \gtrless 0), \qquad \text{if } K < 0;$$

$$a = \sqrt{2\Omega_1}, \qquad \text{if } K = 0;$$

respectively; while the constant α is specified by the initial data.

REMARK 1. In the case of $K < 0$ with $\Omega_2 \neq 0$, by replacing the constant α with $\pm \log(2\alpha^2/a)$ $(\Omega_2 \gtrless 0)$, we have the other appearance of r:

$$r = e^{-\mu t}\sqrt{\alpha^2 e^{\pm 2\sqrt{-K}t} + (a/2\alpha)^2 e^{\mp\sqrt{-K}t} + b} \qquad (\Omega_2 \gtrless 0).$$

REMARK 2. Let $\Omega_2 \to 0$ in the case with $\Omega_2 \neq 0$. Then, $a \to \Omega_1/K = b$ if $K > 0$, $a \to \mp \Omega_1/K = \mp b (\Omega_1 \gtrless 0)$ if $K < 0$ and $b \to 0$ if $K = 0$; accordingly the angle φ in each case of K converges to a constant. And, in view of that for $K > 0$ with $a = b = \Omega_1/K$:

$$a \sin(2\sqrt{K}t + \alpha) + b = \frac{2\Omega_1}{K}\sin^2(\sqrt{K}t + \tfrac{1}{2}\alpha + \tfrac{1}{4}\pi);$$

and for $K < 0$ with $a = \mp b = \mp \Omega_1/K$ $(\Omega_1 \gtrless 0)$:

$$a \cosh(2\sqrt{-K}t + \alpha) + b = -\frac{2\Omega_1}{K}\sinh^2(\sqrt{-K}t + \tfrac{1}{2}\alpha) \qquad (\Omega_1 > 0),$$

$$a \cosh(2\sqrt{-K}t + \alpha) + b = \frac{2\Omega_1}{K}\cosh^2(\sqrt{-K}t + \tfrac{1}{2}\alpha) \qquad (\Omega_1 < 0),$$

the particle's distance r in each case converges respectively to that (up to the sign) in each case with $\Omega_2 = 0$ except the case of $K < 0$ with $\Omega_1 = 0$. However we can avoid the peculiarity by means of the appearance of r in the remark 1 (the case of $K < 0$ with $\Omega_2 \neq 0$). In fact, for $a = \mp b = \mp \Omega_1/K$ $(\Omega_1 \gtrless 0)$, the terms in the root lead to

$$\alpha^2 e^{\pm 2\sqrt{-K}t} + \frac{a^2}{4\alpha^2} e^{\mp 2\sqrt{-K}t} + b = \left(\alpha e^{\pm\sqrt{-K}t} + \frac{\Omega_1}{2\alpha K} e^{\mp\sqrt{-K}t}\right)^2 \qquad (\Omega_2 \gtrless 0),$$

which turns into

$$a^2 \sinh^2(\sqrt{-K}t + \gamma), \qquad \text{if } \Omega_1 > 0,$$

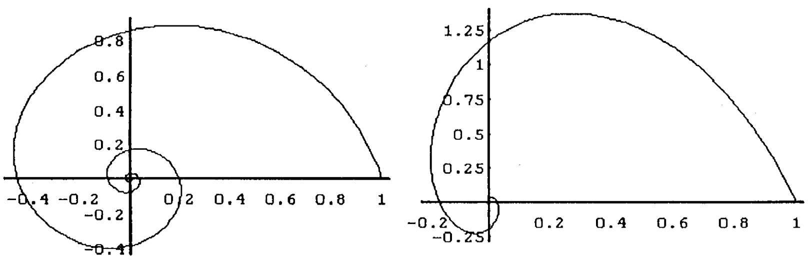
$$a^2 \cosh^2 (\sqrt{-K}\, t + \gamma), \qquad \text{if } \Omega_1 < 0,$$

where $a = \sqrt{\mp (2\Omega_1/k)}$ $(\Omega_1 \gtrless 0)$ and $\gamma = \pm \log (2\alpha/a)$ $(\Omega_2 \gtrless 0)$. Thus the respective motions with $\Omega_2 = 0$ can be regarded as the limiting case: $\Omega_2 \to 0$ of that with $\Omega_2 \neq 0$.

Let a single particle of mass $m = 1$, moving against a medium with retarding force constant $\beta = 2$, have the initial position $(r_0, \varphi_0) = (1, 0)$ and velocity $(\dot{r}_0, \dot{\varphi}_0) = (-1, 5)$. Then $\Omega_2 = 5$, and μ, ω, K, Ω_1 are determined according to the following central force constants σ:

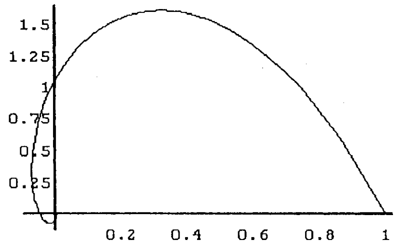
σ	β	μ	ω	K	Ω_1
16	2	1	4	15	20
4	2	1	2	3	14
2	2	1	$\sqrt{2}$	1	13
1	2	1	1	0	12.5
0.2	2	1	$1/\sqrt{5}$	-0.8	12.1
0	2	1	0	-1	12

For the values of σ and $\beta = 2$, the trajectories of the particle determined in the theorem are as follows.

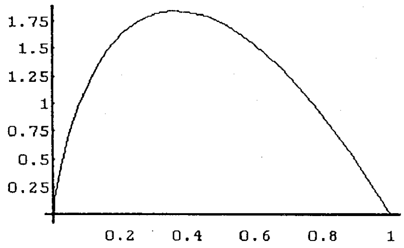


$\sigma = 16 \quad \beta = 2$ $\qquad$ $\sigma = 4 \quad \beta = 2$

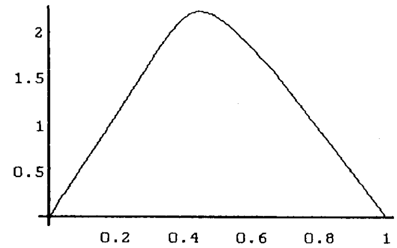
Figure: The motions of a particle under a central force



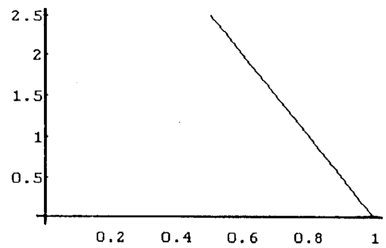
$\sigma=2 \quad \beta=2$



$\sigma=1 \quad \beta=2$



$\sigma=0.2 \quad \beta=2$



$\sigma=0 \quad \beta=2$

Figure (continued)

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