COMBINED ESTIMATION AND SUBOPTIMAL CONTROL FOR NONLINEAR SYSTEMS

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SYNOPSIS

This paper presents a method of combining an estimation and a suboptimal control for nonlinear systems. This method is divided into three parts of estimation, judgement and control. At the first part, all states of the system are estimated under free control by such nonlinear observer as an extended linear observer. In the next part, the estimated values are judged by a judging function whether they are close enough to the true state values. Using the estimated values as the approximated values of state, in the last part, the nonlinear system is controlled by the suboptimal control policy. The policy is derived by applying a formal linearization theory on the function space and a linear control theory.

The remainder of the paper is devoted to the digital simulations of simple example, whose system is a nonlinear control system of one synchronous machine with an infinite bus.

1. INTRODUCTION

Optimal nonlinear combined estimation and control is a major unsolved problem in control theory. We propose to divide this problem into three parts. Those are the estimation part, the judging part and the control part. The estimated values obtained at the estimation part are checked by the judging function, which is stated in the companion paper [1] .

The judging part, in which the judging function is involved, judges whether the differences between the estimated values and the true values are neglegibly small. After this judgement, the control policy is applied by regarding these estimates obtained last as the initial values of the system to be controlled. If the system were to be controlled before the accurate estimates had been obtained, the system would be forced into the unexpected state.

The nonlinear control policy discussed in this paper is based on the augmented linearization method [2] . The original nonlinear system is transformed into the linearized system over the functional space spanned by the polynomials. That is, the nonlinear system equation is expanded into the Taylor's series about the steady state and the augmented linearized system is constructed by regarding each higher order term of the series as a new state variable. Then the linear optimal control theory is applied to the resulting linearized system to get the suboptimal control policy for the nonlinear system.

2. PROBLEM FORMULATION

Let us consider the nonlinear system described by the following differential equation

$$\dot{x}(t) = f(x,\ u,\ t) \tag{1}$$

where x represents the n-dimensional state vector, u is the m-dimensional control vector.

The measurement equation of the system is given as

$$y(t) = h(x) \tag{2}$$

where y is the r-dimensional vector of measurements, f and h are both nonlinear vector valued functions with respective dimensions.

We wish to find the control which minimizes

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the cost function

$$J = \int_{t_0}^{\infty} (x^T Q x + u^T R u)\, dt \tag{3}$$

where

$$x(t_0) = x_0, \tag{4}$$

Q is an $n \times n$ nonnegative definite matrix and R is an $m \times m$ positive definite matrix.

We consider the combined estimation and control problem by dividing it into the following three parts.

[Estimation Part]

The first thing we must do is to synthesize an estimator such as an observer which estimates the free system, that is $u = 0$ at Eq.(1).

[Judging Part]

The estimated values obtained by the observer are checked by the judging function (Companion Paper [1]) whether these values are close enough to the true values of the system. When we have got the accurate estimates $\hat{x}(t_0)$ at time t_0, we proceed to the next part.

[Control Part]

If we let $x(t_0) = \hat{x}(t_0)$ at Eq.(4), we can constitute the suboptimal control $u(t)$ $(t \geqq t_0)$. The way of constitution is discussed in the next section (see [2]).

3. SUBOPTIMAL CONTROL

Consider the nonlinear differential system where the control enters in a linear fashion

$$\dot{x}(t) = f(x) + G(x) u(t) \tag{5}$$

with the cost function (3).

We assume that $f(0) = 0$ and $G(0) \neq 0$. And the initial state of the system is assumed to be known, i. e. $x(t_0) = x_0$.

At first, we define the following set W_i as

$$W_i = \{x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} \mid l_1 + l_2 + \cdots + l_n = i\}$$
$$x(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$$

where l_j and i are nonnegative integers.

If we let $\rho(n, i)$ be the number of elements of W_i, we may observe that

$$\rho(n, i) = (i+1)(i+2) \cdots\cdots (i+n-1) / (n-1)!.$$

We then introduce the vector $S_r(t)$

$$S_r(t) = [\omega_r(1) \quad \omega_r(2) \quad \cdots\cdots \quad \omega_r(\rho(n, r))]^T \tag{6}$$

where $\omega_r(j) \in W_r$, whose elements are arranged properly and the following vector

$$Z_r(t) = [S_1(t)^T S_2(t)^T \cdots\cdots S_r(t)^T]^T \tag{7}$$

is also introduced.

It is convenient for us to rewrite Eq.(7) in terms of each element as

$$Z_r(t) = [z_1(t) \quad z_2(t) \quad \cdots\cdots \quad z_p(t)]^T,$$
$$p = \sum_{j=1}^{r} \rho(n, j)$$

where $z_i(t) \in W_j \quad (j = 1 \sim r)$.

On the assumption that the control is restricted to a linear combination of $z_i(t)$, the goal is to find the optimal control of the form

$$u = K Z_r \tag{8}$$

for the control problem described by Eqs.(5), (3) and (4).

Without loss of generality, we consider the problem of scalar control for simplicity, that is, the control is given as

$$u = \sum_{i=1}^{p} k_i z_i(t).$$

We can easily extend it to the problem when u is m-dimensional.

From Eq.(6) we see that

$$x(t) = S_1(t).$$

If we substitute this equation into Eq.(5), we get

$$\dot{S}_1(t) = f(S_1) + G(S_1) u(t). \tag{9}$$

We should note that

$$f_i(S_1) + g_i(S_1) u = [f_i(S_1) + \{g_i(S_1) - g_i(0)\} u] + g_i(0) u \qquad (i = 1 \sim n)$$

where

$$f = [f_1 \quad f_2 \quad \cdots\cdots \quad f_n]^T$$
$$G = [g_1 \quad g_2 \quad \cdots\cdots \quad g_n]^T$$

Therefore Eq.(9) is described by the form

$$\dot{S}_1(t) = \begin{bmatrix} f_1(S_1) + [g_1(S_1) - g_1(0)] \sum_{i=1}^{p} k_i z_i \\ \vdots \\ f_n(S_1) + [g_n(S_1) - g_n(0)] \sum_{i=1}^{p} k_i z_i \end{bmatrix} + \begin{bmatrix} g_1(0) \\ \vdots \\ g_n(0) \end{bmatrix} u$$

If ξ is the element of $W_q (q > 1)$, that is, $\xi \in S_q$, we may observe that

$$\frac{d}{dt}\xi = \frac{d}{dt} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n} \qquad (l_1 + l_2 + \cdots + l_n = q)$$
$$= \sum_{i=1}^{n} l_i x_1^{l_1} \cdots x_i^{l_i - 1} \cdots x_n^{l_n} [f_i(x) + g_i(x) u].$$

From the Taylor expansion theory, we obtain

$$f_i(x)+g_i(x)u = \sum_{l=0}^{\infty} \sum_{l_1+\cdots+l_n=l} \left\{ \frac{\partial^l f_i}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}}\bigg|_{x=0} + \frac{\partial^l g_i}{\partial x_1^{l_1}\cdots\partial x_n^{l_n}}\bigg|_{x=0} \sum_{j=1}^{p} k_i z_j \right\} \cdot \prod_{m=1}^{n} \frac{x_m^{l_m}}{l_m!} .$$

So we see that $\frac{d}{dt}\xi$ is described by the linear combination of the elements in $S_{q'}(q \leq q' < \infty)$. If we put our attentions only on the elements in S_1 to S_r, that is, if we truncate at Z_r, we obtain after some arrangements

$$\dot{Z}_r(t) = \begin{bmatrix} A_{11}^r(K) & A_{12}^r(K) \\ 0 & A_{22}^r(K) \end{bmatrix} Z_r(t) + \begin{bmatrix} g_1(0) \\ \vdots \\ g_n(0) \\ 0 \end{bmatrix} u(t)$$

$$= A_r(K)Z_r(t) + B_r u(t) \tag{10}$$

where

$$K = [k_1 \quad k_2 \quad \cdots\cdots \quad k_p].$$

Let's call Eq.(10) the approximated equation at r^{th} order. Naturally the cost function for Eq.(10) must be rewritten as

$$J' = \int_{t_0}^{\infty} (Z_r^T Q_r Z_r + u^T R u) dt \tag{11}$$

where

$$Q_r = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}.$$

The optimal control law for a constant linear system with a quadratic performance index, which is expressed by Eqs.(10) and (11), is well known and given as

$$u = -R^{-1} B_r^T P_r Z_r \tag{12}$$

where P_r is the solution of the following Riccati equation

$$-P_r A_r - A_r^T P_r + P_r B_r R^{-1} B_r^T P_r - Q_r = 0. \tag{13}$$

Though A_r is not constant, let us use Eqs.(12) and (13) as our suboptimal control law [2].

From Eqs.(8) and (12), we see that

$$KZ_r = -R^{-1} B_r^T P_r Z_r$$

to obtain

$$K = -R^{-1} B_r^T P_r.$$

P_r is solved as a solution of Eq.(13), because Eq.(13) is reduced to the quadratic equation with respect to P_r only. Applying the resulting control law to the given nonlinear problem of Eqs. (5), (3) and (4), the solution of the suboptimal control problem has been completed.

In a similar fashion which was used to obtain Eq.(10), the dynamic equation for Z_{r+1} is given as

$$\dot{Z}_{r+1}(t) = \begin{bmatrix} A_{11}^{r+1} & A_{12}^{r+1} \\ 0 & A_{22}^{r+1} \end{bmatrix} Z_{r+1}(t) + \begin{bmatrix} B_r \\ 0 \end{bmatrix} u(t)$$

$$= A_{r+1} Z_{r+1}(t) + B_{r+1} u(t)$$

where

$$A_{11}^{r+1} = A_r,$$

and we have

$$-P_{r+1}A_{r+1} - A_{r+1}^T P_{r+1} + P_{r+1} B_{r+1} R^{-1} B_{r+1}^T P_{r+1} - Q_{r+1} = 0 . \tag{14}$$

By substituting A_{r+1}, B_{r+1},

$$P_{r+1} = \begin{bmatrix} P_{11}^{r+1} & P_{12}^{r+1} \\ P_{12}^{r+1\,T} & P_{22}^{r+1} \end{bmatrix} \text{and}$$

$$Q_{r+1} = \begin{bmatrix} Q_r & 0 \\ 0 & 0 \end{bmatrix} \qquad (Q_1 = Q)$$

into Eq.(14) and giving attention to P_{11}^{r+1}, we obtain

$$-P_{11}^{r+1} A_r - A_r^T P_{11}^{r+1} + P_{11}^{r+1} B_r R^{-1} B_r^T P_{11}^{r+1} - Q_r = 0. \tag{15}$$

By comparing Eq.(15) with Eq.(13), we shall see that

$$P_{11}^{r+1} = P_r .$$

That is, P_{11}^{r+1} which is the sub-matrix of P_{r+1} is the same as P_r already calculated. This means that we have only to calculate P_{12}^{r+1} and P_{22}^{r+1} to raise the degree of approximation from r^{th} to $r+1^{th}$.

It is remarkable that the rate of increase of calculation is not sudden but proportional.

4. EXAMPLE

Let us consider the nonlinear optimal control problem for one synchronous machine with an infinite bus. The dynamics are described by the equation

$$M\ddot{\delta} + D\dot{\delta} + K(1+u)\sin\delta = P_{in}$$

where

δ ; load angle, M; moment of inertia,
D; damping coefficient, K; maximum output,
P_{in}; mechanical input.

Assume that we can observe the speed deviation $\dot{\delta}$. The problem is to be reduced to the form of Eq.(5) as follows.

x is 2-dimensional state vector and u is 1-dimensional control vector,

$$f_1 = x_2$$

$$f_2 = -\frac{K}{M}\sin(x_1+\delta_\infty) - \frac{D}{M}x_2 + \frac{P_{\text{in}}}{M}$$

and

$$g_1 = 0\ ,\quad g_2 = -\frac{K}{M}\sin(x_1+\delta_\infty)$$

where

$$x_1 = \delta - \delta_\infty,\qquad x_2 = \dot{\delta}\,,$$

$\delta_\infty = \sin^{-1}(P_{\text{in}}/K)$ is the load angle in the steady state.

We wish to find the control which minimizes the cost function

$$J = \int_{t_0}^{\infty} (x_1^2 + x_2^2 + u^2)\, dt\,,$$

that is, $Q = I_2$ (unit matrix) and $R = 1$, where t_0 means the start time of control.

System constants used here are

$$M = 0.0265,\ \ D = 0.005,\ \ K = 1.0,\ \ P_{\text{in}} = 0.8$$

in per unit.

We synthesize the extended linear observer, being considered up to the second order in discretization, [1], [4] with the parameters

$T = 1/60$ (sec) (measurement period)

$C_{1|0} = 0.1I,\ \ V_k = 10^{-5}$ for all k.

We have applied the judging function $I'(t)$ described by Eq.(15) in the companion paper [1] under quite the same conditions.

The degree of the approximation for control is 2nd order, that is, $r = 2$ for Z_r. The Riccati equation (13) was easily solved by the Newton's method.

Two cases are investigated when the machine remains in the hunting region or in the step-out region.

[Hunting Region]

The initial values of the states are

$$x_1(0) = -0.327,\qquad x_2(0) = 0.2\ .$$

Control characteristics with two different $I'(t_0)$'s are shown in Fig.1. The one is the case when we have controlled the system after good estimates have been obtained, that is, the time t_0 at which the control starts is settled when $I'(t)$ has become less than the threshold value $\varepsilon' = 10^{-6}$ [1]. This start point is shown as P_1 in Fig.1. Another point P_2 has been taken before $I'(t)$ becomes less than ε'.

[Step-Out Region]

The initial values of the states are

$$x_1(0) = -0.722,\qquad x_2(0) = 0.205.$$

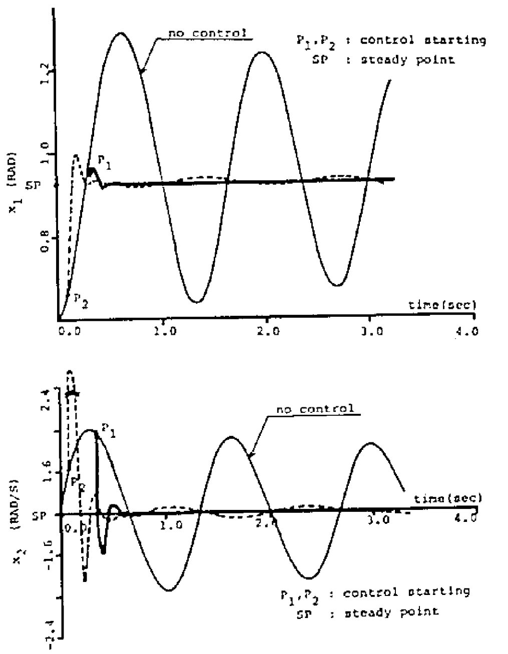

Fig. 1 Control characteristics of x in Hunting Region.
$I'(0.30) = 0.929\times10^{-6}$ at P_1,
$I'(0.10) = 0.28\times10^{-4}$ at P_2.

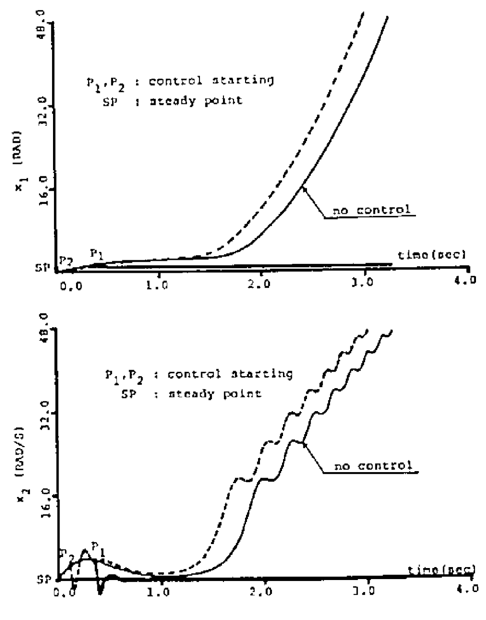

Fig. 2 Control characteristics of x in Step-Out Region
$I'(0.317) = 0.9\times10^{-6}$ at P_1,
$I'(0.10) = 0.59\times10^{-4}$ at P_2.

Control characteristics with different $I'(t_0)$'s are shown in Fig.2. The way of settling t_0 is the same as in the case of Hunting Region.

5. CONCLUSION

In general, up to now, the estimation problem and the control problem have been investigated separately. The important feature of this paper is that the estimation problem is changed to the control one at the time t_0 which is decided with the aid of the judging function. Needless to say, the time t_0 must be chosen carefully.

As for the nonlinear control policy proposed in this paper, we have obtained the promising results by the computer simulation, although the power system example demonstrated here is considered to have relatively high nonlinearities.

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7. REFERENCES

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