PARAMETER ESTIMATION BASED ON GROUPED OR CONTINUOUS DATA FOR TRUNCATED EXPONENTIAL DISTRIBUTIONS

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ABSTRACT

Parameter estimation based on truncated data is dealt with; the data are assumed to obey truncated exponential distributions with a variety of truncation time—a_1 data are obtained by truncation time b_1, a_2 data are obtained by truncation time b_2 and so on, whereas the underlying distribution is the same exponential one. The purpose of the present paper is to give existence conditions of the maximum likelihood estimators (MLEs) and to show some properties of the MLEs in two cases: 1) the grouped and truncated data are given (that is, the data each express the number of the data value falling in a corresponding subinterval), 2) the continuous and truncated data are given.

1. INTRODUCTION

We are concerned with the problem of nonexistence of the MLE and the properties of the MLE for the truncated exponential distribution. Although similar problem has been discussed by Deemer and Votaw (1), this is based on continuous data. In contrast, ours is based on grouped data. Further, it is slightly more general.

Let us introduce the outline of the problem. Suppose that the industrial products in lots are successively shipped and that only the number of failures are reported at the predetermined inspection time. The purpose is to estimate the parameter of the underlying distribution dominating the number of failures.

Next, let us introduce some notations used in the sequel. Denote by τ and s the shipment interval and the shipment times, respectively. Not limiting $s = 1$ is the reason why we said in the above that our case is more general. In addition, denote by T_{end} the time passed from the beginning of the 1st shipment to the end of the inspection.

In the next section, we will give the existence condition of the MLE and the asymptotic variance in the groped data case. In Section 3, we will give those in the continuous data case for comparison. In Section 4 we will give Monte Carlo experiments in both cases, and in the last section summarize them.

2. GROUPED AND TRUNCATED DATA CASE

Divide the truncation interval $[0,\ T_{end}-(i-1)\tau]$ in the ith shipment into non-overlapping subintervals of length τ/g, where g is a positive integer. And denote by $r_j^{(i)}$ the number of the products failed in the subinterval $((j-1)\tau/g,\ j\tau/g]$ on the ith shipment. We assume that T_{end} may be expressed as $\eta\tau/g$ with some positive integer η $(\geq (s-1)g+1)$. If time t, which had passed by a product led to failure since its shipment, obeys a distribution whose density function $f(t;\boldsymbol{\theta})$ depending on a parameter $\boldsymbol{\theta}$, then the log-likelihood function is the following:

$$\ln L_{tg}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sum_{i=1}^{s} \sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} \ln \left\{ \frac{\displaystyle\int_{(j-1)\tau/g}^{j\tau/g} f(t;\boldsymbol{\theta})\mathrm{d}t}{\displaystyle\int_{0}^{T_{end}-(i-1)\tau} f(t;\boldsymbol{\theta})\mathrm{d}t} \right\}. \tag{2. 1}$$

Let us consider a case, where $f(t;\boldsymbol{\theta})$ in (2. 1) is the density function of an exponential distribution, that is, $c\mathrm{e}^{-ct}$. Here, c (> 0) is a parameter. Setting

$$\Delta t \stackrel{\text{def}}{=} \tau/g, \quad t_j \stackrel{\text{def}}{=} j\Delta t, \quad \tau_i \stackrel{\text{def}}{=} T_{end} - (i-1)\tau,$$

we obtain

$$\ln L_{tg}(c) = \sum_{i=1}^{s} \sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} \left\{ -ct_{j-1} + \ln\left(1 - \mathrm{e}^{-c\Delta t}\right) - \ln\left(1 - \mathrm{e}^{-c\tau_i}\right) \right\}. \tag{2. 2}$$

In connection with this, the following lemma holds.

Lemma 2.1 We set

$$n_i \stackrel{\text{def}}{=} \sum_{j=1}^{\eta-(i-1)g} r_j^{(i)}, \quad N \stackrel{\text{def}}{=} \sum_{i=1}^{s} n_i, \quad \tilde{t}_a \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{s} \sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} \left(\frac{t_{j-1} + t_j}{2} \right).$$

If

$$0 < \tilde{t}_a < \frac{1}{2N}\sum_{i=1}^{s} n_i\tau_i,$$

there exists a solution of $\dfrac{\partial \ln L_{tg}}{\partial c}(c) = 0$ and it is the MLE. If not, the MLE does not exist.

■

Proof. From (2. 2)

$$\frac{\partial \ln L_{tg}}{\partial c}(c) = \sum_{i=1}^{s}\sum_{j=1}^{\eta-(i-1)g} r_j^{(i)}\left\{-t_{j-1} + \frac{\Delta t}{\mathrm{e}^{c\Delta t}-1} - \frac{\tau_i}{\mathrm{e}^{c\tau_i}-1}\right\}.$$

According to Lemma 2.2, the right-hand side of the equation above is a strictly decreasing function of c (> 0) if $\tau_i > \Delta t$. Since

$$\lim_{c\to+0}\left\{\frac{\Delta t}{\mathrm{e}^{c\Delta t}-1} - \frac{\tau_i}{\mathrm{e}^{c\tau_i}-1}\right\} = -\frac{1}{2}\Delta t + \frac{1}{2}\tau_i,$$

$$\lim_{c\to+0}\frac{\partial \ln L_{tg}}{\partial c}(c) = N\left[\frac{1}{2N}\sum_{i=1}^{s} n_i\tau_i - \tilde{t}_a\right].$$

On the other hand

$$\lim_{c\to+\infty}\frac{\partial \ln L_{tg}}{\partial c}(c) = -\sum_{i=1}^{s}\sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} t_{j-1} < 0.$$

In addition, $\dfrac{\partial \ln L_{tg}}{\partial c}(c)$ is continuous. Consequently, there exists a solution $c = c_{tg0}$ (> 0) satisfying $\dfrac{\partial \ln L_{tg}}{\partial c}(c) = 0$ if $0 < \tilde{t}_a < \dfrac{1}{2N}\sum_{i=1}^{s} n_i\tau_i$, and $\ln L_{tg}(c) \uparrow \sup_{c>0}\{\ln L_{tg}(c)\}$ as $c \downarrow 0$ if $\tilde{t}_a \geq \dfrac{1}{2N}\sum_{i=1}^{s} n_i\tau_i$. □

Lemma 2.2 The function

$$g(c) \stackrel{\text{def}}{=} \frac{\Delta t}{\mathrm{e}^{c\Delta t}-1} - \frac{\tau_i}{\mathrm{e}^{c\tau_i}-1} \qquad (0 < \Delta t < \tau_i)$$

strictly decreases in $(0, \infty)$. ■

Proof. Because that

$$g'(c) = \frac{1}{c^2}\left\{-\frac{(c\Delta t)^2\mathrm{e}^{c\Delta t}}{(\mathrm{e}^{c\Delta t}-1)^2} + \frac{(c\tau_i)^2\mathrm{e}^{c\tau_i}}{(\mathrm{e}^{c\tau_i}-1)^2}\right\},$$

it suffices for $g'(c) < 0$ $(c > 0)$ holding that $h(x) \stackrel{\text{def}}{=} \dfrac{x^2\mathrm{e}^x}{(\mathrm{e}^x-1)^2}$ is strictly increasing in $(0, \infty)$. Since

$$h'(x) = \frac{x\mathrm{e}^x}{(\mathrm{e}^x-1)^3}\left\{(2-x)\mathrm{e}^x - (2+x)\right\},$$

we set $u(x) \stackrel{\text{def}}{=} (2-x)\mathrm{e}^x - (2+x)$ and investigate this. By differentiating $u(x)$ up to twice we can see $u''(x) < 0$ for $x > 0$ and $u'(0) = 0$. Thus $u'(x) < 0$. In analogy, $u(x) < 0$. Consequently, $h(x)$ is strictly decreasing in $(0, \infty)$. Therefore, $g(c)$ is a strictly decreasing function. □

Next, let us consider the asymptotic property of the MLE, say $\hat{c}_{tg}$.

Denote by $p_i(j; c)$ the probability that a product on the ith shipment fails in the interval $((j-1)\tau/g,\ j\tau/g]$. It is written in the form

$$p_i(j;c) = \frac{\mathrm{e}^{-c(j-1)\tau/g} - \mathrm{e}^{-cj\tau/g}}{1-\mathrm{e}^{-c\tau_i}} = \frac{\mathrm{e}^{-ct_{j-1}}(1-\mathrm{e}^{-c\Delta t})}{1-\mathrm{e}^{-c\tau_i}} \qquad (1 \le j \le \eta-(i-1)g).$$

If we set

$$I_i(c) \stackrel{\text{def}}{=} \sum_{j=1}^{\eta-(i-1)g} \left(\frac{\partial \ln p_i(j;c)}{\partial c}\right)^2 p_i(j;c),$$

this can be rewritten by

$$I_i(c) = \frac{(\Delta t)^2 \mathrm{e}^{c\Delta t}}{(\mathrm{e}^{c\Delta t}-1)^2} - \frac{\tau_i^2 \mathrm{e}^{c\tau_i}}{(\mathrm{e}^{c\tau_i}-1)^2}$$

since

$$\sum_{j=1}^{\eta-(i-1)g} p_i(j;c) = 1, \qquad \frac{\partial^2 \ln p_i(j;c)}{\partial c^2} = -\frac{(\Delta t)^2 \mathrm{e}^{c\Delta t}}{(\mathrm{e}^{c\Delta t}-1)^2} + \frac{\tau_i^2 \mathrm{e}^{c\tau_i}}{(\mathrm{e}^{c\tau_i}-1)^2}.$$

The number $r_j^{(i)}$ in (2. 2) can be expressed as follows: assume that $X_l^{(i)}$ is a random variable, which takes a value j when the lth product on the ith shipment fails in the interval $((j-1)\tau/g,\ j\tau/g]$, and denote by $\#A$ the cardinal number of a set A. Then, $r_j^{(i)}$ can be expressed by $\#\{X_l^{(i)} | j = X_l^{(i)},\ l = 1, \ldots, n_i\}$.

Using these expressions, the log-likelihood function can be rewritten in the form

$$\ln L_{tg}(c) = \sum_{i=1}^{s} \sum_{l=1}^{n_i} \ln p_i(X_l^{(i)}; c). \tag{2. 3}$$

From the likelihood equation, the expansion of $\dfrac{1}{\sqrt{N}} \dfrac{\partial \ln L_{tg}}{\partial c}(\hat{c}_{tg})$ around c, and (2. 3), we obtain

$$\sqrt{N}(\hat{c}_{tg} - c) = \frac{\dfrac{1}{\sqrt{N}} \dfrac{\partial \ln L_{tg}}{\partial c}(c)}{-\dfrac{1}{N} \dfrac{\partial^2 \ln L_{tg}}{\partial c^2}(\tilde{c})} = \frac{\displaystyle\sum_{i=1}^{s} \sqrt{\frac{n_i}{N}} \frac{1}{\sqrt{n_i}} \sum_{l=1}^{n_i} \frac{\partial \ln p_i(X_l^{(i)}; c)}{\partial c}}{\displaystyle\sum_{i=1}^{s} \frac{n_i}{N} I_i(\tilde{c})}.$$

Here, $|\tilde{c}-c| \leq |\hat{c}_{tg}-c| \to 0$ a. s. $(N \to \infty)$.

Thus, if we assume that $n_i/N \to \gamma_i$ a. s. $(N \to \infty)$ and use the notation $\mathcal{N}(a,b)$ that stands for a normal distribution with mean a and variance b, we can see

$$\sqrt{N}(\hat{c}_{tg}-c) \to \mathcal{N}\left(0, \left[\frac{(\Delta t)^2 \mathrm{e}^{c\Delta t}}{(\mathrm{e}^{c\Delta t}-1)^2} - \sum_{i=1}^{s} \gamma_i \frac{\tau_i^2 \mathrm{e}^{c\tau_i}}{(\mathrm{e}^{c\tau_i}-1)^2}\right]^{-1}\right) \text{ in law} \qquad (N \to \infty).$$

3. CONTINUOUS AND TRUNCATED DATA CASE

In Section 2 we discussed the case in which data were given by only the number $r_j^{(i)}$ of products failed. In this section we consider the case in which failure time is given.

Denote by $t_l^{(i)}$ $(l=1,\ldots,n_i)$ failure time of the products failed by τ_i in the ith shipment.

The log-likelihood function is defined by

$$\ln L_t(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sum_{i=1}^{s}\sum_{l=1}^{n_i} \ln \left\{ \frac{f(t_l^{(i)};\boldsymbol{\theta})}{\int_0^{T_{end}-(i-1)\tau} f(t;\boldsymbol{\theta})\mathrm{d}t} \right\}. \tag{3. 1}$$

Let us consider of a case again, where $f(t;\boldsymbol{\theta})$ in (3. 1) is the density function of an exponential distribution. Setting

$$\bar{t}^{(i)} \stackrel{\text{def}}{=} \frac{1}{n_i}\sum_{l=1}^{n_i} t_l^{(i)},$$

we obtain

$$\ln L_t(c) = \sum_{i=1}^{s} n_i \left\{ \ln c - c\bar{t}^{(i)} - \ln\left(1-\mathrm{e}^{-c\tau_i}\right)\right\}. \tag{3. 2}$$

In connection with this, the following lemma holds.

Lemma 3.1 We set

$$\bar{t}_a \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^{s} n_i \bar{t}^{(i)}.$$

If

$$0 < \bar{t}_a < \frac{1}{2N}\sum_{i=1}^{s} n_i \tau_i,$$

there exists a solution of $\dfrac{\partial \ln L_t}{\partial c}(c) = 0$ and it is the MLE. If not, the MLE does not exist. ■

Proof. From (3. 2)

$$\frac{\partial \ln L_t}{\partial c}(c) = N\left[\sum_{i=1}^{s} \frac{n_i}{N}\left\{c^{-1} - \frac{\tau_i}{\mathrm{e}^{c\tau_i}-1}\right\} - \bar{t}_a\right].$$

According to Lemma 3.2, the expression in { } on the right-hand side of the equation above is a strictly decreasing function of c (> 0).

Since

$$\lim_{c\to+0}\left\{c^{-1}-\frac{\tau_i}{\mathrm{e}^{c\tau_i}-1}\right\}=\frac{1}{2}\tau_i,$$

$$\lim_{c\to+0}\frac{\partial \ln L_t}{\partial c}(c)=N\left[\frac{1}{2N}\sum_{i=1}^{s}n_i\tau_i-\bar{t}_a\right].$$

On the other hand

$$\lim_{c\to+\infty}\frac{\partial \ln L_t}{\partial c}(c)=-N\bar{t}_a<0.$$

In addition, $\dfrac{\partial \ln L_t}{\partial c}(c)$ is continuous. Consequently, there exists a solution $c = c_{t0}$ (> 0) satisfying $\dfrac{\partial \ln L_t}{\partial c}(c) = 0$ if $0 < \bar{t}_a < \dfrac{1}{2N}\displaystyle\sum_{i=1}^{s} n_i\tau_i$, and $\ln L_t(c) \uparrow \sup\limits_{c>0}\{\ln L_t(c)\}$ as $c \downarrow 0$ if $\bar{t}_a \geq \dfrac{1}{2N}\displaystyle\sum_{i=1}^{s} n_i\tau_i$. □

Lemma 3.2 The function

$$v(c)\overset{\mathrm{def}}{=}\frac{1}{c}-\frac{\tau_i}{\mathrm{e}^{c\tau_i}-1}\qquad(\tau_i\neq 0)$$

strictly decreases in $(0,\infty)$. ■

Proof. By differentiating $v(c)$ we obtain

$$v'(c)=\frac{\mathrm{e}^{c\tau_i}\left\{2+(c\tau_i)^2-(\mathrm{e}^{c\tau_i}+\mathrm{e}^{-c\tau_i})\right\}}{c^2\left(\mathrm{e}^{c\tau_i}-1\right)^2}.$$

Using Maclaurin expansion of e^x, we can easily show that { } part on the right-hand side of the equation is negative. Thus, $v(c)$ is a strictly decreasing function in $(0,\infty)$. □

Next, let us consider the asymptotic property of the MLE, say $\hat{c}_t$. A similar discussion as in Section 2 leads us to

$$\sqrt{N}(\hat{c}_t-c)\to\mathcal{N}\left(0,\left[\frac{1}{c^2}-\sum_{i=1}^{s}\gamma_i\frac{\tau_i^2\mathrm{e}^{c\tau_i}}{(\mathrm{e}^{c\tau_i}-1)^2}\right]^{-1}\right)\ \text{in law}\qquad(N\to\infty).$$

4. Monte Carlo Studies

The purpose in this section is to indicate some properties of the MLEs by means of Monte Carlo simulation experiments. Before it, however, we state about easy iterative schemes to calculate the maximum likelihood estimates.

4. 1. Iterative Schemes

Setting $\dfrac{\partial \ln L_{tg}}{\partial c}(c) = 0$ and using (2. 1) and $f(t;\boldsymbol{\theta}) = ce^{-ct}$, we obtain

$$\frac{1}{c} = \frac{1}{N}\sum_{i=1}^{s}\sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} E[T|(j-1)\tau/g < T \leq j\tau/g] + \sum_{i=1}^{s}\frac{n_i}{N}\frac{\tau_i}{e^{c\tau_i}-1}. \tag{4. 3}$$

From this, we can get an iterative scheme

$$\frac{1}{c^{(k+1)}} = \frac{1}{N}\sum_{i=1}^{s}\sum_{j=1}^{\eta-(i-1)g} r_j^{(i)} E_{c^{(k)}}[T|(j-1)\tau/g < T \leq j\tau/g] + \sum_{i=1}^{s}\frac{n_i}{N}\frac{\tau_i}{\exp[c^{(k)}\tau_i]-1}. \tag{4. 4}$$

Here, $E[\cdot|\cdot]$ and $E_{c^{(k)}}[\cdot|\cdot]$ mean the conditional expectations, in particular, the latter clearly shows that $c = c^{(k)}$. Similarly, (3. 1) gives an iterative scheme

$$\frac{1}{c^{(k+1)}} = \frac{1}{N}\sum_{i=1}^{s}\sum_{l=1}^{n_i} t_l^{(i)} + \sum_{i=1}^{s}\frac{n_i}{N}\frac{\tau_i}{\exp[c^{(k)}\tau_i]-1}. \tag{4. 5}$$

In both schemes, the second terms in the right-hand side arise from the fact that the data are truncated.

When the assumption in Lemma 2.1 is satisfied, the successive approximations generated by (4. 4) converge linearly to $1/\hat{c}_{tg}$. This can be shown as follows. Since that $\hat{c}_{tg}$ satisfies (4. 3) and

$$E_{c^{(k)}}[T|(j-1)\tau/g < T \leq j\tau/g] = \frac{\Delta t}{1-\exp[c^{(k)}\Delta t]} + (j-1)\Delta t + \frac{1}{c^{(k)}},$$

$$\frac{1}{c^{(k+1)}} - \frac{1}{\hat{c}_{tg}} = \left[1 - \frac{1}{N}\sum_{i=1}^{s}\sum_{j=1}^{\eta-(i-1)g} r_j^{(i)}\frac{(\tilde{c}\Delta t)^2 e^{\tilde{c}\Delta t}}{(1-e^{\tilde{c}\Delta t})^2} + \sum_{i=1}^{s}\frac{n_i}{N}\frac{(\tilde{c}\tau_i)^2 e^{\tilde{c}\tau_i}}{(1-e^{\tilde{c}\tau_i})^2}\right]\left(\frac{1}{c^{(k)}} - \frac{1}{\hat{c}_{tg}}\right).$$

Here, $\left|\dfrac{1}{\tilde{c}} - \dfrac{1}{\hat{c}_{tg}}\right| < \left|\dfrac{1}{c^{(k)}} - \dfrac{1}{\hat{c}_{tg}}\right|$. This equation and the property of $h(x)$ in Lemma 2.2 yield

$$\left|\frac{1}{c^{(k+1)}} - \frac{1}{\hat{c}_{tg}}\right| < K\left|\frac{1}{c^{(k)}} - \frac{1}{\hat{c}_{tg}}\right| \qquad (K \text{ is a constant such that } 0 < K < 1).$$

In analogy, when the assumption in Lemma 3. 1 is satisfied, the successive approximations generated by (4. 5) converge linearly to $1/\hat{c}_t$.

4. 2. Simulation Conditions

Case 1: The shipment times, s, was fixed at 2, and the shipment interval τ was fixed at 0.5. The two variables T_{end} and Δt were set at 1 and 0.1, respectively. The properties of the

MLEs were investigated for a variety of combinations of sample size and the limiting ratio γ_1 of n_1 to N. The combinations involved seven sample sizes (10, 20, 40, 80, 160, 640, 2560) and three ratios (.7, .6, .5).

Case 2: The shipment interval τ was fixed at 0.4. The properties of the MLEs were investigated for a variety of combinations of Δt and the shipment times. The combinations involved three values of Δt (0.1, 0.2, 0.4) and four sorts of the shipment times (1, 2, 3, 4). Corresponding to each shipment times, the limiting ratios of n_i to N were set as Tab. 1.

Table 1: Limiting ratios of n_i to N

s	2	3	4
$\gamma_1, \ldots, \gamma_s$	.63, .37	.44, .35, .21	.34, .29, .23, .14

Common settings: In each experiment, 10000 independent pseudo-random samples were considered, except the samples where the MLE did not exist.

4. 3. Simulation Results

Almost all the results are expressed in tabular form. The numbers in parenthesis indicate the results for grouped data. Three notations MSE, NE rate and $\bar{n}_i$ stand for mean square error, non-existence rate of the MLE, and the average of n_i, respectively. In Tab. 4, CO means the continuous case.

Case 1: The results concerning the MLEs are shown in Tab. 3. It indicates the following: when N is 10 or 20, the biases of the MLEs are positive and the non-existence rate of the MLE are about 20%. When $N >= 640$, the values of $N \times$MSE are nearly equal to the asymptotic variance (Tab. 2).

Table 2: Asymptotic variance

γ_1	.70	.60	.50
Asymptotic variance	16.2(16.4)	17.9(18.2)	20.0(20.4)

Table 3: Properties of the MLE when $s = 2$

γ_1	N	bias	variance	$N\times$MSE	NE rate
.70	10	+.551(+.582)	1.27(1.29)	15.8(16.3)	.22(.23)
	20	+.233(+.244)	.631(.649)	13.7(14.2)	.13(.14)
	40	+.080(+.087)	.348(.356)	14.2(14.5)	.07(.07)
	80	+.017(+.021)	.132(.186)	14.6(14.9)	.03(.03)
	160	−.022(−.018)	.106(.108)	17.1(17.4)	.01(.01)
	640	−.035(−.032)	.023(.024)	15.8(15.5)	.00(.00)
	2560	−.036(−.033)	.005(.005)	16.1(15.9)	.00(.00)
.60	10	+.610(+.651)	1.39(1.41)	17.7(18.3)	.23(.25)
	20	+.267(+.288)	.687(.701)	15.2(15.7)	.15(.16)
	40	+.098(+.106)	.371(.379)	15.2(15.6)	.08(.08)
	80	+.024(+.029)	.197(.201)	15.8(16.2)	.04(.04)
	160	−.019(−.015)	.114(.116)	18.3(18.6)	.01(.01)
	640	−.036(−.033)	.026(.027)	17.4(17.8)	.00(.00)
	2560	−.037(−.034)	.006(.006)	17.6(17.3)	.00(.00)
.50	10	+.687(+.731)	1.58(1.60)	20.5(21.3)	.25(.26)
	20	+.318(+.337)	.764(.786)	17.3(18.0)	.17(.17)
	40	+.111(+.126)	.411(.418)	16.9(17.4)	.08(.09)
	80	+.029(+.033)	.219(.225)	17.6(18.0)	.04(.04)
	160	−.016(−.011)	.123(.126)	19.8(20.1)	.02(.02)
	640	−.039(−.035)	.028(.029)	19.1(19.4)	.00(.00)
	2560	−.039(−.036)	.006(.006)	20.3(19.8)	.00(.00)

Case 2: We show the results concerning the MLEs in Tab. 4. In this table we can see that all the variates decrease as s increases or Δt decreases. Besides, the table indicates the tendency that the difference between grouped and continuous cases becomes smaller as s increases or Δt decreases. Note that the increase of s means the prolongation of T_{end} because that τ is fixed.

Table 4: Properties of the MLE when $s = 1,\ 2,\ 3,\ 4$

s	Δt	bias	variance	$N\times$MSE	NE rate
1	.4	$*$	$*$	$*$	1
	.2	+.891	1.41	88.1	.313
	.1	+.616	1.12	59.8	.276
	CO	+.542	1.10	55.6	.247
2	.4	+.378	.815	38.3	.183
	.2	+.217	.539	23.4	.141
	.1	+.188	.521	22.3	.125
	CO	+.174	.511	21.7	.120
3	.4	+.101	.392	16.1	.066
	.2	+.081	.325	13.3	.064
	.1	+.077	.316	12.9	.063
	CO	+.066	.321	13.0	.056
4	.4	+.046	.249	10.1	.031
	.2	+.031	.230	9.23	.023
	.1	+.030	.223	9.06	.023
	CO	+.029	.226	9.07	.023

5. SUMMARY

We first stated the conditions under which the MLEs may be given for grouped and truncated data or continuous and truncated data when shipment occurs repeatedly, and second investigated asymptotic properties of the MLEs and their properties in finite samples.

With respect to the case where products are shipped only one time, serious estimating problems in an exponential distribution have been discussed by Deemer and Votaw (1), and those in a Weibull distribution have been discussed by Mittal and Dahiya (2). Deemer and Votaw have given a condition about the existence of the MLE: Let T_r be the truncation time, $\bar{t}$ the average of failure time. Then, the MLE exists if $0 < \bar{t} < \dfrac{1}{2}T_r$, and the MLE does not exist if $\bar{t} \geq \dfrac{1}{2}T_r$. On the other hand, Mittal and Dahiya have given a conjecture about such a condition.

Comparing this with the results in Section 2 or Section 3, we explain about them. The expression $\frac{1}{N}\sum_{i=1}^{s} n_i \tau_i$ is the average of truncation time because that n_i and τ_i are the total number of failed products and the truncation time on the ith shipment, respectively. If $\frac{t_{j-1}+t_j}{2}$ is chosen as the representative value of failure time in the subinterval, $\tilde{t}_a$ means an approximate value to the average of failure time. On the other hand $\bar{t}_a$ is the average of failure time on all the shipments since $\bar{t}^{(i)}$ is the average of failure time on the ith shipment. Summarizing the things above, we can say as follows: if we replace the truncation time in Deember's result with the average one, we obtain the result in Section 3. Besides the replacement, if we replace the average of failure time with the approximate value, we obtain the result in Section 2.

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