EASY ESTIMATION BY A NEW PARAMETERIZATION FOR THE THREE-PARAMETER LOGNORMAL DISTRIBUTION

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A new parameterization and algorithm are proposed for seeking the primary relative maximum of the likelihood function in the three-parameter lognormal distribution. The parameterization yields the dimension reduction of the three-parameter estimation problem to a two-parameter estimation problem on the basis of an extended lognormal distribution. The algorithm provides the way of seeking the profile of an object function in the two-parameter estimation problem. It is simple and numerically stable because it is constructed on the basis of the bisection method. The profile clearly and easily shows whether a primary relative maximum exists or not, and also gives a primary relative maximum certainly if it exists.

Keywords: Extended lognormal distribution; Dimension reduction; Primary relative maximum; Local maximum likelihood estimate; Embedded problem

1 INTRODUCTION

The three-parameter lognormal distribution is one of the most important distributions in many fields. With a variable x and three parameters α, β and γ, the probability density function is expressed by

$$f(x;\alpha,\beta,\gamma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}(x-\alpha)\beta}\exp\left[-\frac{\{\ln((x-\alpha)/\gamma)\}^2}{2\beta^2}\right], \quad x>\alpha,\ \beta>0,\ \gamma>0 \qquad (1.1)$$

and the likelihood function is expressed by $L(\alpha,\beta,\gamma) \stackrel{\text{def}}{=} \prod_{i=1}^{n} f(x_i;\alpha,\beta,\gamma)$. Here, x_i $(1 \leq i \leq n)$ stand for independent observations. Without loss of generality, we assume $x_1 > x_2 \geq \cdots \geq x_{n-1} > x_n$.

Since $\ln(X-\alpha)$ obeys a normal distribution if a random variable X obeys a lognormal distribution, $L(\alpha,\beta,\gamma)$ achieves its maximum at a point $(\alpha_0, \hat{\beta}(\alpha_0), \hat{\gamma}(\alpha_0))$ provided that α

is fixed to α_0, where

$$\hat{\beta}(\alpha) \stackrel{\text{def}}{=} \sqrt{\frac{1}{n}\sum_{i=1}^{n}\{\ln(x_i-\alpha)-\ln\hat{\gamma}(\alpha)\}^2} \quad \text{and} \quad \hat{\gamma}(\alpha) \stackrel{\text{def}}{=} \exp\left[\frac{1}{n}\sum_{i=1}^{n}\ln(x_i-\alpha)\right].$$

Consequently, if we want to obtain the maximum likelihood estimate, it suffices to find an α such that $\hat{L}(\alpha) \stackrel{\text{def}}{=} L(\alpha, \hat{\beta}(\alpha), \hat{\gamma}(\alpha))$ achieves its maximum. However, because that $\hat{L}(\alpha) \to \infty$ as $\alpha \to x_n - 0$, $L(\alpha, \beta, \gamma)$ becomes unbounded. Furthermore, the other parameters then lead to inadmissible values.

Hill (1963), using the Bayes theorem, has given a statistical implication of $(\hat{\alpha}, \hat{\beta}(\hat{\alpha}), \hat{\gamma}(\hat{\alpha}))$ at which $L(\alpha, \beta, \gamma)$ has its maximum in the region except the singular region; for a small $\delta > 0$, at $\alpha = \hat{\alpha}$ has $\hat{L}(\alpha)$ its relative and absolute maximum under the condition $x_n - \alpha > \delta$. The point $(\hat{\alpha}, \hat{\beta}(\hat{\alpha}), \hat{\gamma}(\hat{\alpha}))$ is used instead of the maximum likelihood estimate, and it is called the primary relative maximum (PRM) or the local maximum likelihood estimate of the likelihood function.

Displaying $\hat{L}(\alpha)$ is an effective way for finding $\hat{\alpha}$, but the search may be difficult because that the shape of $\hat{L}(\alpha)$ is complicated in some cases depending on data sets (Cheng and Iles, 1990; Hill, 1963; Johnson, Kotz and Balakrishnan, 1994). For example, $\hat{L}(\alpha)$ for Data 3 (Table I) increases very slowly as α becomes small, furthermore it begins to oscillate when α becomes sufficiently small due to numerical errors (Fig. 1). On the other hand, $\hat{L}(\alpha)$ for Data 4 (Table 1) attains its maximum at a point which is closely near to the end-point of the domain of definition (Fig. 2). In this case it is necessary to magnify carefully the graph to a big scale not to miss its maximum. Besides, if an iterative solver like Newton's method is used to find the maximum, some difficulties can happen. This is also one of the reasons why many researchers tackled this estimation problem.

Here is an approximate position for Figure 1

Here is an approximate position for Figure 2

For problems to seek $\hat{\alpha}$, that is, the one-parameter estimation problems for the lognormal distribution, Wingo (1975, 1976, 1984) has proposed a computing method to avoid the singular range $x_n - \alpha \leq \delta$ by adopting a penalty function. On the other hand, for problems to seek simultaneously $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$, where $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is a PRM, that is, the three-parameter estimation problems (Lambert, 1964) for the distribution, Munro and Wixley (1970) have proposed a parameterization to improve the convergency of many iterative methods (Eastham, LaRiccia and Schuenemeyer, 1987; Hirose, 1997). As seen now, there are two ways for dealing with the parameter estimation of the distribution for complete data. Besides these, Giesbrecht and Kempthorne (1976) have proposed replacing complete data with grouped data to avoid the singularity described above.

The use of Munro and Wixley's parameterization, that is, the substitutions of $\alpha = \mu - \sigma/\lambda$, $\beta = \lambda$ and $\gamma = \sigma/\lambda$ into (1.1) yield

$$f(x; \mu - \sigma/\lambda, \lambda, \sigma/\lambda) = \frac{1}{\sqrt{2\pi}\{\sigma + \lambda(x - \mu)\}} \exp\left[-\frac{\{\ln(\sigma + \lambda(x - \mu)) - \ln \sigma\}^2}{2\lambda^2} \right]. \quad (1.2)$$

This can be extended by allowing $x < \mu - \sigma/\lambda$, then the generalization permits λ to be negative. In this way, we obtain the density function for the extended lognormal distribution permitting that $\lambda \neq 0$ and $\sigma > 0$. Let $\tilde{f}(x; \lambda, \mu, \sigma)$ be this density function and $\tilde{L}(\lambda, \mu, \sigma)$ the likelihood function. Cheng and Iles (1990) have shown that as $\lambda \to 0$, $\tilde{f}(x; \lambda, \mu, \sigma)$ leads to the normal distribution with mean μ and variance σ^2, which is called the embedded distribution. They have also investigated tests of statistical hypothesis to see whether the embedded model should be used. Consequently, Munro and Wixley's parameterization can not only improve the convergency of many iterative methods but also cope with the embedded problem.

In the present article we propose a new reparameterization of the extended lognormal distribution to change the three-parameter estimation problem to a two-parameter estimation problem. Because that the reparameterization also permits λ to be estimated negatively, it can cope with even data that cause the embedded problem. In addition, on the two-parameter estimation problem we propose an algorithm to obtain stably the profile of an object function. This makes it possible to seek certainly a PRM if it exists or to show clearly it does not exist.

In Section 2 we describe the reparameterization and an object function to be maximized, and give a theorem, which is useful to construct the algorithm. In Section 3 we introduce the algorithm to obtain the profile of the object function and a PRM. In Section 4 we challenge some estimation examples and perform Monte Carlo simulation experiments. A discussion and summary are given in the last two sections.

2 TWO-PARAMETER ESTIMATION

In this section we analyze a function maximized to find a PRM. First of all we introduce the function.

Set $\tau = \sigma - \lambda\mu$ and $s = \ln\sigma$, and define $\bar{f}(x;\lambda,\tau,s) \stackrel{\text{def}}{=} \tilde{f}(x;\lambda,(\mathrm{e}^s-\tau)/\lambda,\mathrm{e}^s)$:

$$\bar{f}(x;\lambda,\tau,s) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}(\lambda x+\tau)} \exp\left[-\frac{\{\ln(\lambda x+\tau)-s\}^2}{2\lambda^2}\right], \qquad \lambda \neq 0.$$

By arranging $\ln \bar{L}(\lambda,\tau,s) \stackrel{\text{def}}{=} \sum_{i=1}^n \ln \bar{f}(x_i;\lambda,\tau,s)$, we obtain

$$\begin{aligned}\ln \bar{L}(\lambda,\tau,s) &= -\frac{n}{2\lambda^2}\left\{s-\frac{1}{n}\sum_{i=1}^n \ln(\lambda x_i+\tau)\right\}^2 - n\ln\sqrt{2\pi} \\ &\quad + \frac{1}{2n\lambda^2}\left\{\sum_{i=1}^n \ln(\lambda x_i+\tau)\right\}^2 - \frac{1}{2\lambda^2}\sum_{i=1}^n\{\ln(\lambda x_i+\tau)\}^2 - \sum_{i=1}^n \ln(\lambda x_i+\tau).\end{aligned}$$

Only the first term depends on s in the right-hand side of the above equation. And this term has the maximum value 0 when $s = (1/n)\sum_{i=1}^n \ln(\lambda x_i+\tau)$. Hence it suffices to maximize the sum of the third, the fourth and the fifth terms in the equation. Expressing the sum by

$F(\lambda,\tau)$, let us deal with it:

$$F(\lambda,\tau) \stackrel{\text{def}}{=} \frac{1}{2n\lambda^2}\left\{\sum_{i=1}^{n}\ln(\lambda x_i+\tau)\right\}^2 - \frac{1}{2\lambda^2}\sum_{i=1}^{n}\{\ln(\lambda x_i+\tau)\}^2 - \sum_{i=1}^{n}\ln(\lambda x_i+\tau).$$

Next, we introduce a useful theorem for constructing an algorithm searching for a PRM

Theorem We set $\bar{x} \stackrel{\text{def}}{=} (1/n)\sum_{i=1}^{n} x_i$, and define $\tau_U^+(\lambda)$ and $\tau_U^-(\lambda)$ as follows:

$$\tau_U^+(\lambda) \stackrel{\text{def}}{=} -\lambda x_n\left(\frac{1-\bar{x}/x_n \mathrm{e}^{-\lambda^2}}{1-\mathrm{e}^{-\lambda^2}}\right) \text{ for } \lambda>0, \quad \tau_U^-(\lambda) \stackrel{\text{def}}{=} -\lambda x_1\left(\frac{1-\bar{x}/x_1 \mathrm{e}^{-\lambda^2}}{1-\mathrm{e}^{-\lambda^2}}\right) \text{ for } \lambda<0.$$

Then, the following statements hold.

1) $\dfrac{\partial F}{\partial \tau}(\lambda,\tau_U^+(\lambda))<0 \quad \text{for } \lambda>0, \qquad \dfrac{\partial F}{\partial \tau}(\lambda,\tau_U^-(\lambda))<0 \quad \text{for } \lambda<0.$

2) $\displaystyle\lim_{\tau\to-\lambda x_n+0}\frac{\partial F}{\partial \tau}(\lambda,\tau)=+\infty \text{ for } \lambda>0, \qquad \lim_{\tau\to-\lambda x_1+0}\frac{\partial F}{\partial \tau}(\lambda,\tau)=+\infty \text{ for } \lambda<0.$

3) $\dfrac{\partial F}{\partial \tau}(\lambda,\tau)<0$ for any point (λ,τ) that satisfies $\dfrac{\partial^2 F}{\partial \tau^2}(\lambda,\tau)=0.$

4) $\displaystyle\lim_{\lambda\to+\infty}F(\lambda,\tau_U^+(\lambda))=+\infty, \qquad \lim_{\lambda\to-\infty}F(\lambda,\tau_U^-(\lambda))=+\infty.$

5) $F_0(\tau) \stackrel{\text{def}}{=} \displaystyle\lim_{\lambda\to\pm 0}F(\lambda,\tau)$ achieves the relative maximum when

$$\tau=\tau^* \stackrel{\text{def}}{=} \frac{1}{n}\sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(x_i-x_j)^2}.$$

■

Proof. For $\lambda>0$, we can obtain

$$\frac{\partial F}{\partial \tau}(\lambda,\tau) < \frac{1}{\lambda^2}\left(\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\right)\left(\ln\frac{\lambda\bar{x}+\tau}{\lambda x_n+\tau}-\lambda^2\right) \tag{2.1}$$

by $(A.1)$, Jensen's inequality and

$$\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\left(\ln\frac{\lambda\bar{x}+\tau}{\lambda x_i+\tau}-\lambda^2\right) < \left(\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\right)\left(\ln\frac{\lambda\bar{x}+\tau}{\lambda x_n+\tau}-\lambda^2\right).$$

By setting the right-hand side of (2.1) equals 0 and arranging it, we can see $\tau=\tau_U^+(\lambda)$. This leads to the inequality in 1). The proof for $\lambda<0$ is similar.

The statements in 2), 4) and 5) are obtained by direct calculations.

When $\dfrac{\partial^2 F}{\partial \tau^2}(\lambda, \tau) = 0$, $(A.2)$ in Appendix is equivalent to

$$\frac{1}{n\lambda^2}\left\{\sum_{i=1}^{n}\ln(\lambda x_i+\tau)\right\} = \frac{1}{\lambda^2}\left[\frac{\left(\sum_{i=1}^{n}\dfrac{1}{\lambda x_i+\tau}\right)^2}{n\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}} + \frac{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}\ln(\lambda x_i+\tau)}{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}} - 1\right] + 1.$$

The substitution of this into $(A.1)$ yields

$$\frac{\partial F}{\partial \tau}(\lambda,\tau) = \frac{1}{\lambda^2}\left(\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\right)\left[\frac{n}{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{(\lambda x_i+\tau)^2}\right\}\right.$$
$$\left. - \frac{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}\ln\dfrac{1}{\lambda x_i+\tau}}{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)^2}} + \frac{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)}\ln\dfrac{1}{\lambda x_i+\tau}}{\sum_{i=1}^{n}\dfrac{1}{(\lambda x_i+\tau)}}\right].$$

In the bracket, the first term is negative. Furthermore, we can see the sum of the second and third ones is also negative by noting that $\dfrac{\sum_i a_i \ln a_i}{\sum_i a_i} \le \dfrac{\sum_i a_i^2 \ln a_i}{\sum_i a_i^2}$ holds for any $a_i(> 0)$, with equality if and only if all a_i's are equal. □

The statements 1) and 2) indicate that for each $\lambda > 0$ there exists a solution, say $\tau_0(\lambda)$, of $\frac{\partial F}{\partial \tau}(\lambda,\tau) = 0$ in $(-\lambda x_n, \tau_U^+(\lambda))$, and for each $\lambda < 0$ it exists in $(-\lambda x_1, \tau_U^-(\lambda))$. On the other hand, $\frac{\partial F}{\partial \tau}(\lambda,\tau) < 0$ for each $\lambda > 0$ and any $\tau \ge \tau_U^+(\lambda)$ because of 1) and (2.1), and it holds for each $\lambda < 0$ and any $\tau \ge \tau_U^-(\lambda)$ because of similar reasons. Furthermore, $\frac{\partial F}{\partial \tau}(\lambda,\tau) < 0$ holds for each $\lambda > 0$ and $\tau \in (\tau_0(\lambda), \tau_U^+(\lambda))$ and for each $\lambda < 0$ and $\tau \in (\tau_0(\lambda), \tau_U^-(\lambda))$ since 1) and 3). Thus, $\tau_0(\lambda)$ is the unique solution of $\frac{\partial F}{\partial \tau}(\lambda,\tau) = 0$ for each λ. In addition, from these facts, 1) and 4), $\lim_{\lambda\to\pm\infty} F(\lambda, \tau_0(\lambda)) = \infty$. The statement 5) will be used at the beginning in the algorithm stated below. Finally, note that the intervals in which a $\tau_0(\lambda)$ exits, that is, $(-\lambda x_n, \tau_U^+(\lambda))$ and $(-\lambda x_1, \tau_U^-(\lambda))$ become rapidly narrower as $|\lambda|$ becomes larger. For instance, when $\lambda = 6$, the width of $(-\lambda x_n, \tau_U^+(\lambda))$ is $\frac{6}{e^{36}-1}(\bar{x} - x_n)$.

3 AN ALGORITHM FOR SEEKING THE PROFILE OF F

The theorem can be used to seek the profile of $F(\lambda, \tau)$ concerning λ with the bisection method. In $\lambda > 0$, the procedure is written as follows:

1) $\tau \leftarrow \delta_0$, $\lambda \leftarrow \varepsilon_0 > 0$.

2) If $\lambda > \lambda^+_{max}$, end. Otherwise, $\tau_{min} \leftarrow -\lambda x_n$, $\tau_{max} \leftarrow \tau^+_U(\lambda)$.

3) If $\tau_{min} < \tau < \tau_{max}$ and $\dfrac{\partial F}{\partial \tau}(\lambda, \tau) > 0$, $\tau_{min} \leftarrow \tau$. If $\tau_{min} < \tau < \tau_{max}$ and $\dfrac{\partial F}{\partial \tau}(\lambda, \tau) \leq 0$, $\tau_{max} \leftarrow \tau$.

4) If $\dfrac{\partial F}{\partial \tau}(\lambda, (\tau_{min} + \tau_{max})/2) > 0$, then $\tau_{min} \leftarrow (\tau_{min} + \tau_{max})/2$. Otherwise, $\tau_{max} \leftarrow (\tau_{min} + \tau_{max})/2$.

5) If $(\tau_{max} - \tau_{min})/|\tau_{max}| > \varepsilon_1$, then go to 4). Otherwise, $\tau \leftarrow \tau_{max}$.

6) If $\left|\dfrac{\partial F}{\partial \tau}(\lambda, \tau)\right| < \varepsilon_2$, then record $(\lambda, \tau, F(\lambda, \tau))$, $\lambda \leftarrow \lambda + \Delta\lambda$ and go to 2). Otherwise, end.

In $\lambda < 0$, replace 1), 2) and 6) with 1′), 2′) and 6′), respectively:

1′) $\tau \leftarrow \delta_0$, $\lambda \leftarrow -\varepsilon_0$.

2′) If $\lambda < \lambda^-_{min}$, end. Otherwise, $\tau_{min} \leftarrow -\lambda x_1$, $\tau_{max} \leftarrow \tau^-_U(\lambda)$.

6′) If $\left|\dfrac{\partial F}{\partial \tau}(\lambda, \tau)\right| < \varepsilon_2$, then record $(\lambda, \tau, F(\lambda, \tau))$, $\lambda \leftarrow \lambda - \Delta\lambda$ and go to 2′). Otherwise, end.

Here, δ_0, ε_0, ε_1, ε_2, λ^+_{max}, $\Delta\lambda$ and λ^-_{min} are preassigned constants for the procedure. The way of determining them will be explained in the next section.

Using $\{(\lambda, F(\lambda, \tau))\}$ in the records in 6) and 6′), we can plot the profile of $F(\lambda, \tau)$. In addition, if a PRM exists and we set $\Delta\lambda$ at a sufficiently small positive value, we can immediately get the extreme point of $F(\lambda, \tau)$ with high accuracy.

4 COMPUTATIONAL EXPERIMENTS

In this section we give searching examples and Monte Carlo studies. The examples include the three types of cases: 1) the λ coordinate of a PRM is positive, 2) that is negative, 3) no PRM exists. The Monte Carlo studies give a correlation among the value of the population parameter λ, the rate at that λ is positively estimated and the existence rate of a PRM.

TABLE I Data sets.

Data 1: fatigue life in hours of 10 bearings (Cohen *et al.*, 1985; McCool, 1974)									
152.7	172.0	172.5	173.3	193.0	204.7	216.5	234.9	262.6	422.6
Data 2: times to failure of vehicles (Cheng and Iles, 1990; Bilikan, Moore and Petrick, 1979)									
184	250	439	444	450	478	487	524	688	850
1048	1280	1364	1488	1513	1860	1947	1991	2200	2446
Data 3: strengths of 15 cm fibres (Cheng and Iles, 1990; Smith and Naylor, 1987)									
0.37	0.40	0.70	0.75	0.80	0.81	0.83	0.86	0.92	0.92
0.94	0.95	0.98	1.03	1.06	1.06	1.08	1.09	1.10	1.10
1.13	1.14	1.15	1.17	1.20	1.20	1.21	1.22	1.25	1.28
1.28	1.29	1.29	1.30	1.35	1.35	1.37	1.37	1.38	1.40
1.40	1.42	1.43	1.51	1.53	1.61				
Data 4: Menon's data example (Menon, 1963)									
$e^{-6.824}$	$e^{-3.506}$	$e^{-2.64}$	$e^{-1.686}$	$e^{-1.064}$	$e^{-0.832}$	$e^{-0.758}$	$e^{-0.754}$	$e^{-0.684}$	$e^{-0.438}$
$e^{-0.41}$	$e^{-0.216}$	$e^{-0.03}$	$e^{0.032}$	$e^{0.438}$	$e^{0.716}$	$e^{1.262}$	$e^{1.954}$	$e^{2.208}$	$e^{4.054}$
Data 5: pollution data (Chen and Balakrishnan, 1995; Steen and Stickler, 1976)									
109	111	154	200	282	327	336	482	718	900
918	1045	1082	1345	1415	1918	2120	5900	6091	53600

Data 6: an artificial data set generated in Monte Carlo simulation				
−0.912527	−0.905886	−0.836045	−0.382619	−0.319501
0.030242	0.326860	2.325620	4.333967	5.663170

4.1 Searching Examples

Using the algorithm in Section 3, we seek PRMs and profiles of $F(\lambda, \tau)$, that is, $F(\lambda, \tau_0(\lambda))$ for six data sets. They are indicated in Table I. Data 1 has been introduced as an example to fail to find the PRM in (Cohen, Whitten and Ding, 1985). Data 2 and 3 have been introduced as difficult examples to seek the PRMs in (Cheng and Iles, 1990). Data 4 and 5 are examples in which $\hat{L}(\alpha)$ attains its maximum at a point closely near to the end-point of the domain of definition. We picked up Data 3 and 4 for the introduction in Section 1. Data 6 is an artificial data set generated in Monte Carlo simulation when the true values of λ, μ and σ are set at 0.4, 0 and 1, respectively.

At first the preassigned constants of the algorithm are selected in the following way. We set $\Delta\lambda = 0.05$ so as to obtain a crude range in which the λ coordinate of a PRM lies. Since zero is not included in the domain of definition of the parameter λ, we set $\varepsilon_0 = 0.05$ (> 0) to exclude it from the search range. Then, it is appropriate to set $\delta_0 = \tau^*$ because of 5) in the theorem. As shown in the last part of Section 2, the search range of λ is sufficiently wide when λ^+_{max} and λ^-_{min} are set at 6 and -6, respectively. For each λ, the $\tau_0(\lambda)$ can be calculated to an accuracy of the order of the machine epsilon on a used computer. But even if the calculation is performed with such an accuracy, $|\frac{\partial F}{\partial \tau}(\lambda, \tau)|$ may not take a value sufficiently close to 0 for a large $|\lambda|$ and an approximate value of $\tau_0(\lambda)$. That is because $\frac{\partial F}{\partial \tau}(\lambda, \tau)$ varies considerably in $(-\lambda x_n, \tau^+_U(\lambda))$ or $(-\lambda x_1, \tau^-_U(\lambda))$ when $|\lambda|$ is large. Thus, we set that $\varepsilon_1 = 10^{-14}$ and $\varepsilon_2 = 0.01$.

Under this setting, we have obtained the profiles of F. Each one has a shape drawn with a solid curve on Fig. 3. Dotted curves express $F(\lambda, \tau^+_U(\lambda))$ or $F(\lambda, \tau^-_U(\lambda))$. In the figure we can see that solid and dotted curves almost overlap each other in the intervals from -2 or 2 up to about -4 or 4 in λ, respectively.

The profile for Data 1 indicates that the PRM exists and its λ coordinate is around 1. In fact, we can know that the λ coordinate of the PRM is in $(0.85, 0.95)$ and $\tau_0(0.85) = -121.0106$ from the record

$$\begin{aligned}\{(\lambda, \tau, F(\lambda, \tau))\} &= \{\ldots, (.85, -121.0106, -43.4512), (.90, -129.4756, -43.4380), \\ &\qquad (.95, -137.8374, -43.4433), \ldots\}\end{aligned}$$

obtained by the algorithm. Thus, if we perform only the part of the algorithm in $\lambda > 0$ again after resetting that $\varepsilon_0 = 0.85$, $\delta_0 = -121.0106$, $\Delta\lambda = 5 \times 10^{-4}$ and $\lambda^+_{max} = 0.95$, we can know the coordinate of the PRM more precisely, which is $(\lambda, \tau) = (0.9095, -131.0716)$. By means of a similar procedure, we obtain the results in Table II. The profiles for Data 2, 4 and 5 indicate a similar situation to that for Data 1. The profile for Data 3 differs from these in the point that the λ coordinate of the PRM is negative. The last one is the profile for Data 6. It clearly shows that no PRM exists.

TABLE II Estimates of λ and τ.

	Data				
	1	2	3	4	5
λ	.9095	.7030	-0.2955	1.9065	2.5135
τ	-131.0716	28.3203	.5984	.0126	-272.6434

Here is an approximate position for Figure 3

4.2 Monte Carlo Studies

Varying the value of the parameter λ in Monte Carlo simulation, we investigate the existence rate of a PRM and the rate at that λ is positively estimated. In addition, we seek the successful rate in finding a PRM with Munro and Wixley's parameterization for comparison's sake.

The simulation conditions are as follows: The sample number n is set at 10, 15 or 20. The parameter λ is set at 0.01, 0.25, 0.5, 0.75, …, 1.75 or 2.0, whereas the other parameters μ and σ are fixed at 0 and 1, respectively. For each combination of values of n and λ, 1000 independent pseudo-random samples are considered.

We judge whether a PRM exists or not for each data set by tracing $\{(\lambda, F(\lambda, \tau))\}$ in each record.

TABLE III Existence rate of a PRM.

		λ								
		.01	.25	.50	.75	1.0	1.25	1.5	1.75	2.0
	10	.98	.97	.96	.91	.83	.71	.56	.40	.28
n	15	1.0	1.0	1.0	1.0	.98	.96	.87	.73	.54
	20	1.0	1.0	1.0	1.0	1.0	1.0	.98	.93	.80

TABLE IV Rate at that λ is positively estimated.

		λ								
		.01	.25	.50	.75	1.0	1.25	1.5	1.75	2.0
	10	.49	.72	.85	.94	.97	.99	.98	.99	.99
n	15	.51	.79	.93	.99	1.0	1.0	1.0	1.0	1.0
	20	.51	.85	.98	1.0	1.0	1.0	1.0	1.0	1.0

The simulation results for our parameterization are shown in Table III and IV. From Table III we can see that the existence rate of a PRM becomes lower as λ becomes larger.

On the other hand, from Table IV we can see that the rate at that λ is positively estimated becomes higher as λ becomes larger.

TABLE V   Successful rate in finding a PRM with Munro and Wixley's parameterization.

		λ								
		.01	.25	.50	.75	1.0	1.25	1.5	1.75	2.0
	10	.98	.96	.94	.87	.74	.58	.42	.27	.17
n	15	1.0	1.0	.99	.97	.85	.67	.42	.24	.13
	20	1.0	1.0	1.0	.99	.91	.69	.39	.18	.08

The result for Munro and Wixley's parameterization is shown in Table V. In the parameterization, even when λ is set at a value, it needs to solve a two-dimensional non-linear equation not a scalar equation. For it we used Newton's method. Besides, we adopted a similar algorithm to the one in Section 2. That is, we set that $\lambda \leftarrow \pm 0.05$, $\mu \leftarrow \bar{x}$ and $\sigma \leftarrow \sqrt{(1/n)\sum_{i=1}^{n}(x_i - \bar{x})^2}$ at the first stage (this corresponds to 1) or 1′)), and chose, as the initial values of μ and σ at the present stage, the values of them obtained at the previous stage (this corresponds to 3)). Comparing Table III and V, we can see that the failure rate in finding a PRM becomes higher as λ becomes larger from 0.5. In addition, these tables indicate that, although the existence rate is low when λ is large in the case of $n = 10$, it is easier to find a PRM than the other cases if it exists.

5 Discussion

We discuss about $F(\lambda, \tau_0(\lambda))$ when $|\lambda|$ is large. For each λ, $\tau_0(\lambda)$ lies in the interval $(-\lambda x_n, \tau_U^+(\lambda))$ or $(-\lambda x_1, \tau_U^-(\lambda))$, and as $|\lambda| \to \infty$ the widths of the intervals approach 0 at a speed of $O(\mathrm{e}^{-\lambda^2 + \ln|\lambda|})$. Thus, $\tau_0(\lambda)$ approaches $-\lambda x_n$ or $-\lambda x_1$ at a speed equal to or faster than the order. Our question is how $F(\lambda, \tau_0(\lambda))$ behaves then. If $|\frac{\partial F}{\partial \tau}(\lambda, \tau)|$ does not increase rapidly for any τ in $(\tau_0(\lambda), \tau_U^+(\lambda))$ or $(\tau_0(\lambda), \tau_U^-(\lambda))$ as $|\lambda| \to \infty$, we can tell, with the mean value theorem, that $F(\lambda, \tau_0(\lambda))$ behaves similarly to $F(\lambda, \tau_U^+(\lambda))$ or $F(\lambda, \tau_U^-(\lambda))$ for large $|\lambda|$. However, the $|\frac{\partial F}{\partial \tau}(\lambda, \tau)|$ may increase rapidly because $|\frac{\partial F}{\partial \tau}(\lambda, \tau_U^+(\lambda))|$ and $|\frac{\partial F}{\partial \tau}(\lambda, \tau_U^-(\lambda))|$ increase at a speed of $O(\mathrm{e}^{\lambda^2 - \ln|\lambda|})$.

We set that $\tau^+ = -\lambda x_n + g(\lambda)$ for $\lambda > 0$ and $\tau^- = -\lambda x_1 + g(\lambda)$ for $\lambda < 0$, where $g(\lambda) > 0$ and $g(\lambda) \to 0$ $(\lambda \to \pm\infty)$. Then, the following holds. As $|\lambda| \to \infty$

1) $F(\lambda, \tau^+) \to -\infty$, $F(\lambda, \tau^-) \to -\infty$ if $g(\lambda) = o(\mathrm{e}^{-|\lambda|^\eta})$ for $\eta > 2$,

2) $F(\lambda, \tau^+) \to -\infty$, $F(\lambda, \tau^-) \to -\infty$ if $g(\lambda) = C_1\mathrm{e}^{-C_2\lambda^2 + o(\lambda^2)}$ for $C_1 > 0$, $C_2 \geq \frac{2n}{n-1}$,

3) $F(\lambda, \tau^+) \to \infty$, $F(\lambda, \tau^-) \to \infty$ if $g(\lambda) = C_1\mathrm{e}^{-C_2\lambda^2 + o(\lambda^2)}$ for $C_1 > 0$, $0 < C_2 < \frac{2n}{n-1}$.

This can be proved by standard mathematical calculations.

From 3), $\lim_{\lambda \to \pm\infty} F(\lambda, \tau_0(\lambda)) = \infty$ and the fact that $\tau_0(\lambda)$ lies in $(-\lambda x_n, \tau_U^+(\lambda))$ or $(-\lambda x_1, \tau_U^-(\lambda))$,

$$\lambda x_n + \tau_0(\lambda) \sim C_1\mathrm{e}^{-C_2'\lambda^2 + o(\lambda^2)} \quad \text{or} \quad \lambda x_1 + \tau_0(\lambda) \sim C_1\mathrm{e}^{-C_2'\lambda^2 + o(\lambda^2)}$$

for large $|\lambda|$. Here, $1 \leq C_2' < \frac{2n}{n-1}$. Consequently, $F(\lambda, \tau_0(\lambda))$ is considered to behave similarly to $F(\lambda, \tau_U^+)$ or $F(\lambda, \tau_U^-)$ for large $|\lambda|$. That is, we may regard $F(\lambda, \tau_U^+(\lambda))$ or $F(\lambda, \tau_U^-(\lambda))$ as $F(\lambda, \tau_0(\lambda))$ for large $|\lambda|$. See also the examples on Fig. 3.

6 SUMMARY

We have proposed a reparameterization of the extended lognormal distribution for the parameter estimation. The reparameterization changes the three-parameter estimation problem to a two-parameter estimation problem, and enables us to cope with extensive data sets including those which cause the embedded problem. On the two-parameter estimation problem, we have made an algorithm to seek the profile of an object function. The algorithm is simple and makes it possible to seek the profile stably. The profile clearly shows whether a PRM exists or not, and if it exists, we can obtain it certainly from the record for drawing the profile. In fact, we have illustrated that the reparameterization and the algorithm go well for the six data sets including ones introduced as difficult examples to estimate the PRMs in other articles.

By means of Monte Carlo simulation, we have investigated the existence rate of a PRM while varying the value of the parameter λ. From the simulation result it has become clear that the rate largely falls down when λ goes through from 1.25 to 2.0. In such cases that the degree of skewness is high and no MLE exists, there is a possibility that dealing with data as grouped data can help us cope with the difficulty (Giesbrecht and Kempthorne, 1976). This possibility was not pursued in the present paper because it is beyond its purpose. We have investigated the rate at that λ is positively estimated on data governed by (1.2). The simulation result has shown the rate is less than 1 in almost all the sample number when λ is less than 1. This indicates the necessity for our or Munro and Wixley's parameterization, which permits λ to be negative, since this generalization makes it possible to cope with the embedded problem.

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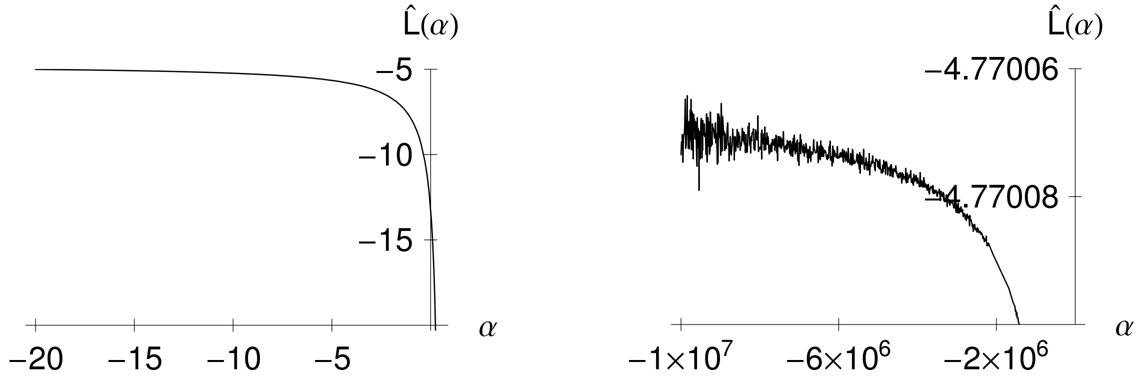
APPENDIX

The following is the first and second derivatives of the object function F with respect to τ:

$$\frac{\partial F}{\partial \tau}(\lambda,\tau) = \frac{1}{n\lambda^2}\left\{\sum_{i=1}^{n}\ln(\lambda x_i+\tau)\right\}\left\{\sum_{j=1}^{n}\frac{1}{\lambda x_j+\tau}\right\}$$
$$-\frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\ln(\lambda x_i+\tau)-\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}, \qquad (A.1)$$

$$\frac{\partial^2 F}{\partial \tau^2}(\lambda,\tau) = \frac{1}{n\lambda^2}\left[\left(\sum_{i=1}^{n}\frac{1}{\lambda x_i+\tau}\right)^2-\left\{\sum_{j=1}^{n}\ln(\lambda x_j+\tau)\right\}\left\{\sum_{j=1}^{n}\frac{1}{(\lambda x_j+\tau)^2}\right\}\right.$$
$$\left.+\sum_{i=1}^{n}\frac{n}{(\lambda x_i+\tau)^2}\ln(\lambda x_i+\tau)-\sum_{i=1}^{n}\frac{n}{(\lambda x_i+\tau)^2}\right]+\sum_{i=1}^{n}\frac{1}{(\lambda x_i+\tau)^2}. \qquad (A.2)$$

The author's name: Yoshio Komori. Figure number: 1



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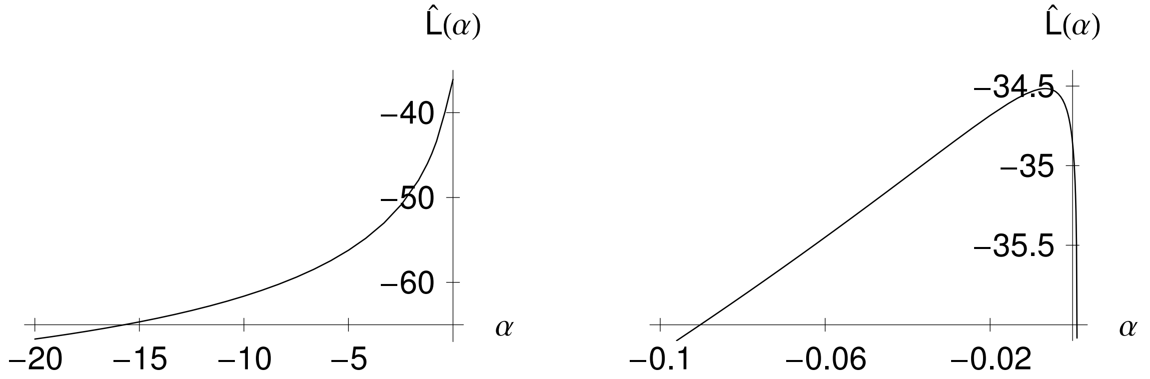

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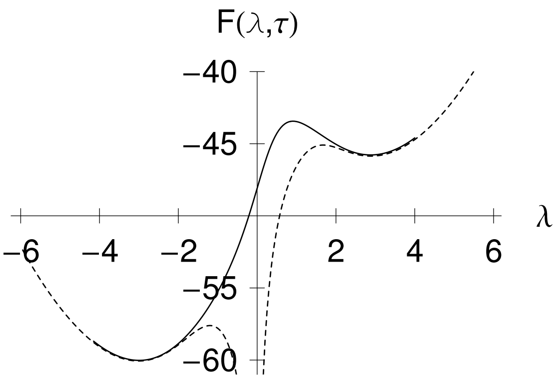

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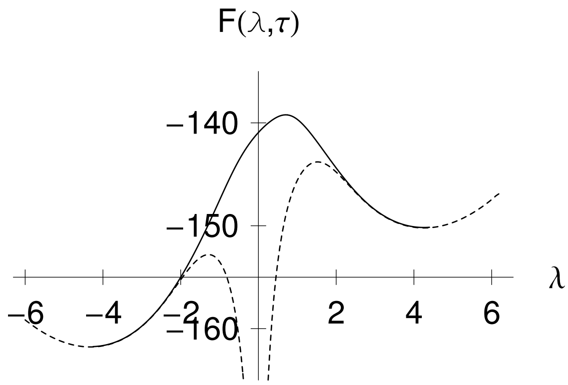

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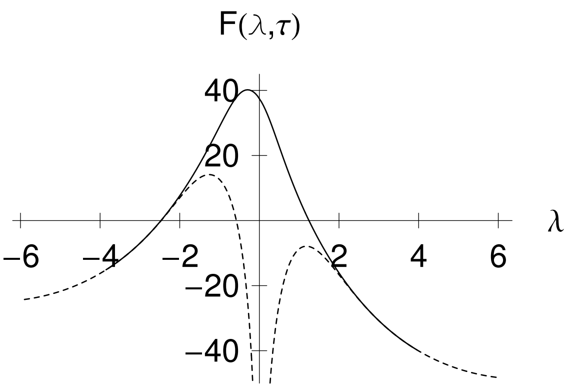

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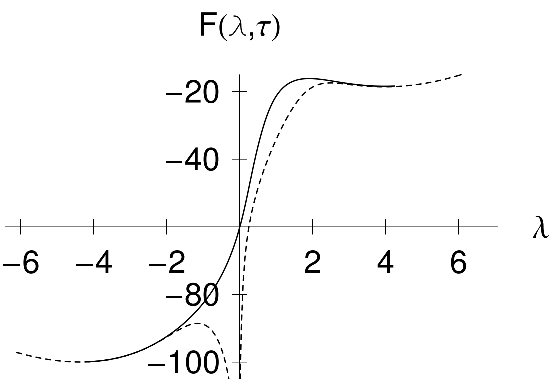

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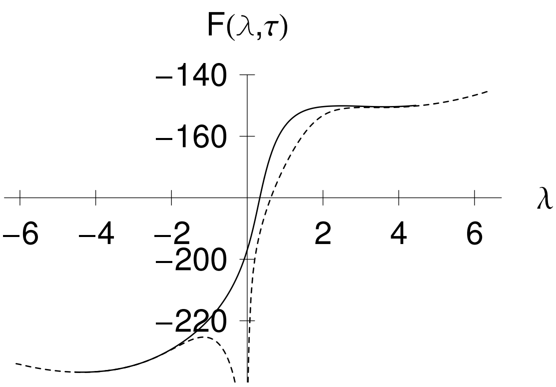

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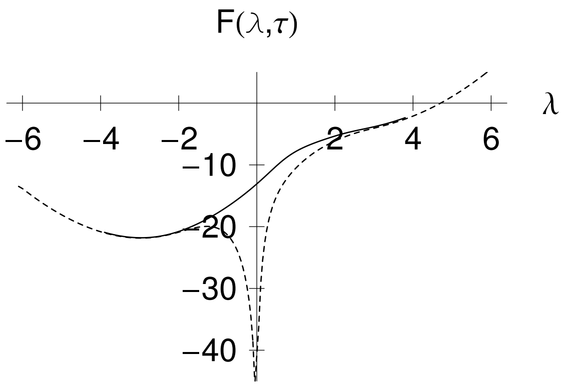

Data 6

This is a list of figure captions.

Figure 1

$\hat{L}(\alpha)$ for Data 3.

Figure 2

$\hat{L}(\alpha)$ for Data 4.

Figure 3

Profiles of $F(\lambda, \tau)$. Solid curves indicate $F(\lambda, \tau_0(\lambda))$. Dotted curves indicate $F(\lambda, \tau_U^+(\lambda))$ or $F(\lambda, \tau_U^-(\lambda))$. These are included for comparison.