Stability of Augmented Staggered Method for Electromagnetic and Structural Coupled Problem

Yoshikazu Tanaka, Tomoyoshi Horie, Tomoya Niho, Eiji Shintaku, and Yukio Fujimoto

***Abstract*—Numerical instability occurs in analyses of an electromagnetic and structural coupled problem or a magnetic damping problem when the matrix equations of structure and eddy current are solved alternately. In this paper, an augmented staggered method is proposed for the coupled analysis. The stability of this method is demonstrated for the magnetically damped vibration by finite element analyses. According to the characteristic equation obtained from the recurrence relation of this time integration scheme, it is confirmed that this method is unconditionally stable for the intensity of magnetic field and the size of time increment.**

***Index Terms*—Augmented staggered method, electromagnetic and structural coupled problem, numerical instability, stability analysis.**

I. Introduction

In ANALYSES of an electromagnetic and structural coupled problem or magnetic damping problem, two types of coupling methods are employed: one is the simultaneous method or tight-coupling method in which equations of eddy current and structure are solved simultaneously, and the other is the staggered method or loose-coupling method in which these equations are solved alternately. Although the staggered method is attractive in computing cost rather than the simultaneous method, results of the staggered method diverge as the result of numerical instability under the analysis conditions of higher magnetic field and larger time increment. Therefore, stabilized staggered methods are desirable for the coupled analysis.

The staggered methods have been applied to the magnetic damping problems of TEAM problem 12 by Turner and Hua [1] and 16 by Takagi [2]. Niho *et al.* [3] have examined the stability of a staggered method using finite element in time. Horie *et al.* [4] have examined and compared the stability of several staggered methods using the characteristic equation of the recurrence relation. Park and Felippa [5] have shown that the staggered methods constructed by the reformulated governing equations are unconditionally stable for the fluid-structure interaction analysis. However, the stabilized staggered methods for the magnetic damping problem have not been proposed yet. In this paper, a new staggered method is proposed based on the reformulation of governing equations of eddy current and structure. Numerical stability is also discussed for the magnetic damping problem using the characteristic equation of the recurrence relation.

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Y. Tanaka, E. Shintaku, and Y. Fujimoto are with the Department of Social and Environmental Engineering, Graduate School of Engineering, Hiroshima University, Hiroshima 739-8527, Japan (e-mail: yoshi@naoe.hiroshima-u.ac.jp; shintaku@naoe.hiroshima-u.ac.jp; fujimoto@naoe.hiroshima-u.ac.jp).

T. Horie and T. Niho are with the Department of Mechanical System Engineering, Kyushu Institute of Technology, Fukuoka 820-8502, Japan (e-mail: horie@mse.kyutech.ac.jp; niho@mse.kyutech.ac.jp).

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II. Analysis Method of the Coupled Problem

A. Modal Equations of the Coupled Problem

In the analysis of the electromagnetic and structural coupled problem, combining the finite-element equations of motion and eddy current is needed. The matrix equation of structure is expressed using displacement $\boldsymbol{u}$ and the normal component of the current vector potential $\boldsymbol{T}$ by

$$\mathbf{M}\ddot{\boldsymbol{u}} + \mathbf{K}\boldsymbol{u} = \boldsymbol{f} + \mathbf{C}_s\boldsymbol{T} \tag{1}$$

where $\mathbf{M}, \mathbf{K}, \mathbf{C}_s$, and $\boldsymbol{f}$ are the mass matrix, the stiffness matrix, the coupling submatrix by electromagnetic force, and the external force, respectively. The matrix equation of eddy current is expressed by

$$\mathbf{U}\dot{\boldsymbol{T}} + \mathbf{R}\boldsymbol{T} = \dot{\boldsymbol{b}} + \mathbf{C}_e\dot{\boldsymbol{u}} \tag{2}$$

where $\mathbf{U}, \mathbf{R}, \mathbf{C}_e$, and $\dot{\boldsymbol{b}}$ are the inductance matrix, the resistance matrix, the coupling submatrix by electromotive force, and the change of external magnetic field, respectively. Since the numerical instability is associated with vibration modes, it is convenient to use decoupled ordinary equations of (1) and (2) in modal coordinates for discussion. The equation of motion in the modal coordinate $\boldsymbol{\xi}$ results in

$$\ddot{\boldsymbol{\xi}} + \boldsymbol{\Omega}\boldsymbol{\xi} = \boldsymbol{\Phi}_s^T\mathbf{C}_s\boldsymbol{\Phi}_e\boldsymbol{\zeta} + \boldsymbol{\Phi}_s^T\boldsymbol{f} \tag{3}$$

where $\boldsymbol{\Omega}$ is the stiffness matrix diagonalized by the modal matrix of structure $\boldsymbol{\Phi}_s$, and $\boldsymbol{\zeta}$ is the modal coordinate of eddy current. Similarly, the equation of eddy current in the modal coordinate $\boldsymbol{\zeta}$ is

$$\dot{\boldsymbol{\zeta}} + \boldsymbol{\Gamma}\boldsymbol{\zeta} = \boldsymbol{\Phi}_e^T\mathbf{C}_e\boldsymbol{\Phi}_s\dot{\boldsymbol{\xi}} + \boldsymbol{\Phi}_e^T\dot{\boldsymbol{b}} \tag{4}$$

where $\boldsymbol{\Gamma}$ is the resistance matrix diagonalized by the modal matrix of the eddy current $\boldsymbol{\Phi}_e$. The diagonal elements of $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ are angular frequencies of structure and eigenvalues of eddy current, respectively.

B. Conventional Staggered Methods

In the conventional staggered methods, Newmark's β method and the Crank–Nicolson method are applied to (3) and (4), respectively. To construct the staggered methods, the electromotive force in (4) is estimated using the velocity of previous time step or the approximated velocity using forward difference. Iterations can be added for each time step. Generally, the results

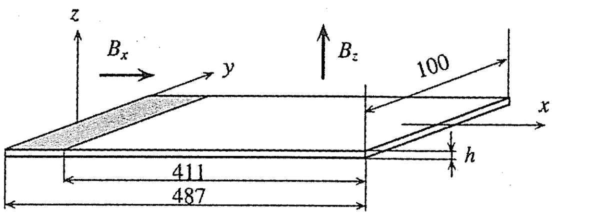

Fig. 1. Schematic diagram of a bending plate in a steady magnetic field B_x.

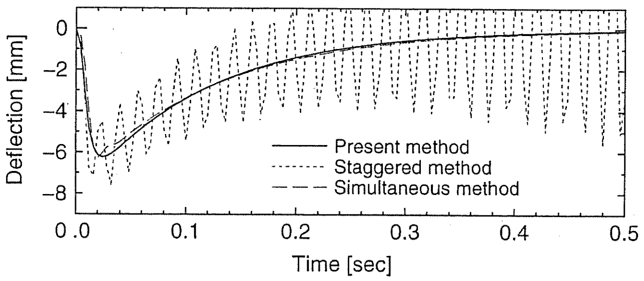

Fig. 2. Deflection of the plate at the free end under the conditions of $B_x = 2.0$ T and $\Delta t = 4$ ms.

of the conventional staggered method can diverge more easily when a higher magnetic field and higher vibration modes are used.

C. Augmented Staggered Method

Substituting ζ in (4) into (3) leads to a reformulated equation of motion for the coupled problem between eddy current and structural vibration such as

$$\ddot{\xi} + \Omega\xi - \boldsymbol{\Phi}_s^T \mathbf{C}_s \boldsymbol{\Phi}_e \boldsymbol{\Gamma}^{-1} \boldsymbol{\Phi}_e^T \mathbf{C}_e \boldsymbol{\Phi}_s \dot{\xi} = -\boldsymbol{\Phi}_s^T \mathbf{C}_s \boldsymbol{\Phi}_e \boldsymbol{\Gamma}^{-1} \dot{\zeta} + \boldsymbol{\Phi}_s^T \mathbf{C}_s \boldsymbol{\Phi}_e \boldsymbol{\Gamma}^{-1} \boldsymbol{\Phi}_e^T \dot{b} + \boldsymbol{\Phi}_s^T f. \quad (5)$$

Compared with equation of motion (3), a damping term is added in the left hand side of the equation. Therefore, stabilizing effect is expected when (4) and (5) are solved alternately.

III. Finite-Element Analysis of the Coupled Problem

A. Analysis Model

Fig. 1 shows the analysis model of a copper plate in a steady magnetic field B_x [1]. The interaction between the eddy current, which is induced by the transient magnetic field B_z, and the steady magnetic field B_x causes bending deformation. While the plate is vibrating, the electromotive force by the deformation velocity and B_x influences the eddy current. Solving this problem with different B_x and time increment Δt, the numerical stability will be discussed in the following.

B. Analysis Results

Fig. 2 shows the deflection calculated by the augmented staggered method. The result of the staggered method in which the electromotive force is evaluated using the previous time step velocity, and that of the simultaneous method are also shown for comparison. Vibration mode 1, which is the fundamental bending mode, and all eddy-current modes are included in all methods. According to Fig. 2, while the result of the conventional staggered method shows numerical instability, the augmented method is stable in the same manner as the simultaneous method.

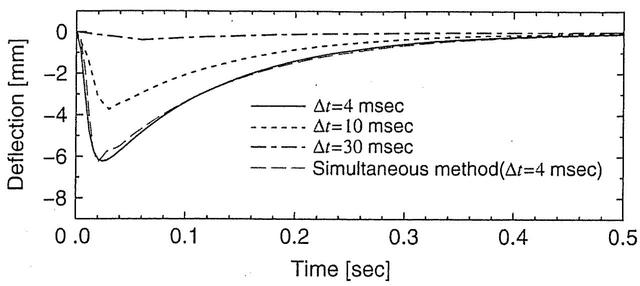

Fig. 3. Deflection by the augmented staggered method with different time increments when $B_x = 2.0$ T.

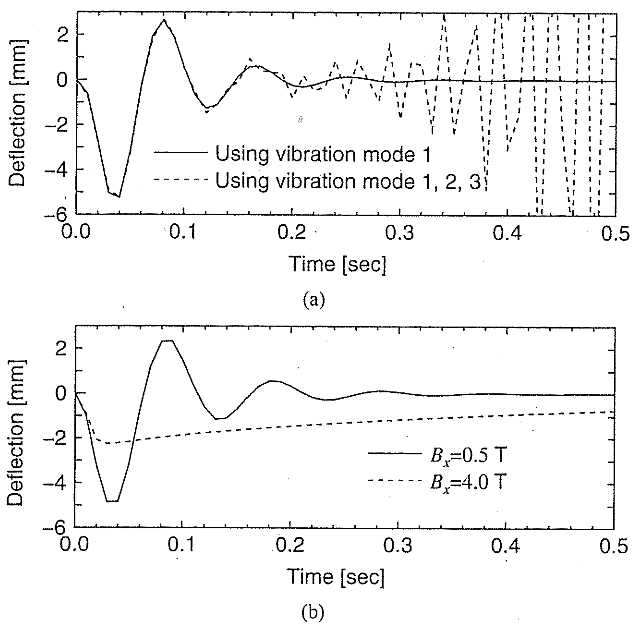

Fig. 4. Deflection with different vibration modes. (a) Conventional staggered method with $B_x = 0.5$ T and $\Delta t = 10$ ms. (b) Augmented staggered method with all vibration modes and $\Delta t = 10$ ms.

Fig. 3 is the results of the augmented staggered method with large time increments. Although the critical time increment of the staggered method was 7.48 ms for $B_x = 2.0$ T, the numerical instability does not appear in the augmented method even if the time increment is larger than 7.48 ms. Thus, this method is more stable than the conventional staggered method. It is noted that the amplitudes become smaller because of the error caused by large time increment.

Fig. 4 shows the results with more vibration modes. According to the result of the conventional staggered method, numerical instability occurs in a higher vibration mode such as the mode 2 or mode 3 because of the smaller critical time increment. On the other hand, the results of the augmented method are stable even for all vibration modes and a higher magnetic field.

IV. Stability Analysis of the Augmented Staggered Method

A. Stability-Analysis Method

Stability of the augmented staggered method is discussed in this section using recurrence relation of the time integration scheme. For each vibration mode and eddy current mode, the recurrence relation of the coupled system is expressed such as

$$\begin{Bmatrix} \ddot{\xi}_{t+\Delta t} \\ \dot{\xi}_{t+\Delta t} \\ \xi_{t+\Delta t} \\ \zeta_{t+\Delta t} \end{Bmatrix} = \mathbf{A} \begin{Bmatrix} \ddot{\xi}_{t} \\ \dot{\xi}_{t} \\ \xi_{t} \\ \zeta_{t} \end{Bmatrix} + L_1 f_{t+\Delta t} + \boldsymbol{L}_2 \dot{B}_{t+\theta\Delta t}. \quad (6)$$

The stability of time integration scheme is examined based on the eigenvalue λ of the coupled integration operator $\mathbf{A}$. In just the same manner as in the conventional stability analysis, the coupled system is stable when $|\lambda| \leq 1$, while it is unstable when $|\lambda| > 1$ [4].

B. Characteristics Equation

For an eddy-current mode and a vibration mode, (4) and (5) reduce to

$$\dot{\zeta} + \frac{R}{U}\zeta = \frac{C_e}{U}\dot{\xi} - \frac{1}{U}\dot{b} \quad (7)$$

and

$$\ddot{\xi} + \frac{k}{m}\xi - \frac{C_e C_s}{mR}\dot{\xi} = -\frac{C_s}{m}\frac{U}{R}\dot{\zeta} - \frac{C_s}{mR}\dot{b}. \quad (8)$$

Applying the Crank–Nicolson method to (7) and Newmark's β method to (8), recurrence relation such as (6) is obtained. The characteristic equation obtained from the recurrence relation is expressed by

$$\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0 \quad (9)$$

where constants $c_3, c_2, c_1,$ and c_0 can be calculated using $k/m, R/U, C_sC_e/mU$, and θ such as

$$c_3 = -2 + \frac{1}{d_1} + \frac{\frac{R}{U}\theta}{d_2}\left(\frac{1}{\theta} - 1\right) - \frac{1}{\Delta t d_2} + \frac{\frac{C_eC_s}{mR}}{\Delta t\omega^2 d_1} - \frac{\frac{C_eC_s}{mU}}{2\Delta t^2\omega^2 \frac{R}{U} d_1 d_2}$$

$$c_2 = 1 + \frac{2\frac{R}{U}\theta}{d_2}\left(-\frac{1}{\theta} + 1\right) + \frac{\frac{R}{U}\theta}{d_1 d_2}\left(\frac{1}{\theta} - 1\right) - \frac{\frac{C_eC_s}{mR}}{\Delta t\omega^2 d_1} + \frac{2}{\Delta t d_2} - \frac{1}{\Delta t d_1 d_2} - \frac{\frac{C_eC_s}{mR}}{\Delta t^2\omega^2 d_1 d_2} + \frac{\frac{C_eC_s}{mU}}{2\Delta t^2\omega^2\frac{R}{U} d_1 d_2} + \frac{\frac{C_eC_s}{mR}\frac{R}{U}}{\Delta t\omega^2 d_1 d_2} - \frac{\frac{C_eC_s}{mR}\frac{R}{U}\theta}{\Delta t\omega^2 d_1 d_2}$$

$$c_1 = -\frac{1}{\Delta t d_2} + \frac{\frac{R}{U}\theta}{d_2}\left(\frac{1}{\theta} - 1\right) + \frac{\frac{C_eC_s}{mR}}{\Delta t^2\omega^2 d_1 d_2} + \frac{\frac{C_eC_s}{mR}}{2\Delta t^2\omega^2\frac{R}{U} d_1 d_2} - \frac{\frac{C_eC_s}{mR}\frac{R}{U}}{\Delta t\omega^2 d_1 d_2} + \frac{\frac{C_eC_s}{mR}\frac{R}{U}\theta}{\Delta t\omega^2 d_1 d_2}$$

$$c_0 = -\frac{\frac{C_eC_s}{mU}}{2\Delta t^2\omega^2\frac{R}{U} d_1 d_2}$$

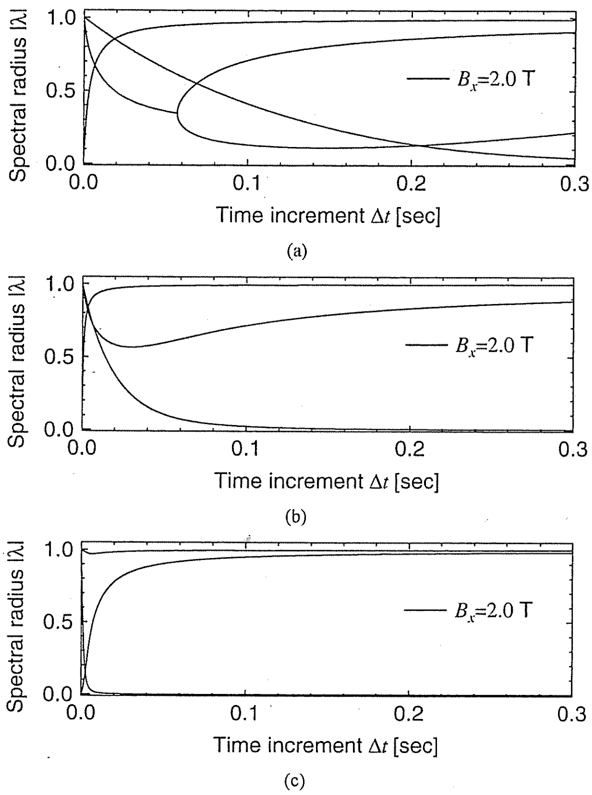

Fig. 5. Change of spectral radius $|\lambda|$ with time increment Δt. (a) Vibration mode 1 (bending mode of angular frequency $\omega = 6.70 \times 10^1$ rad/s). (b) Vibration mode 2 (bending mode of $\omega = 4.20 \times 10^2$ rad/s). (c) Vibration mode 3 (torsional mode of $\omega = 6.30 \times 10^2$ rad/s).

where

$$d_1 = \frac{1}{4} + \frac{1}{\Delta t^2\omega^2} + \frac{\frac{C_eC_s}{mR}}{2\Delta t\omega^2}$$
$$d_2 = \frac{1}{\Delta t} + \frac{R}{U}\theta.$$

θ is the parameter of the Crank–Nicolson method and the backward difference method. Since the parameters k/m, R/U, and C_eC_s/mU can be expressed as the function of young's modulus E, mass density ρ, electric conductivity κ, plate thickness h, and steady magnetic field B_x, the solution λ of (9) are the function of E, ρ, κ, h, B_x, and Δt similar to the coupling intensity parameter [6].

C. Discussion of Stability

Fig. 5 shows the relation between Δt and $|\lambda|$ for the coupled problem shown in Fig. 1 when $\theta = 1/2$ and $B_x = 2.0$ T. Since the spectral radius $|\lambda|$ is always less than 1 for the time increment Δt of less than 300 ms, numerical instability does not occur in the result of the augmented staggered method as shown in Fig. 3. Further discussion on the case with $\Delta t \to \infty$ or $B_x \to \infty$ are described in the following.

When Δt approaches infinity, (9) reduces to

$$\lambda^4 + \left(1 + \frac{1}{\theta}\right)\lambda^3 + \left(\frac{2}{\theta} - 1\right)\lambda^2 + \left(\frac{1}{\theta} - 1\right)\lambda = 0 \quad (10)$$

for any vibration and eddy-current modes. Substituting $\theta = 1$ into (10) for the backward difference method yields

$$\lambda + 2\lambda^2 + \lambda^3 = 0. \tag{11}$$

Similarly, in the case of the Crank–Nicolson method with $\theta = 1/2$, (10) becomes

$$1 + 3\lambda + 3\lambda^2 + \lambda^3 = 0. \tag{12}$$

Since the solutions of (11) and (12) satisfy the stability condition of $|\lambda| \leq 1$, then the augmented method is unconditionally stable for any Δt and any vibration and eddy-current modes.

In the case of $B_x \to \infty$, (9) reduces to

$$\begin{aligned}\lambda^4 &+ \left(\frac{\frac{R}{U}}{\frac{1}{\Delta t} + \frac{R}{U}\theta} - \frac{\frac{R}{U}\theta}{\frac{1}{\Delta t} + \frac{R}{U}\theta} - \frac{2}{\Delta t\left(\frac{1}{\Delta t} + \frac{R}{U}\theta\right)}\right)\lambda^3 \\ &+ \left(\frac{1}{\Delta t\left(\frac{1}{\Delta t} + \frac{R}{U}\theta\right)} - 1\right)\lambda^2 \\ &+ \left(-\frac{\frac{R}{U}}{\frac{1}{\Delta t} + \frac{R}{U}\theta} + \frac{\frac{R}{U}\theta}{\frac{1}{\Delta t} + \frac{R}{U}\theta} + \frac{2}{\Delta t\left(\frac{1}{\Delta t} + \frac{R}{U}\theta\right)}\right)\lambda \\ &- \frac{1}{\Delta t\left(\frac{1}{\Delta t} + \frac{R}{U}\theta\right)} = 0.\end{aligned} \tag{13}$$

The solutions λ are obtained for the backward difference method ($\theta = 1$) as

$$\lambda = -1, 1, \frac{-j}{-j + \sqrt{\Delta t}\sqrt{\frac{R}{U}}}, \quad \frac{j}{j + \sqrt{\Delta t}\sqrt{\frac{R}{U}}}. \tag{14}$$

Therefore, the stability condition of $|\lambda| \leq 1$ is always satisfied. As for the Crank–Nicolson method ($\theta = 1/2$), the solutions become

$$\begin{aligned}\lambda = -1, 1, &\frac{4 - \Delta t\frac{R}{U} + \sqrt{\Delta t}\sqrt{\frac{R}{U}}\sqrt{-16 + \Delta t\frac{R}{U}}}{2\left(2 + \Delta t\frac{R}{U}\right)} \\ &\frac{4 - \Delta t\frac{R}{U} - \sqrt{\Delta t}\sqrt{\frac{R}{U}}\sqrt{-16 + \Delta t\frac{R}{U}}}{2\left(2 + \Delta t\frac{R}{U}\right)}.\end{aligned} \tag{15}$$

To satisfy the stability condition of $|\lambda| \leq 1$

$$0 < \Delta t\frac{R}{U} \tag{16}$$

should be valid. It is true because Δt and R/U are always positive.

Then, it is proved that the augmented staggered method is unconditionally stable for any $\Delta t, B_x$, vibration modes and eddy current modes.

V. Conclusion

The augmented-staggered method is proposed for the electromagnetic and structural coupled problem or the magnetic damping problem. Finite-element coupled analyses were performed, and the stable solutions were obtained using this method. According to the stability analysis using the characteristic equation of the coupled system, the present method is unconditionally stable to the intensity of magnetic field and size of time increment for any vibration modes and eddy current modes.

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