Bipancyclic properties of Cayley graphs generated by transpositions

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Abstract

Cycle is one of the most fundamental graph classes. For a given graph, it is interesting to find cycles of various lengths as subgraphs in the graph. The Cayley graph $\mathbf{Cay}(\mathfrak{S}_n, S)$ on the symmetric group has an important role for the study of Cayley graphs as interconnection networks. In this paper, we show that the Cayley graph generated by a transposition set is vertex-bipancyclic if and only if it is not the star graph. We also provide a necessary and sufficient condition for $\mathbf{Cay}(\mathfrak{S}_n, S)$ to be edge-bipancyclic.

Key words: Cayley graph, bipancyclicity, transposition tree

1 Introduction

A graph G with $n \geq 3$ vertices is called *pancyclic* if it contains an l-cycle for every $3 \leq l \leq n$, where l-cycle means a cycle of length l. We say that G is *vertex-pancyclic* if, for each vertex v of G and for every integer l with $3 \leq l \leq n$, there is an l-cycle that contains v. Furthermore, a graph is called *edge-pancyclic* if every edge lies on an l-cycle for every $3 \leq l \leq n$. Since a bipartite graph has no odd cycle, a bipartite graph with n vertices is *bipancyclic* if it has an l-cycle for every even $4 \leq l \leq n$. For bipartite graphs, *vertex-bipancyclicity* and *edge-bipancyclicity* are defined similarly. From these definitions, every edge-(bi)pancyclic graph is vertex-(bi)pancyclic, and every vertex-(bi)pancyclic graph is (bi)pancyclic.

An interconnection network is usually modeled by a graph, and pancyclicity is related to the cycle embedding problem. In order to use parallel or distributed algorithms for rings and paths, it is desired that a given network topology contains cycles of various lengths. Pancyclicity has been investigated for many interconnection networks such as arrangement graphs [5], pancake graphs [13], cube-connected cycles [7], butterfly graphs [10] and so on.

Cayley graphs have been an important class of graphs in the study of interconnection networks for parallel and distributed computing [1,4,9,12,15]. In particular, the star graph, which belongs to the class of Cayley graphs, has been widely studied as an interconnection network topology by many researchers. The star graph is defined as a Cayley graph with respect to the symmetric group, and its generator set is a subset of transpositions. The bubble-sort graph is also known as a class of Cayley graphs on the symmetric group and its generator set is a subset of transpositions. Pancyclicity for star graphs was studied in [11], and for bubble-sort graphs in [14].

This paper focuses on Cayley graphs generated by a subset of transpositions. This class of Cayley graphs includes the star graphs and bubble-sort graphs. Recently, some topological properties of this class have been studied [2,3]. We show that a Cayley graph generated by a transposition set is vertex-bipancyclic if and only if it is not a star graph. We also show a necessary and sufficient condition for such Cayley graphs to be edge-bipancyclic. Our results generalize the results of [11] and [14].

2 Preliminaries

Let Γ be a group and Δ a subset of Γ such that it does not include the identity element and $\Delta = \Delta^{-1}$. The *Cayley graph* of Γ with respect to Δ, denoted by $\mathbf{Cay}(\Gamma, \Delta)$, has vertex set Γ, and there is an edge $(\gamma, \gamma\tau)$ for each $\gamma \in \Gamma$ and $\tau \in \Delta$. The symmetric group on $\{1, 2, \ldots, n\}$ is denoted by $\mathfrak{S}_n$. A Cayley graph of the permutation group Γ generated by some subset of permutations S is denoted by $\mathbf{Cay}(\Gamma, S)$.

Let S be a subset of transpositions on $\{1, 2, \ldots, n\}$. The *transposition graph* of S, denoted by $T(S)$, is a graph with vertex set $\{1, 2, \ldots, n\}$ and two vertices i and j are adjacent if and only if the transposition (i, j) is in S. On the contrary, for a given graph G with vertex set $\{1, 2, \ldots, n\}$, we define the set $S(G)$ of transpositions by $S(G) = \{(i, j) \mid (i, j) \in E(G)\}$. A vertex x of $\mathbf{Cay}(\mathfrak{S}_n, S)$ is labeled by a string $x = x_1x_2 \cdots x_n$, and the ith label of x is denoted by x_i. By the definition of Cayley graphs, if a transposition (i, j) is in S, a vertex $x = x_1x_2 \cdots x_n$ is adjacent to $y = y_1y_2 \cdots y_n$ when $x_i = y_j$, $x_j = y_i$ and $x_k = y_k$ for all $k \neq i, j$. In this case, we say that the edge $e = (x, y)$ is an (i, j)-edge.

For an (i,j)-edge e and $k \neq i,j$, we denote by $e[k]$ the kth label of the two ends of e.

For a transposition set S which generates a permutation group, the following lemma holds:

Lemma 1 ([8],pp.52) *Let S be a set of transpositions from $\mathfrak{S}_n$. Then S is a generating set for $\mathfrak{S}_n$ if and only if $T(S)$ is connected.*

From the Lemma 1, this paper treats the Cayley graphs generated by transpositions which induce a connected graph.

Assume that S is a subset of transpositions, and (i,j) and (s,t) are disjoint transpositions. In other words, edges (i,j) and (s,t) have no common vertex in the transposition graph $T(S)$. Then any (i,j)-edge is contained some 4-cycle in the Cayley graph $\mathbf{Cay}(\mathfrak{S}_n, S)$. In fact, for an (i,j)-edge $e=(u,v)$ of $\mathbf{Cay}(\mathfrak{S}_n, S)$, there exists a 4-cycle $uvxy$, where u and v are adjacent to y and x by the (s,t)-edges, respectively. Then the edge $f=(x,y)$ is an (i,j)-edge. From this observation, for an (i,j)-edge e, the edge f is called the *coupled pair-edge* of e with respect to (s,t), and the pair of (s,t)-edges (u,y) and (v,x) are called the *coupler*. For two disjoint transpositions (i,j) and (s,t), the coupled pair-edge of e with respect to (s,t) is uniquely determined.

Let $uvxy$ be a 4-cycle and $e=(u,v)$ and $f=(x,y)$ be coupled pair-edges. Let C_1 and C_2 be cycles such that $V(C_1)\cap V(C_2)=\emptyset$ and C_1 and C_2 have the edge e and the edge f, respectively. We can construct a cycle of length $|V(C_1)|+|V(C_2)|$ from C_1 and C_2 by adding the coupler of e and f, and removing e and f. This procedure is said that C_1 and C_2 are *merged* by pair-edges e and f.

A sequence of cycles $C_1, C_2, \ldots, C_n$ is a *merge sequence* if $V(C_i)\cap V(C_j)=\emptyset$ for $i\neq j$ and there are coupled pair-edges $e_i \in C_i$ and $f_{i+1}\in C_{i+1}$ for all $1\leq i\leq n-1$. For a merge sequence, we can construct a cycle of length $\sum_{i=1}^{n}|C_i|$ by merging the cycles consecutively.

An edge e is a *pendant edge* if it is incident to an end-vertex. A tree T is a *star* $K_{1,n-1}$ if every two edges are adjacent. A tree T is a *double-star* if it contains exactly one non pendant edge. The following lemma guarantees that a transposition tree has a non-adjacent edge pair when it is not a star.

Lemma 2 *Let T be a tree with n vertices. Then, there is a pair of non-adjacent edges in T if and only if T is not a star $K_{1,n-1}$.*

If $T(S)$ is a tree, we say that it is a *transposition tree*. If $T(S)$ is a star $K_{1,n-1}$, then $\mathbf{Cay}(\mathfrak{S}_n, S)$ is called the *n-dimensional star graph* and is denoted by ST_n. If $T(S)$ is a path with n vertices, then $\mathbf{Cay}(\mathfrak{S}_n, S)$ is called the *n-dimensional* *bubble-sort graph* and is denoted by BS_n.

Pancyclicity of graphs are widely investigated by many researchers. The following lemmas are known results for pancyclicity of the Cayley graphs generated by transpositions.

Lemma 3 ([11]) *Every edge in the star graph ST_n is contained in an l-cycle for every even $6 \leq l \leq n!$.*

Lemma 4 ([14]) *For $n \geq 4$, the bubble-sort graph BS_n is vertex-bipancyclic.*

Lemma 5 ([14]) *For $n \geq 5$, the bubble-sort graph BS_n is edge-bipancyclic.*

A bipartite graph is *hamiltonian laceable* if it has a Hamiltonian path between any pair of vertices in different partite sets.

Theorem 6 ([16]) *For any transposition tree T, the Cayley graph $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ is Hamiltonian laceable for $n \geq 4$.*

The next theorem indicates that the labeling of the transposition graph does not affect the structure of the Cayley graph generated by the transposition graph.

Theorem 7 ([6]) *Let S and S' be two sets of transpositions on $\{1, 2, \dots, n\}$. The Cayley graphs $\mathbf{Cay}(\mathfrak{S}_n, S)$ and $\mathbf{Cay}(\mathfrak{S}_n, S')$ are isomorphic if and only if $T(S)$ and $T(S')$ are isomorphic.*

From the definition of $\mathbf{Cay}(\mathfrak{S}_n, S)$ and Theorem 7, we obtain the following result.

Proposition 8 *Let T be a graph with vertex set $\{1, 2, \dots, n\}$. We choose an end-vertex t, and let r be the vertex adjacent to t. For $i = 1, 2, \dots, n$, let V_i be the set of vertices of $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ such that $v_t = i$. Then, each induced subgraphs $\langle V_i \rangle$ are isomorphic to $\mathbf{Cay}(\mathfrak{S}_{n-1}, S')$, where $S' = S(T) \setminus \{(r, t)\}$.*

Theorem 9 *Let T be a transposition tree on $\{1, 2, \dots, n\}$ and suppose that (r, t) is a pendant edge of T. For any $k \leq n$, if there exists a cycle C of length at least $(k-1)(n-1)! + 2$ in $\mathbf{Cay}(\mathfrak{S}_n, S(T))$, then C contains at least k edges generated by the transposition (r, t).*

PROOF. Without loss of generality, we may assume t is an end-vertex in T. Since (r, t) is a pendant edge, we can partition $V(\mathbf{Cay}(\mathfrak{S}_n, S(T)))$ into $V_1 \cup V_2 \cup \cdots \cup V_n$, such that $v \in V_i$ if $v_t = i$ for each $1 \leq i \leq n$. We assume that there exists a cycle C of length $l \geq (k-1)(n-1)! + 2$. Since C has $l \geq (k-1)(n-1)! + 2$ vertices, C must include at least k vertices such that each vertex belongs to k distinct partitions. To connect two vertices belonging to different partitions $V_i, V_j, i \neq j$, we must use (r,t)-edges since all edges that connect vertices belonging to different partitions are (r,t)-edges. Therefore, C contains at least k (r,t)-edges. $\square$

3 Hamilton cycles in Cayley graphs

We introduce the notion of 2-edge hamiltonian. This property is very important to construct desired cycles in Section 4. A graph is *2-edge hamiltonian* if, for any two edges, it has a Hamilton cycle that contains these two edges. In this section, we show that Cayley graphs $\mathbf{Cay}(\mathfrak{S}_n, S)$ are 2-edge hamiltonian. We consider first the case of the star graphs, after that we consider the cases of other Cayley graphs.

Lemma 10 *The star graph ST_n is 2-edge hamiltonian for $n \geq 3$.*

PROOF. Prove by induction on n. When $n = 3$, ST_3 is isomorphic to a 6-cycle.

When $n = 4$, from the symmetry of the star graph, it is sufficient to show that there exists a set of Hamilton cycles $C_1, C_2, \ldots, C_k$ for some k such that every Hamilton cycle includes an edge $(1234, 2134)$ and $\bigcup_{i=1}^{k} E(C_i)$ includes all $(1,2)$-edges and $(1,3)$-edges. The following two Hamilton cycles C_1 and C_2 can form a desired set.

$$\begin{aligned} C_1 = {} & 1234, 2134, 3124, 4123, 1423, 2413, 4213, 1243, \\ & 2143, 3142, 4132, 1432, 3412, 4312, 1342, 2341, \\ & 3241, 4231, 2431, 3421, 4321, 1324, 2314, 3214, \end{aligned}$$

and

$$\begin{aligned} C_2 = {} & 1234, 2134, 3124, 1324, 2314, 3214, 4213, 1243, \\ & 2143, 4123, 1423, 2413, 3412, 4312, 1342, 3142, \\ & 4132, 1432, 2431, 3421, 4321, 2341, 3241, 4231. \end{aligned}$$

Assume that the statement holds for some integer $k \geq 4$. Let α, β be edges in ST_{k+1}. By the symmetry of the star $K_{1,k}$, we may assume that α is a $(1,2)$-edge and if the generators of α and β are the same, β can be a $(1,2)$-edge. Otherwise, we may assume that β is a $(1,3)$-edge.

Let $V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ be a partition of $V(ST_{k+1})$ such that $v \in V_i$ if $v_{k+1} = i$. From Proposition 8, each induced subgraph $\langle V_i \rangle$ is isomorphic to ST_k. If $\alpha[k+1] = \beta[k+1]$ in ST_{k+1}, by the induction hypothesis, there is a Hamilton cycle C

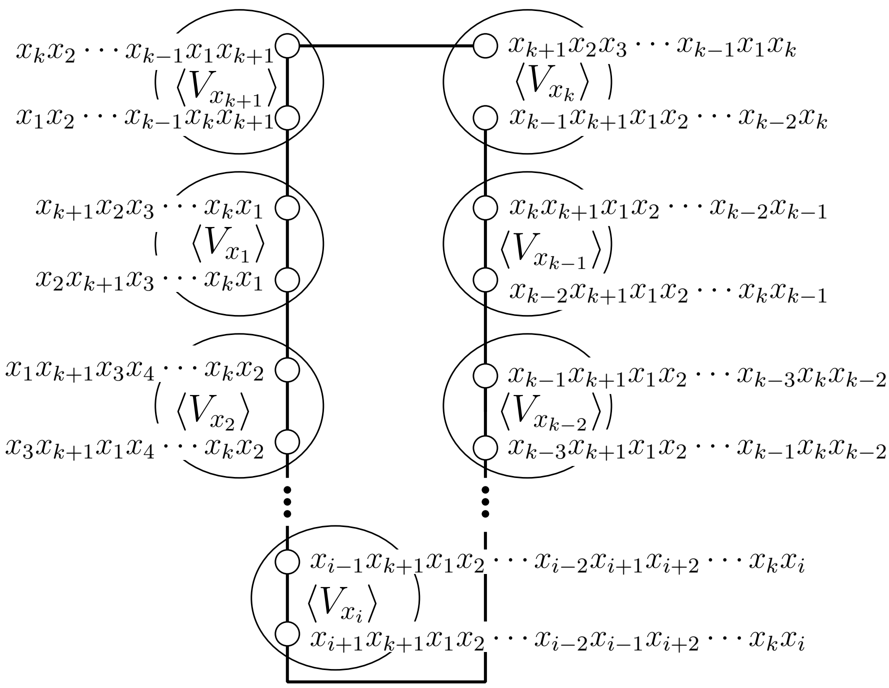

Fig. 1. A path P in ST_{k+1}

that contains edges α and β in $\langle V_{\alpha[k+1]} \rangle$. From the cycle C, we choose a $(1,k)$-edge $x = (x_1 x_2 \cdots x_{k+1}, x_k x_2 x_3 \cdots x_{k-1} x_1 x_{k+1})$ such that $x[k+1] = \alpha[k+1]$, where x is neither α nor β. By Theorem 9, such edge exists.

A path P on ST_{k+1} that contains an edge x shown in Figure 1 can be defined based on the edge x.

The path P contains exactly one edge e_{x_i} from each $\langle V_{x_i} \rangle$ where $1 \leq i \leq k-1$. By the induction hypothesis, there exists a Hamilton cycle in each $\langle V_{x_i} \rangle$ that includes e_{x_i}. A path P' of $k \times k! + 2$ vertices can be obtained by replacing e_{x_i} by the corresponding Hamilton cycle in $\langle V_{x_i} \rangle$ for $1 \leq i \leq k-1$ and for $\langle V_{x_{k+1}} \rangle$, we choose a cycle C as a Hamilton cycle.

Both end-vertices of P' are included in $\langle V_{x_k} \rangle$ and no internal vertices of P' is contained in $\langle V_{x_k} \rangle$. Since the distance between two end-vertices of P' in $\langle V_{x_k} \rangle$ is odd and by Theorem 6, there exists a path Q that contains all vertices in $\langle V_{x_k} \rangle$ and end-vertices are the same as end-vertices of P'. Connecting P' and Q leads to a Hamilton cycle in ST_{k+1} that contains edges α and β.

Let $\alpha[k+1] \neq \beta[k+1]$ in ST_{k+1}. If β is a $(1,3)$-edge, we choose a $(1,k)$-edge x such that $x_1 = \beta[k+1]$ in $\langle V_{\alpha[k+1]} \rangle$. In other case, that is, if β is a $(1,2)$-edge, we choose a $(1,k)$-edge x such that $x_2 = \beta[k+1]$. By the induction, there exist such edges and a Hamilton cycle in $\langle V_{x_{k+1}} \rangle$ that includes edges α and x. Then, make a path P similar to the case $\alpha[k+1] = \beta[k+1]$.

To complete the proof, we must show that β is not contained in P. If β is contained in P and β is a $(1,3)$-edge, then x_2 must be $\beta[k+1]$. However, we assumed $x_1 = \beta[k+1]$ and therefore a contradiction occurs. Similarly, another case contradicts our assumption, thus β is not contained in P.

We will pay attention to the construction of a Hamilton cycle in $\langle V_{\beta[k+1]}\rangle$. By induction hypothesis, we can obtain a Hamilton cycle in $\langle V_{\beta[k+1]}\rangle$ that contains an edge β and the corresponding edge in P. A Hamilton path Q in $\langle V_{x_k}\rangle$ exists and therefore a Hamilton cycle in ST_{k+1} that contains edges α and β is obtained. $\square$

Theorem 11 *Let T be a tree with the vertex set $\{1, 2, \ldots, n\}$. Then $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ is 2-edge hamiltonian for $n \geq 3$.*

PROOF. If T is a star, the theorem follows from Lemma 10. If T is a path, it was already proved that BS_n is 2-edge hamiltonian [14]. Hence we assume that T is neither a star nor a path.

Prove by induction on n. When $n = 3$, a tree of three vertices is a path (and a star). When $n = 4$, a tree of four vertices is either a star or a path. For some positive integer $n = k \geq 4$, we assume that the statement holds, and consider the case $n = k + 1$.

Let T be a tree of $k + 1$ vertices such that it is neither a star nor a path. Let $\alpha = (u, v)$ and $\beta = (x, y)$ be any edges in $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$. Without loss of generality, we may assume that $u = 12\cdots k(k + 1)$.

Since T is not a path, it has at least 3 pendant edges. Thus there is at least one pendant edge e such that it corresponds to the generators of neither α nor β. From Theorem 7, we can assume that $e = (k, k + 1)$, where $k + 1$ is an end-vertex of T. Moreover, since T is not a star, there is at least one end-vertex which is not adjacent to the vertex k. We assume that one of those vertex is 1 and a vertex adjacent to 1 is 2. Other vertices in T are labeled arbitrarily. As a result, the tree T has pendant edges $(1, 2)$ and $(k, k + 1)$ and $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ has $(1, 2)$-edges and $(k, k + 1)$-edges.

Let $V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ be the partition of the vertex set of $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ such that $v \in V_i$ if $v_{k+1} = i$. By Proposition 8, the induced subgraph $\langle V_i\rangle$ is isomorphic to $\mathbf{Cay}(\mathfrak{S}_k, S(T) \setminus \{(k, k + 1)\}$.

To construct a Hamilton cycle, we choose some edges by the following procedure.

(1) For $i = 1, 2, \ldots, k+1$, let e_i be a $(1, 2)$-edge in $\langle V_i\rangle$ such that $e_1[k] = k+1$ and $e_i[k] = i - 1$ for $i \geq 2$.

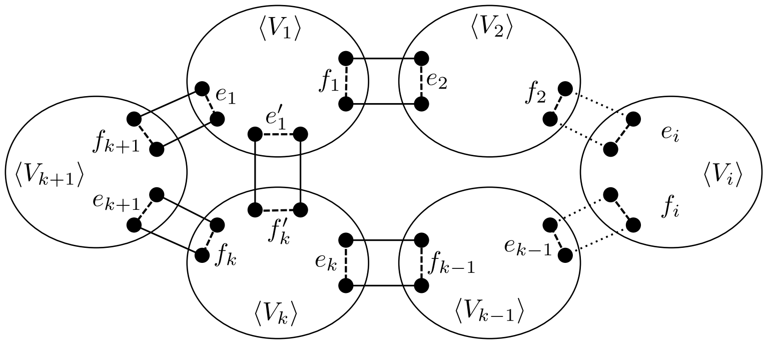

Fig. 2. Edges e_i and f_i in $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$.

(2) For $i = 1, 2, \ldots, k$, let f_i be the coupled pair-edge of e_{i+1} with respect to $(k, k+1)$, and for $i = k+1$, f_{k+1} the coupled pair-edge of e_1 with respect to $(k, k+1)$. By the definition of e_i, f_i is a $(1, 2)$-edge in $\langle V_i\rangle$.

(3) Let e_1' be a $(1, 2)$-edge in $\langle V_1\rangle$ such that $e_1'[k] = k$, and let f_k' be the coupled pair-edge of e_1' with respect to $(k, k+1)$.

Figure 2 illustrates the $2k+4$ edges in the subgraphs $\langle V_i\rangle, i = 1, 2, \cdots, k+1$.

Since the transposition $(k, k+1)$ is not a generator of α and β, $u_{k+1} = v_{k+1}$ and $x_{k+1} = y_{k+1}$. Hence it is sufficient to prove two cases for $u_{k+1} = x_{k+1}$ and $u_{k+1} \neq x_{k+1}$.

Case1: $u_{k+1} = x_{k+1}$

In this case, the edges α and β are in $\langle V_{k+1}\rangle$. By the induction hypothesis, there is a Hamilton cycle C_{k+1} of $\langle V_{k+1}\rangle$ that contains edges α and β.

Since $k+1 \geq 5$ and Theorem 9, C_{k+1} has a $(1, 2)$-edge f_{k+1}' that is neither α nor β. Let $p = f_{k+1}'[k]$ and e_p' be the coupled pair-edge of f_{k+1}' with respect to $(k, k+1)$ in $\langle V_p\rangle$. In $\langle V_p\rangle$, there exists a Hamilton cycle C_p that contains edges e_p' and f_p.

For $i \in \{1, 2, \ldots, k+1\} \setminus \{1, k+1, p, k\}$, $\langle V_i\rangle$ has a Hamilton cycle C_i in $\langle V_i\rangle$ such that it contains e_i and f_i. If $p \neq 1$, let C_1 be a Hamilton cycle in $\langle V_1\rangle$ that contains e_1' and f_1. If $p \neq k$, let C_k be a cycle in $\langle V_k\rangle$ that contains e_k and f_k'.

For cycles C_i for $1 \leq i \leq k+1$, we define a merge sequence as follows:

$$C_{k+1}, C_p, C_{p+1}, \ldots, C_k, C_1, C_2, \ldots, C_{p-1},$$

and thus we can construct a Hamilton cycle of $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ that contains α and β. Figure 3 shows the construction of a Hamilton cycle when $\alpha[k+1] = \beta[k+1]$.

Case2: $u_{k+1} \neq x_{k+1}$

Let $x_{k+1} = y_{k+1} = t \neq k+1$. If $t = 1$, from the induction hypothesis, there exists a Hamilton cycle C_{k+1} in $\langle V_{k+1}\rangle$ that contains α and e_{k+1}. For $2 \leq i \leq k$, we can obtain Hamilton cycles C_i including edges e_i and f_i. A Hamilton cycle C_1 in $\langle V_1\rangle$ contains edges β and f_1 which is a coupled

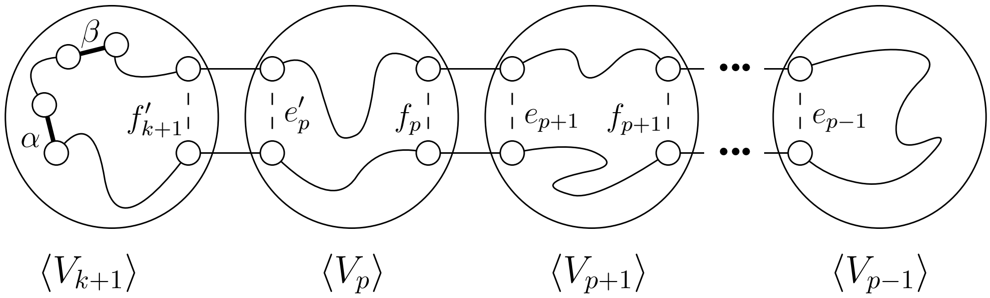

Fig. 3. A Hamilton cycle in $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ that contains given edges α and β.

pair-edge of e_2 with respect to $(k, k+1)$. A merge sequence is defined as follows:

$$C_1, C_2, \ldots, C_k, C_{k+1}.$$

If $t = k$, from the induction hypothesis, there exists a Hamilton cycle C_{k+1} in $\langle V_{k+1}\rangle$ that contains α and f_{k+1}. For $2 \leq i \leq k-1$, we can obtain Hamilton cycles C_i including edges e_i and f_i. A Hamilton cycle C_1 in $\langle V_1\rangle$ that contains edges e_1 and f_1 which are coupled pair-edges of f_{k+1} and e_2, respectively, with respect to $(k, k+1)$. A Hamilton cycle C_k in $\langle V_k\rangle$ that contains edges β and e_k which is a coupled pair-edge of f_{k-1} with respect to $(k, k+1)$. A merge sequence is defined as follows:

$$C_{k+1}, C_1, C_2, \ldots, C_k.$$

If $t \neq 1$ and $t \neq k$, we can obtain a desired Hamilton cycle by the following cycles.

- Let γ be a $(1,2)$-edge in $\langle V_{t+1}\rangle$ such that $\gamma[k] = t-1$, and γ' be the coupled pair-edge of γ with respect to $(k, k+1)$ in $\langle V_{t-1}\rangle$. By the induction hypothesis, there is a Hamilton cycle C_{t+1} of $\langle V_{t+1}\rangle$ that contains edges f_{t+1} and γ, and a Hamilton cycle C_{t-1} of $\langle V_{t-1}\rangle$ that contains e_{t-1} and γ'.
- Let δ be a $(1,2)$-edge in $\langle V_t\rangle$ such that $\delta[k] = 1$ and it is different from β (by Theorem 9, such edge exists). By the induction hypothesis, there is a Hamilton cycle C_t of $\langle V_t\rangle$ that contains β and δ.
- Let δ' be the coupled pair-edge of δ with respect to $(k, k+1)$ in $\langle V_1\rangle$. By the induction hypothesis, there is a Hamilton cycle C_1 of $\langle V_1\rangle$ that contains f_1 and δ'.

Then we can define a merge sequence $C_{k+1}, C_k, \ldots, C_{t+1}, C_{t-1}, C_{t-2}, \ldots, C_1, C_t$. Figure 4 shows the constructed Hamilton cycle of $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ that contains α and β when $\alpha[k+1] \neq \beta[k+1]$. □

When the transposition graph G is not a tree, we consider the spanning tree that includes two edges that corresponds to the transpositions of edges in the Cayley graph. Then from Theorem 11, the following result is obtained.

Corollary 12 *Let G be a connected graph with vertex set $\{1, 2, \ldots, n\}$. Then $\mathbf{Cay}(\mathfrak{S}_n, S(G))$ is 2-edge hamiltonian for $n \geq 3$.*

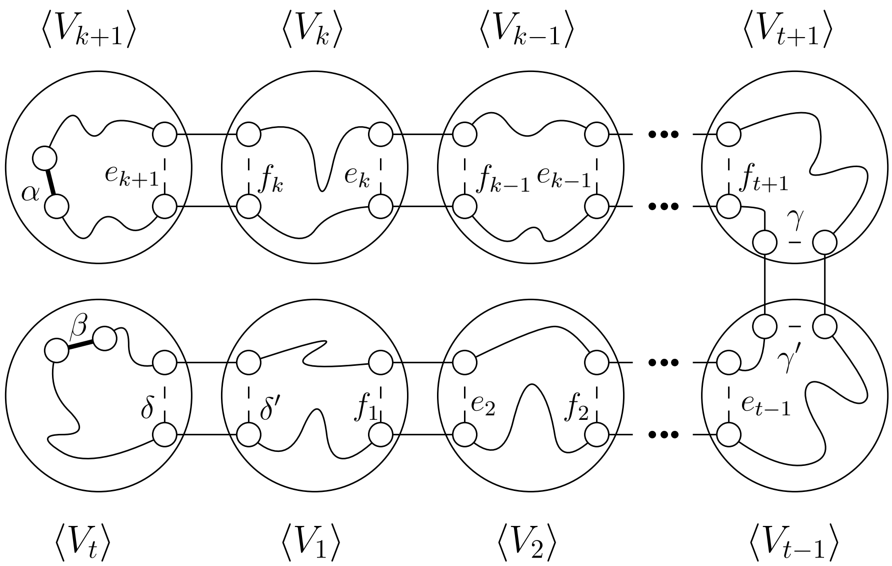

Fig. 4. A Hamilton cycle in $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$ that contains given edges α, β.

4 Bipancyclicity of Cayley graphs

This section shows that the bipancyclic property of the Cayley graph generated by a transposition set. Now we prove the case when the transposition graph is a tree. When the transposition graph is not a tree, it is sufficient to prove the bipancyclic property by considering a spanning tree of the transposition graph.

Theorem 13 *For a tree T with vertex set $\{1, 2, \ldots, n\}$ such that $n \geq 3$, every edge lies on an even cycle of length $l \geq 6$ in the Cayley graph $\mathbf{Cay}(\mathfrak{S}_n, S(T))$.*

PROOF. Since every Cayley graph is vertex-transitive, it is sufficient to prove for $n-1$ edges incident to the vertex $x = 123\cdots n$. Prove by induction on n. For $n = 3$, $\mathbf{Cay}(\mathfrak{S}_3, S(T))$ is ST_3 and forms a 6-cycle. For $n = 4$, the tree with 4 vertices is either a star or a path. In both cases, the statement has been shown in [11] and [14], respectively. We assume that the statement holds for all $n \leq k$. Let T be a tree with $k+1$ vertices. If T is a star $K_{1,k}$, then the statement holds from Lemma 3. We consider other cases. We choose an end-vertex in T and put it a label $k+1$. The vertex adjacent to the vertex $k+1$ is labeled by k. Since T is not a star, there is a pendant edge that is not incident to the vertex k. The end-vertex of such edge may be labeled by 1 and another vertex that is incident to the edge may be labeled by 2. Other vertices in T are labeled arbitrarily. By those labeling, $(1, 2)$-edges and $(k, k+1)$-edges are in $\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T))$.

The vertex set $V(\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T)))$ is partitioned into $V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ such that $v \in V_i$ if $v_{k+1} = i$. From Proposition 8, each $\langle V_i\rangle$ is isomorphic to the Cayley graph $\mathbf{Cay}(\mathfrak{S}_k, S(T) \setminus \{(k, k+1)\})$. By the induction hypothesis, for each edge e adjacent to x except $(k, k+1)$-edge is contained in an l-cycle, $6 \le l \le k!$.

Now we suppose that $k! + 2 \le l \le (k+1)!$. Let $l = qk! + r$ where $1 \le q \le k+1$, $0 \le r < k!$. Although the method of constructing cycles described below can not handle the case when $r = 4$, therefore we show that case after the general method is shown.

Let C_{k+1} be a Hamilton cycle in $\langle V_{k+1} \rangle$ that contains edges e and e_{k+1} as mentioned in Theorem 11. Similarly, let $C_k, C_{k-1}, \ldots, C_{k-q+2}$ be Hamilton cycles in $\langle V_k \rangle, \langle V_{k-1} \rangle, \ldots, \langle V_{k-q+2} \rangle$, respectively, such that each cycle C_i contains f_i and e_i. From Theorem 11, those cycles exist. By the assumption, there exists a r-cycle C_{k-q+1} that contains an edge f_{k-q+1} in $\langle V_{k-q+1} \rangle$ where $r \ge 6$. By the case $r = 2$, we choose an edge f_{k-q+1} as C_{k-q+1}. By merging cycles $C_{k+1}, C_k, C_{k-1}, \ldots, C_{k-q+2}, C_{k-q+1}$, we can construct a cycle of length $l = qk! + r$ where $r \ne 4$ that contains an edge e.

Next, we show the existence of cycles of length $l = qk!+4$ where $1 \le q < k+1$. Let C_{k+1} be a $(k! - 2)$-cycle which contains an edge e. By the induction hypothesis, such cycle exists. From Theorem 9, there exists a $(1, 2)$-edge f different from e in C_{k+1}. Let $p = f[k]$ and e_p be a coupled pair-edge of f with respect to $(k, k+1)$ in $\langle V_p \rangle$.

(1) If $q \le k+1-p$, let $C_p, C_{p+1}, \ldots, C_{p+q-2}$ be Hamilton cycles in $\langle V_p \rangle, \langle V_{p+1} \rangle, \ldots, \langle V_{p+q-2} \rangle$, respectively, such that each cycle C_i contains f_i and e_i as mentioned in Theorem 11, and C_{p+q-1} be a 6-cycle which contains e_{p+q-1} in $\langle V_{p+q-1} \rangle$. By the induction hypothesis and Theorem 11, those cycles exist. By merging cycles $C_{k+1}, C_p, C_{p+1}, \ldots, C_{p+q-1}$, we can construct a $(qk! + 4)$-cycle that contains an edge e.

(2) If $q > k+1-p$, let $C_p, C_{p+1}, \ldots, C_{k-1}, C_2, C_3, \ldots, C_{q-k+p-2}$ be Hamilton cycles in $\langle V_p \rangle, \langle V_{p+1} \rangle, \ldots, \langle V_{k-1} \rangle, \langle V_2 \rangle, \langle V_3 \rangle, \ldots, \langle V_{q-k+p-2} \rangle$, respectively, such that each cycle C_i contains f_i and e_i as mentioned in Theorem 11, and C_k be a Hamilton cycle in $\langle V_k \rangle$ which includes edges f'_k and e_k. Let C_1 be a Hamilton cycle in $\langle V_1 \rangle$ which includes edges e'_1 and f_1 and $C_{q-k+p-1}$ be a 6-cycle which contains $e_{q-k+q-1}$ in $\langle V_{q-k+p-1} \rangle$. By the induction hypothesis and Theorem 11, those cycles exist. By merging cycles $C_{k+1}, C_p, C_{p+1}, \ldots, C_k, C_1, C_2, \ldots, C_{q-k+p-1}$, we can construct a $(qk!+4)$-cycle that contains an edge e.

For a $(k, k+1)$-edge that is adjacent to x, we partition the vertex set $V(\mathbf{Cay}(\mathfrak{S}_{k+1}, S(T)))$ into $V_1 \cup V_2 \cup \cdots \cup V_{k+1}$ such that $v \in V_i$ if $v_1 = i$, and apply the above discussion similarly. Therefore, every edge in $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ lies on a cycle of length $l \ge 6$. $\square$

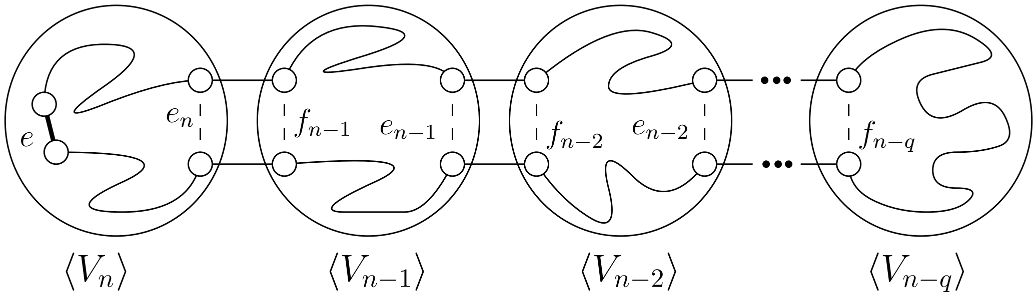

Fig. 5. A $l = q(n-1)! + r$ cycle in $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ that contains an edge e.

Figure 5 shows a cycle of length $l = q(n-1)! + r$, $r \neq 4$ that contains an edge e in $\mathbf{Cay}(\mathfrak{S}_n, S(T))$.

By Theorem13, Lemma 3 and Lemma 5, we obtain the following theorem.

Theorem 14 *If a tree T is neither a star nor a double-star, the Cayley graph $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ is edge-bipancyclic.*

PROOF. Let T be a tree with n vertices, neither a star nor a double-star. Then, for any edge e in T, there exists an edge disjoint from e. It means that for any transposition in $S(T)$, there exists a disjoint transposition, and any edge lies on a 4-cycle derived from such pair of disjoint transpositions. From Theorem 13, we conclude that the statement holds. □

For any pendant edge in double-star, it is easy to verify that there exists an edge disjoint from such edge. From this fact, the following corollary is obtained.

Corollary 15 *Let T be a double-star with n vertices. Then, any edge in $\mathbf{Cay}(\mathfrak{S}_n, S(T))$ generated by a transposition corresponding to some pendant edge in T is contained in even cycle.*

The rest of this section considers the case when the transposition graph is not a tree. Before we state the main theorem, we show an useful lemma.

Lemma 16 *$\mathbf{Cay}(\mathfrak{S}_3, S(K_3))$ is edge-bipancyclic.*

PROOF. $\mathbf{Cay}(\mathfrak{S}_3, S(K_3))$ is isomorphic to complete bipartite graph $K_{3,3}$ and the desired result follows. □

We have the following characterizations of the bipancyclicity of the Cayley graph generated by a transposition set.

Theorem 17 *Let G be a connected graph with $V(G) = \{1, 2, \ldots, n\}$. The Cayley graph* $\mathbf{Cay}(\mathfrak{S}_n, S(G))$ *is edge-bipancyclic if and only if G is neither a star nor a double-star.*

PROOF. From Theorem 13, it is sufficient to show that there exists a 4-cycle which includes the given edge in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$. First, we show the sufficiency of the statement.

Let G be a connected graph with n vertices, neither a star nor a double-star. Let e be a (u, v)-edge in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$. Since G is neither a star nor a double-star, if (u, v) does not lie on some cycle in G, there exists an edge f that is not adjacent to the edge e. Let (u, v) lie on some cycle in G. Then if the length of the cycle is at least 4, there exists an edge f. If the length of the cycle is 3, from Lemma 16, e is contained in some 4-cycle in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$. The necessity is shown by its contrapositive.

If G is a star, $\mathbf{Cay}(\mathfrak{S}_n, S(G))$ does not contain any 4-cycle. If G is a double-star, edges in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$ generated by the transposition which corresponds to the edge in G which connects two non end-vertices are not contained in the 4-cycle. □

Theorem 18 *Let G be a connected graph with $V(G) = \{1, 2, \ldots, n\}$. The Cayley graph* $\mathbf{Cay}(\mathfrak{S}_n, S(G))$ *is vertex-bipancyclic if and only if G is not a star.*

PROOF. From Theorem 17, it is sufficient to show the case when G is a double-star. From Lemma 2, for any vertex v in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$, there exists a 4-cycle that contains v. On the contrary, if there exists a 4-cycle in $\mathbf{Cay}(\mathfrak{S}_n, S(G))$, then there exists a pair of disjoint edges in G. If a graph G is not a star, such pair exists in G. □

From Theorem 17 and Theorem 18, we obtain the next results for the transposition set.

Corollary 19 *Let S be a transposition set on $\{1, 2, \ldots, n\}$. Then the Cayley graph* $\mathbf{Cay}(\mathfrak{S}_n, S)$ *is edge-bipancyclic if and only if the transposition graph $T(S)$ is neither a star nor a double-star.*

Corollary 20 *Let S be a transposition set on $\{1, 2, \ldots, n\}$. Then the Cayley graph* $\mathbf{Cay}(\mathfrak{S}_n, S)$ *is vertex-bipancyclic if and only if the transposition graph $T(S)$ is not a star.*

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