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Bilaterally Flexible Lyapunov Inequalities for Integral Input-to-State Stable Systems*§

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Abstract: This paper investigates stability of interconnection of integral input-to-state stable (iISS) systems. A new tool to verify the stability is proposed by focusing on Lyapunov inequalities each system is to satisfy in accordance with a small-gain-type condition. The purpose of this paper is to extend the technique of "flexible Lyapunov inequalities" developed previously for input-to-state stable(ISS) systems. The achievement is threefold. One is the employment of flexibility for both the systems connected with each other. The former technique only allows the flexibility to appear in one of the mutually connected systems. The second accomplishment is to cover iISS systems. The third is unification of the flexibility in Lyapunov inequalities for iISS and ISS systems. Establishment of the stability is based on explicit construction of $\mathbf{C}^1$ Lyapunov functions.

Keywords: Nonlinear interconnected systems, Integral input-to-state stability, Lyapunov function, Scaled small-gain condition

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1 Introduction

The problem of establishing stability of interconnected systems has attracted a lot of attention in the field of nonlinear systems control for many decades. For a nonlinear system whose behavior cannot be analyzed directly, decomposition into simpler subsystems sometimes permit verification of the overall system property. In the framework of dissipative theory(see [22]), one can derive dissipative properties of interconnected systems from dissipativity of individual subsystems (see, e.g., [6]). One of useful dissipative properties is input-to-state stability(ISS) proposed in [17]. Its storage functions serve as ISS Lyapunov functions. The ISS small-gain theorem proposed by [14] and [21] deals with the feedback interconnection of ISS systems and establishes its stability based on the notion of nonlinear gain when the nonlinear loop gain is less than identity.

The nonlinear gain is computed from dissipation inequalities of the individual subsystems. The inequalities are sometimes referred to as Lyapunov inequalities or Hamilton-Jacobi inequalities. There are two ways to look at the inequalities.

- Given a supply rate, solve the dissipation inequality for the storage function.
- Given a storage function, modify the dissipation inequality maintaining eligible dissipation a system can have.

The former view is essentially the direct approach to the optimal control. The latter fits the idea of Lyapunov redesign and also conforms to the inverse optimal approach[5]. In both situations, when one applies the ISS small-gain theorem, obtaining a successful Lyapunov inequality conforming to a small-gain condition is not a straightforward task. If a large gain is computed from a Lyapunov inequality, it does not satisfy the small-gain condition. If a restrictive supply rate is associated with the satisfaction of the small-gain condition, the system does not accept the supply rate. This fact motivated the author to develop his own idea of flexible Lyapunov inequalities, which provides many Lyapunov inequalities with which a single small-gain-type condition can establish stability of an interconnection of ISS systems (see [8, 9]). The technique introduces a degree of flexibility in choosing supply rates, which is a nonlinear extension of the idea of scaling technique utilized widely in $\mathcal{H}^{\infty}$-type linear robust control[3, 4, 16]. For a scaled $\mathcal{H}^{\infty}$-type formulation in dissipative inequalities, see [7].

This paper is the upgrade of flexible Lyapunov inequalities [8, 9] in the following three points:

- Introduction of flexibility into both the systems connected with each other
- Covering integral input-to-state stability(iISS) property in the technique of flexible Lyapunov inequalities
- Unification of the flexibility technique for iISS and ISS systems

The previous studies in [8, 9] only allow the flexibility to be included in one of the mutually connected systems. The class of iISS is broader than ISS. An ISS system is always iISS. The converse does not hold. The class of iISS systems encompasses more systems of practical importance than the ISS (see [1]).

In this paper, the interval $[0, \infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. The Euclidean norm of a vector in $\mathbb{R}^n$ of dimension n is denoted by $|\cdot|$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ and written as $\gamma \in \mathcal{K}$ if it is a continuous, strictly increasing function satisfying $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$ and written as $\gamma \in \mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function satisfying $\lim_{r\to\infty} \gamma(r) = \infty$.

2 Interconnected system

Consider the nonlinear interconnected system Σ consisting of two subsystems described by

$$\Sigma_0: \ \dot{x}_0 = f_0(t, x_0, u_0, r_0) \tag{1}$$

$$\Sigma_1: \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \tag{2}$$

These two systems are connected with each other through

$$u_0 = x_1, \quad u_1 = x_0 \tag{3}$$

Assume that $f_0(t, 0, 0, 0) = 0$ and $f_1(t, 0, 0, 0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$. We also assume that the functions f_0 and f_1 are piecewise continuous in t, and locally Lipschitz in the other arguments uniformly in t. The state vector of the interconnected system Σ is $x = [x_0^T, x_1^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. The exogenous signals $r_0 \in \mathbb{R}^{b_0}$ and $r_1 \in \mathbb{R}^{b_1}$ form a vector $r = [r_0^T, r_1^T]^T \in \mathbb{R}^b$. We will exploit dissipative property of each system instead of using f_i directly. When we investigate global asymptotic stability of the interconnected system, we suppose that $r_i(t) \equiv 0$, $i = 1, 2$.

3 Bilateral flexibility

The purpose of this paper is to propose a tool to establish the stability of the interconnected system Σ by making use of admissible flexibility in Lyapunov inequalities of the individual subsystems Σ_i. The following theorem is the main result which identifies a form of flexibility in the Lyapunov inequalities which can be associated with a fixed small-gain-type inequality leading to the stability of the interconnected system Σ.

Theorem 1 *For $i = 0, 1$, consider the following functions:*

$$\alpha_i, \sigma_i, \sigma_{ri} \in \mathcal{K} \tag{4}$$

$$\hat{\lambda}_i, \hat{\lambda}_{ri}: \mathbb{R}_+ \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{5}$$

$$\hat{\lambda}_i(s) > 0, \ \forall s \in (0, \infty) \tag{6}$$

$$V_i: \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}, \ \mathbf{C}^1 \tag{7}$$

$$\underline{\alpha}_i(|x_i|) \leq V_i(t, x_i) \leq \overline{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}, t \in \mathbb{R}_+ \tag{8}$$

$$\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty \tag{9}$$

where $\hat{\lambda}_i(s)$ is supposed to be absolutely continuous on any finite interval in $(0, \infty)$. For each $i = 0, 1$, assume that

$$\frac{dV_i}{dt} \leq \hat{\lambda}_i(V_i(t, x_i)) \left[-\alpha_i(|x_i|) + \sigma_i(|x_{1-i}|)\right] + \hat{\lambda}_{ri}(V_i(t, x_i))\sigma_{ri}(|r_i|) \tag{10}$$

holds along the trajectories of the system Σ_i for all $x \in \mathbb{R}^n$, $r_i \in \mathbb{R}^{b_i}$ and $t \in \mathbb{R}_+$. Suppose that there exist real numbers $c_0 > 1$ and $c_1 > 1$ such that

$$c_1\sigma_1 \circ \underline{\alpha}_0^{-1} \circ \overline{\alpha}_0 \circ \alpha_0^{-1} \circ c_0\sigma_0(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \tag{11}$$

is satisfied, then the following properties hold:

(a) If

$$\lim_{s\to\infty} \alpha_1(s) < \infty \ \Rightarrow \ \limsup_{s\to\infty} \hat{\lambda}_1(s) < \infty \tag{12}$$

holds, the equilibrium $x = 0$ of Σ is uniformly globally asymptotically stable (UGAS).

(b) If (12),

$$\left.\begin{array}{l} \lim_{s\to\infty} \alpha_0(s) < \infty \\ \lim_{s\to\infty} \alpha_1(s) < \infty \end{array}\right\} \Rightarrow \ \limsup_{s\to\infty} \hat{\lambda}_0(s) < \infty \tag{13}$$

$$\limsup_{s\to\infty} \frac{\hat{\lambda}_{ri}(s)}{\hat{\lambda}_i(s)} < \infty \tag{14}$$

and one of

(H1) $\lim_{s\to\infty} \alpha_0(s) = \infty$ *and* $\lim_{s\to\infty} \sigma_0(s) < \infty$

(H2) $\lim_{s\to\infty} \sigma_1(s) < \infty$ *and* $\lim_{s\to\infty} \sigma_0(s) < \infty$

are satisfied, the interconnected system Σ is iISS with respect to input r and state x.

(c) If (14) and

(H3) $\lim_{s\to\infty} \alpha_0(s) = \infty$ *and* $\lim_{s\to\infty} \alpha_1(s) = \infty$

are satisfied, the interconnected system Σ is ISS with respect to input r and state x.

Note that (11) implicitly requires

$$\lim_{s\to\infty} \alpha_0(s) = \infty \ \text{ or } \ \infty > \lim_{s\to\infty} \alpha_0(s) > \lim_{s\to\infty} \sigma_0(s) \tag{15}$$

The inverse of α_0 in (11) is not necessarily well defined for the entire $\mathbb{R}_+$. Instead, the condition (11) only needs $\alpha_0^{-1} \circ c_0\sigma_0 \in \mathcal{K}$ for $c_0 > 1$ which exists if and only if (15) holds. When we take $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$, i.e., no flexibilities, Theorem 1 can be viewed as a nonlinear small-gain theorem. The dissipation inequality (10) with $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$ implies that each Σ_i is iISS with respect to input (x_{1-i}, r_i) and state x_i. Hence, Theorem 1 includes the result of [10] as the special case when $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$. In the case of *(H3)*, the dissipation inequality (10) with $\hat{\lambda}_i = \hat{\lambda}_{ri}$ implies that Σ_i is ISS (see [20, 9]). Therefore, the claim *(c)* in Theorem 1 with $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, $i = 0, 1$ reduces to the ISS small-gain theorem in [14] and [21]. Indeed, the condition (11) associated with (10) for $\hat{\lambda}_0 = \hat{\lambda}_1 = 1$ is a nonlinear small-gain condition. It is worth noting that (10) does not restrict Σ_i to being ISS, even under $\hat{\lambda}_i = \hat{\lambda}_{ri} = 1$, if $\lim_{s\to\infty} \alpha_i(s) < \infty$ (see [18, 1]).

The Lyapunov inequalities of the from (10) in connection with the small-gain condition (11) were primarily inspired by the scaled $\mathcal{H}^\infty$-type linear robust control[3, 4, 16]. The flexible parameters $\{\hat{\lambda}_i, \hat{\lambda}_{ri}\}$ correspond to the scaling factors, which are now allowed to be nonlinear for nonlinear systems with not necessarily quadratic supply rates. One may argue that we can often remove $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ from (10) by redefining α_i, σ_i and σ_{ri}. Qualitatively, the two Lyapunov inequalities with and without $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ should be the same. However, each individual technique to remove $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ results in a quantitatively different gain of Σ_i. Some may satisfy the small-gain condition (11), while others may not. The formulation of flexible inequalities absorbs this quantitative variation of the small-gain restriction. Theorem 1 tells that, as far as $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ appear in the form of (10), the functions used in the small-gain-type condition (11) remain unchanged, while we can choose

$\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ from a large variety of functions. Note that this quantitative issue does not arise from cascade interconnection of systems. The primitive idea of the flexible Lyapunov inequalities first appeared in [8] for ISS systems in a restrictive setting. It was extended to a general case of ISS systems by [9]. The result in [9] was not only for ISS systems, but also unilateral, i.e., it was the *(c)* case with $\hat{\lambda}_1 = 1$ in Theorem 1.

Remark 1 The above theorem cannot be explained by individual Lyapunov functions in the form of $\hat{V}_i = \int_0^{V_i} 1/\hat{\lambda}_i(s)ds$ since they are not guaranteed to be integrable and radially unbounded. It is also mentioned that the technique of changing supply functions proposed in [19] is not applicable to iISS systems.

Remark 2 When

$$\lim_{s\to\infty} \alpha_1(s) = \infty \quad \text{or} \quad \infty > \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) \tag{16}$$

holds, we have $\alpha_1^{-1} \circ c_1\sigma_1 \in \mathcal{K}$ for sufficiently small $c_1 > 1$. Then, we obtain

$$c_1\sigma_1\left(\underline{\alpha}_0^{-1}\circ\overline{\alpha}_0\circ\alpha_0^{-1}\circ c_0\sigma_0\circ\underline{\alpha}_1^{-1}\circ\overline{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s)\right) \leq c_1\sigma_1(s),\ \forall s\in\mathbb{R}_+$$

from (11). The property $c_1\sigma_1 \in \mathcal{K}$ in the above inequality implies

$$\underline{\alpha}_0^{-1}\circ\overline{\alpha}_0\circ\alpha_0^{-1}\circ c_0\sigma_0\circ\underline{\alpha}_1^{-1}\circ\overline{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \leq s,\ \forall s\in\mathbb{R}_+$$

Applying $\alpha_0\circ\overline{\alpha}_0^{-1}\circ\underline{\alpha}_0 \in \mathcal{K}$ to both sides from left, we arrive at

$$c_0\sigma_0\circ\underline{\alpha}_1^{-1}\circ\overline{\alpha}_1\circ\alpha_1^{-1}\circ c_1\sigma_1(s) \leq \alpha_0\circ\overline{\alpha}_0^{-1}\circ\underline{\alpha}_0(s),\ \forall s\in\mathbb{R}_+ \tag{17}$$

Thus, in the case of (16), the existence of a pair of $c_0, c_1 > 1$ satisfying (17) is implied by the existence of another pair of $c_0, c_1 > 1$ achieving (11). Therefore, when (15) and (16) hold, the inequalities (11) and (17) are equivalent in the sense of the existence of $c_0, c_1 > 1$.

Remark 3 Arcak et al.[2] considers a time-invariant cascade interconnection in which an iISS system is driven by a globally asymptotically stable (GAS) system. Instead of constructing Lyapunov functions, they take a trajectory-based approach to prove GAS of the cascade under the assumption of trade-off between the convergence rate of the driving subsystem and the growth rate of iISS gain of the driven subsystem. This paper continues pursuing the stability problem of interconnections involving iISS subsystems in order to tackle feedback interconnections defined with $\alpha_1, \alpha_2 \in \mathcal{K}$. However, the focus is not simply on the extention of their philosophy, but rather on the introduction of flexibilities $\hat{\lambda}_i$ and $\hat{\lambda}_{ri}$ and the construction of Lyapunov functions for the whole system in the presence of external signals.

4 Relaxation for static components

This section shows that the stability criterion can be relaxed when one of the subsystems in Σ is static. Suppose that Σ_0 is static. Replace (1) by

$$\Sigma_0:\ v_0 = h_0(t, u_0, r_0) \tag{18}$$

which is connected with Σ_1 through $u_0 = x_1 \in \mathbb{R}^{n_1}$ and $u_1 = v_0 \in \mathbb{R}^{n_0}$. Assume that $h_0(t, 0, 0) = 0$ holds for all $t \in [t_0, \infty)$, $t_0 \geq 0$. Piecewise continuity in t and locally Lipschitzness in (u_0, r_0) which is uniform in t are also assumed for h_0. The state vector of the interconnected system Σ becomes $x = x_1 \in \mathbb{R}^n$, where $n = n_1$. The following theorem demonstrates that (11) can be relaxed.

Theorem 2 *Consider the following functions:*

$$\alpha_i, \sigma_i, \sigma_{ri} \in \mathcal{K}, \ i = 0, 1 \tag{19}$$
$$\hat{\lambda}_0, \hat{\lambda}_{r0} : \mathbb{R}^{n_0} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{20}$$
$$\hat{\lambda}_0(v_0) > 0, \ \forall v_0 \in \mathbb{R}^{n_0} \setminus \{0\} \tag{21}$$
$$\hat{\lambda}_1, \hat{\lambda}_{r1} : \mathbb{R}_+ \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{22}$$
$$\hat{\lambda}_1(s) > 0, \ \forall s \in (0, \infty) \tag{23}$$
$$V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}, \ \mathbf{C}^1 \tag{24}$$
$$\underline{\alpha}_1(|x_1|) \le V_1(t, x_1) \le \overline{\alpha}_1(|x_1|), \quad \forall x_1 \in \mathbb{R}^{n_1}, t \in \mathbb{R}_+ \tag{25}$$
$$\underline{\alpha}_1, \bar{\alpha}_1 \in \mathcal{K}_\infty \tag{26}$$

where $\hat{\lambda}_1(s)$ *is supposed to be absolutely continuous on any finite interval in* $(0, \infty)$*. Assume that the system* Σ_0 *satisfies*

$$0 \le \hat{\lambda}_0(v_0) \left[-\alpha_0(|v_0|) + \sigma_0(|x_1|)\right] + \hat{\lambda}_{r0}(v_0)\sigma_{r0}(|r_0|) \tag{27}$$

for all $x_1 \in \mathbb{R}^{n_1}$*,* $r_0 \in \mathbb{R}^{b_0}$ *and* $t \in \mathbb{R}_+$*, and that*

$$\frac{dV_1}{dt} \le \hat{\lambda}_1(V_1(t, x_1)) \left[-\alpha_1(|x_1|) + \sigma_1(|u_1|)\right] + \hat{\lambda}_{r1}(V_1(t, x_1))\sigma_{r1}(|r_1|) \tag{28}$$

holds along the trajectories of the system Σ_1 *for all* $x_1 \in \mathbb{R}^{n_1}$*,* $u_1 \in \mathbb{R}^{n_0}$*,* $r_1 \in \mathbb{R}^{b_1}$ *and* $t \in \mathbb{R}_+$*. Suppose that there exist real numbers* $c_0 > 1$ *and* $c_1 > 1$ *such that*

$$c_1\sigma_1 \circ \alpha_0^{-1} \circ c_0\sigma_0(s) \le \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \tag{29}$$

is satisfied, then the following properties hold:

(a) If (12) holds, the equilibrium $x = 0$ *of* Σ *is UGAS.*

(b) If (12), (13),

$$\lim_{s \to \infty} \sup_{s \le |v_0|} \frac{\hat{\lambda}_{r0}(v_0)}{\hat{\lambda}_0(v_0)} < \infty \tag{30}$$
$$\limsup_{s \to \infty} \frac{\hat{\lambda}_{r1}(s)}{\hat{\lambda}_1(s)} < \infty \tag{31}$$

and one of (H1) and (H2) are satisfied, the interconnected system Σ *is iISS with respect to input* r *and state* x_1*.*

(c) If (30), (31) and (H3) are satisfied, the interconnected system Σ *is ISS with respect to input* r *and state* x_1*.*

The right-hand side of (27) plays the role of a supply rate of Σ_0 although energy is never stored in static systems. Note that (29) again requires (15). It is worth noting that the property (15) for the pair $\{\alpha_0, \sigma_0\}$ in (27) is not restrictive for the system Σ_0 satisfying the local Lipschitzness when v_0 is not disconnected from r_0 in the sense that $\hat{\lambda}_{r0}(v_0)/\hat{\lambda}_0(v_0) > 0$ holds uniformly in $v_0 \in \mathbb{R}^{n_0} \setminus \{0\}$. To see this, assume that (15) is violated. Then, for each $|r_0| \neq 0$, there exists a constant $U > 0$ such that $\lim_{s\to\infty} \alpha_0(s) \le \sigma_0(U) + \hat{\lambda}_{r0}(v_0)\sigma_{r0}(|r_0|)/\hat{\lambda}_0(v_0)$ is satisfied. Hence, the inequality (27) allows v_0 to be arbitrarily large for u_0 and r_0 selected independently of v_0. This fact contradicts the local Lipschitzness.

The following shows that the stability condition for the interconnected system can be simplified further when the external signal r_0 affecting the static system is absent.

Theorem 3 *Consider functions satisfying*

$$\alpha_0, \sigma_0, \alpha_1, \sigma_1, \sigma_{r1} \in \mathcal{K} \tag{32}$$

$$\hat{\lambda}_0 : \mathbb{R}^{n_0} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{33}$$

$$\hat{\lambda}_0(v_0) > 0, \ \forall v_0 \in \mathbb{R}^{n_0} \backslash \{0\} \tag{34}$$

$$\hat{\lambda}_1, \hat{\lambda}_{r1} : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}_+, \ \mathbf{C}^0 \tag{35}$$

$$\inf_{t \in \mathbb{R}_+} \hat{\lambda}_1(t, x_1) > 0, \quad \forall x_1 \in \mathbb{R}^{n_1} \backslash \{0\} \tag{36}$$

and (24), (25) and (26). Assume that Σ_0 satisfies

$$0 \leq \hat{\lambda}_0(v_0) \left[-\alpha_0(|v_0|) + \sigma_0 |x_1|) \right] \tag{37}$$

for all $x_1 \in \mathbb{R}^{n_1}$ and $t \in \mathbb{R}_+$, and that

$$\frac{dV_1}{dt} \leq \hat{\lambda}_1(t, x_1) \left[-\alpha_1(|x_1|) + \sigma_1(|u_1|) \right] + \hat{\lambda}_{r1}(t, x_1) \sigma_{r1}(|r_1|) \tag{38}$$

holds along the trajectories of the system Σ_1 for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_0}$, $r_1 \in \mathbb{R}^{b_1}$ and $t \in \mathbb{R}_+$. Suppose that there exist real numbers $c_0 > 1$ and $c_1 > 1$ such that

$$c_1 \sigma_1 \circ \alpha_0^{-1} \circ c_0 \sigma_0(s) \leq \alpha_1(s), \ \forall s \in \mathbb{R}_+ \tag{39}$$

is satisfied, then the following properties hold:

(a) The equilibrium $x = 0$ of Σ is UGAS.

(b) If there exists $k \in (-\infty, 1]$ such that

$$\lim_{s \to \infty} \sup_{s \leq |x_1|, t \in \mathbb{R}_+} \frac{\hat{\lambda}_{r1}(t, x_1)}{[\underline{\alpha}_1(|x_1|)]^k} < \infty \tag{40}$$

is satisfied, the interconnected system Σ is iISS with respect to input r_1 and state x_1.

(c) If there exists $k \in (-\infty, 1]$ such that (40) and

$$\lim_{s \to \infty} \inf_{s \leq |x_1|, t \in \mathbb{R}_+} \frac{\hat{\lambda}_1(t, x_1) \alpha_1(|x_1|)}{[\bar{\alpha}_1(|x_1|)]^k} = \infty \tag{41}$$

are satisfied, the interconnected system Σ is ISS with respect to input r_1 and state x_1.

It is worth mentioning that the flexibility in the inequality (37) of the static system Σ_0, has no effect, i.e, (37) implies $0 \leq -\alpha_0(|v_0|) + \sigma_0(|x_1|)$.

5 $\mathbf{C}^1$ Lyapunov functions

5.1 Proof of Theorem 1

First, suppose that $\liminf_{s \to \infty} \hat{\lambda}_i(s) = 0$ and define

$$A_i(s) = \begin{cases} \hat{\lambda}_i(T), & s \in [0, T) \\ \hat{\lambda}_i(s), & s \in [T, \infty) \end{cases}$$

$$B_i(s) = \int_0^s \frac{1}{A_i(t)} dt, \quad W_i(t, x_i) = B_i \circ V_i(t, x_i)$$

$$\underline{\beta}_i(s) = B_i \circ \underline{\alpha}_i(s), \quad \overline{\beta}_i(s) = B_i \circ \overline{\alpha}_i(s)$$

for some $T > 0$. Then, we have

$$
\begin{aligned}
&B_i, \underline{\beta}_i, \overline{\beta}_i \in \mathcal{K}_\infty, \quad \liminf_{s\to\infty} \frac{\hat{\lambda}_i(s)}{A_i(s)} = 1 \\
&\underline{\beta}_i^{-1} \circ \overline{\beta}_i(s) = \underline{\alpha}_i^{-1} \circ \overline{\alpha}_i(s), \quad \forall s \in \mathbb{R}_+ \\
&\frac{\partial W_i}{\partial t} + \frac{\partial W_i}{\partial x_i} f_i = \frac{1}{A_i(V_i)} \left(\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i \right)
\end{aligned}
$$

Thus, we can transform the $\liminf_{s\to\infty} \hat{\lambda}_i(s) = 0$ case into

$$
\liminf_{s\to\infty} \hat{\lambda}_i(s) > 0, \quad i = 0, 1 \tag{42}
$$

via the following substitution.

$$
\begin{aligned}
&V_i \to W_i, \quad \underline{\alpha}_i \to \underline{\beta}_i, \quad \overline{\alpha}_i \to \overline{\beta}_i \\
&\hat{\lambda}_i(s) \to \frac{\hat{\lambda}_i \circ B_i^{-1}(s)}{A_i \circ B_i^{-1}(s)}, \quad \hat{\lambda}_{ri}(s) \to \frac{\hat{\lambda}_{ri} \circ B_i^{-1}(s)}{A_i \circ B_i^{-1}(s)}
\end{aligned}
$$

Note that (14) remains the same under this operation. The rest assumes (42).

Suppose that

$$
\lim_{s\to\infty} \alpha_0(s) = \lim_{s\to\infty} \sigma_0(s) = \infty \text{ and } \infty > \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) \tag{43}
$$

does not hold. In the case of

$$
\lim_{s\to\infty} \alpha_0(s) = \infty \quad \text{and} \quad \lim_{s\to\infty} \alpha_1(s) = \infty \tag{44}
$$

and the case of

$$
\infty = \lim_{s\to\infty} \alpha_0(s) > \lim_{s\to\infty} \sigma_0(s) \text{ and } \lim_{s\to\infty} \alpha_1(s) > \lim_{s\to\infty} \sigma_1(s) , \tag{45}
$$

there exist $\hat{\sigma}_1 \in \mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1 > 1$ such that

$$
\hat{c}_1 \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0 \circ \sigma_0(s) \leq \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \quad \forall s \in \mathbb{R}_+ \tag{46}
$$

$$
\sigma_1(s) \leq \hat{\sigma}_1(s), \ \forall s \in \mathbb{R}_+ \tag{47}
$$

$$
\hat{\alpha}_1(s) \leq \alpha_1(s), \ \forall s \in \mathbb{R}_+ \tag{48}
$$

$$
\lim_{s\to\infty} \hat{\sigma}_1(s) \geq \lim_{s\to\infty} \hat{\alpha}_1(s) \tag{49}
$$

are satisfied with

$$
\hat{\alpha}_1 = \alpha_1 \tag{50}
$$

under the assumption that (11) is achieved with some $c_1, c_2 > 1$. In fact, $\lim_{s\to\infty} \alpha_0(s) > \lim_{s\to\infty} \sigma_0(s)$ guarantees $X := \lim_{s\to\infty} \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0 \circ \sigma_0(s) < \infty$ for sufficiently small $\hat{c}_0 > 1$, which renders (46) and (49) simultaneously achievable since (46) does not impose any constraint on $\hat{\sigma}_1(s)$ for $s > X$. Also note that (49) is achieved by choosing $\hat{\sigma}_1 \in \mathcal{K}_\infty$ if $\lim_{s\to\infty} \alpha_1(s) = \infty$. In the case of

$$
\lim_{s\to\infty} \alpha_0(s) < \infty , \tag{51}
$$

there exist $\hat{\alpha}_1 \in \mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1 > 1$ such that (46), (47), (48) and (49) are satisfied with

$$\hat{\sigma}_1 = \sigma_1 \tag{52}$$

since $\lim_{s\to\infty} \alpha_0^{-1}\circ \hat{c}_0\circ \sigma_0(s) < \infty$ is guaranteed for sufficiently small $\hat{c}_0 > 1$. If none of (44), (45) and (51) holds, the inequalities (46), (47), (48) and (49) are fulfilled with

$$\hat{\alpha}_1 = \alpha_1, \quad \hat{\sigma}_1 = \sigma_1, \quad \hat{c}_i = c_i, \ i = 0, 1 \ . \tag{53}$$

Pick real numbers $\tau_1, \phi \geq 0$ satisfying

$$1 < \tau_1 < \hat{c}_1, \quad \left(\frac{\tau_1}{\hat{c}_1}\right)^{\phi} \leq (\tau_1 - 1)(\hat{c}_0 - 1) \tag{54}$$

Define $\hat{\zeta}_0, \hat{\zeta}_1 \in \mathcal{K}$ as

$$\hat{\zeta}_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)}\sqrt{\frac{\hat{c}_1}{\tau_1}}\left[\hat{\sigma}_1\circ \underline{\alpha}_0^{-1}(s)\right]^{\phi+1} \tag{55}$$

$$\hat{\zeta}_1(s) = \left[\alpha_0\circ \bar{\alpha}_0^{-1}\circ \underline{\alpha}_0\circ\hat{\sigma}_1^{-1}\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]^{\phi} \tag{56}$$

We can always select a continuous function $F_i : s \in \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &F_i(s) > 0, \quad \forall s \in (0, \infty) \\ &F_i(s)\hat{\lambda}_i(s) : \text{non-decreasing on } \mathbb{R}_+ \\ &F_i(s)\hat{\zeta}_i(s) \ : \text{non-decreasing on } \mathbb{R}_+ \end{aligned} \tag{57}$$

$$\lim_{s\to\infty} \hat{\alpha}_1(s) < \infty \ \Rightarrow \ \lim_{s\to\infty} F_i(s) < \infty \tag{58}$$

hold for each $i = 0, 1$. Here, (42) is used for achieving (58) under (57). Let U_i, $i = 0, 1$, denote

$$U_0(s) = \left[F_0\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}(s)\right]\left[\hat{\lambda}_0\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}(s)\right], \ s\in[0, \hat{\sigma}_1(\infty))$$

$$U_1(s) = \begin{cases} \left[F_1\circ\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}(\tau_1 s)\right]\left[\hat{\lambda}_1\circ\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}(\tau_1 s)\right] & , s \in [0, \frac{1}{\tau_1}\hat{\alpha}_1(\infty)) \\ F_1(\infty)\hat{\lambda}_1(\infty) & , s \in [\frac{1}{\tau_1}\hat{\alpha}_1(\infty), \hat{\sigma}_1(\infty)) \end{cases}$$

The properties (12) and (58) for $i = 1$ together with (48), (50) and (53) make sure that U_1 is well-defined. Note that $\hat{\sigma}_1(\infty) \geq (1/\tau_1)\hat{\alpha}_1(\infty)$ holds since (49). Define a non-decreasing continuous function $\nu : [0, \hat{\sigma}_1(\infty)) \to \mathbb{R}_+$ as

$$\nu(s) = U_0(s)U_1(s)$$

Let $\lambda_0, \lambda_1, \lambda_M \in \mathcal{K}$ be given by

$$\lambda_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)}\sqrt{\frac{\hat{c}_1}{\tau_1}}\left[\nu\circ\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]\left[\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]^{\phi+1} \tag{59}$$

$$\begin{aligned} \lambda_1(s) = &\left[\alpha_0\circ\bar{\alpha}_0^{-1}\circ\underline{\alpha}_0\circ\hat{\sigma}_1^{-1}\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right] \\ &\cdot\left[\nu\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]^{\phi} \end{aligned} \tag{60}$$

$$\lambda_{M.0}(s) = F_0(s)\hat{\zeta}_0(s)\left[U_1\circ\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}(s)\right]$$

$$\lambda_{M.1}(s) = F_1(s)\hat{\zeta}_1(s)\left[U_0\circ\frac{1}{\tau_1}\hat{\alpha}_1\circ\overline{\alpha}_1^{-1}(s)\right]$$

The pair of (46) and (54) yields

$$\begin{aligned}&\left[\hat{\sigma}_1\circ\underline{\alpha}_0^{-1}\circ\bar{\alpha}_0\circ\alpha_0^{-1}\circ\hat{c}_0\sigma_0(s)\right]^{\phi+1}\\&\quad\le\frac{1}{\hat{c}_1\tau_1^{\phi}}(\tau_1-1)(\hat{c}_0-1)\left[\hat{\alpha}_1\circ\bar{\alpha}_1^{-1}\circ\underline{\alpha}_1(s)\right]^{\phi+1}\end{aligned}\tag{61}$$

If (13) holds, it follows from (48), (49), (50), (53), (58) and (12) that

$$\alpha_0(\infty)<\infty \text{ and } \hat{\sigma}_1(\infty)<\infty \Rightarrow \lambda_0(\infty)<\infty \text{ and } \lambda_1(\infty)<\infty\tag{62}$$

By virtue of (14), we can pick $C_i>0$ so that $C_i<\liminf_{s\to\infty}\hat{\lambda}_i(s)/\hat{\lambda}_{ri}(s)$ holds. Defining $\tilde{\lambda}_{ri},\tilde{\sigma}_{ri}:\mathbb{R}_+\to\mathbb{R}_+$ by $\tilde{\lambda}_{ri}=C_i\hat{\lambda}_{ri}$ and $\tilde{\sigma}_{ri}=\sigma_{ri}/C_i$, we have

$$\lambda_{M.i}(s)\hat{\lambda}_i(s)+D_i\ge\lambda_{M.i}(s)\tilde{\lambda}_{ri}(s),\quad\forall s\in\mathbb{R}_+\tag{63}$$

for some $R_i>0$, where $D_i=\max_{s\in[0,R_i]}\lambda_{M.i}(s)\tilde{\lambda}_{ri}(s)$. Note that (62) and (63) are not used in the UGAS case of *(a)* since σ_{r1} and σ_{r2} vanish. Thus, (13) and (14) are not necessary for *(a)*. It can be verified that the inequalities (46) and (61) guarantee the existence of $\alpha_{cl}\in\mathcal{K}$ satisfying

$$\begin{aligned}&\lambda_{M.0}(V_0(t,x_0))\hat{\lambda}_0(V_0(t,x_0))\left[-\alpha_0(|x_0|)+\sigma_0(|x_1|)\right]\\&\quad+\lambda_{M.1}(V_1(t,x_1))\hat{\lambda}_1(V_1(t,x_1))\left[-\hat{\alpha}_1(|x_1|)+\hat{\sigma}_1(|x_0|)\right]\\&\qquad\le-\alpha_{cl}(|x|)\end{aligned}\tag{64}$$

Using the additional property (62) for *(b)* and *(c)*, we can also obtain

$$\begin{aligned}&\lambda_{M.0}(V_0(t,x_0))\hat{\lambda}_0(V_0(t,x_0))\left[-\alpha_0(|x_0|)+\sigma_0(|x_1|)+\tilde{\sigma}_{r0}(|r_0|)\right]\\&\quad+\lambda_{M.1}(V_1(t,x_1))\hat{\lambda}_1(V_1(t,x_1))\left[-\hat{\alpha}_1(|x_1|)+\hat{\sigma}_1(|x_0|)+\tilde{\sigma}_{r1}(|r_1|)\right]\\&\qquad\le-\alpha_{cl}(|x|)+\sigma_{cl}(|r|)\end{aligned}\tag{65}$$

for some α_{cl}, $\sigma_{cl}\in\mathcal{K}$ if one of *(H1)*, *(H2)* and *(H3)* holds. In the case of *(H3)*, we can verify that α_{cl} in (65) is of class $\mathcal{K}_\infty$. Now, define

$$V_{cl}(t,x)=\int_0^{V_0(t,x_0)}\lambda_{M.0}(s)ds+\int_0^{V_1(t,x_1)}\lambda_{M.1}(s)ds\tag{66}$$

The property (8) and $\lambda_{M.0},\lambda_{M.1}\in\mathcal{K}$ imply that there exist $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl}\in\mathcal{K}_\infty$ such that $\underline{\alpha}_{cl}(|x|)\le V_{cl}(t,x)\le\overline{\alpha}_{cl}(|x|)$ holds. Due to (10), (63) and (65) with (47) and (48), the property

$$\frac{dV_{cl}}{dt}\le-\alpha_{cl}(|x|)+\sigma_{cl}(|r|)+\sum_{i=0}^{1}D_i\tilde{\sigma}_{ri}(|r_i|)\tag{67}$$

holds along Σ for all $x\in\mathbb{R}^n$, $r\in\mathbb{R}^m$ and $t\in\mathbb{R}_+$. In the *(a)* case, the pair of (10) and (64) with (47) and (48) yields (67) for $r_i(t)\equiv0$, $i=1,2$.

Next, we deal with the situation where (43) holds. Then, there exist $\hat{\sigma}_1\in\mathcal{K}$ and $\hat{c}_0$, $\hat{c}_1>1$ such that (46), (47), (48) and $\lim_{s\to\infty}\hat{c}_1\hat{\sigma}_1(s)=\lim_{s\to\infty}\hat{\alpha}_1(s)$ are satisfied with (50). Define

$$L=\lim_{s\to\infty}\hat{\sigma}_1(s)\tag{68}$$

$$\tau_1(s)=(\underline{\tau}_1+\frac{\hat{c}_1-\underline{\tau}_1}{L}s)s,\quad 1<\underline{\tau}_1<\hat{c}_1\tag{69}$$

$$Q(t)=\frac{1}{\tau_1^{-1}(\hat{c}_1t)-t}\max\left\{\left(\frac{\hat{c}_1}{\underline{\tau}_1(\underline{\tau}_1-1)(\hat{c}_0-1)}-1\right),\left(\frac{\hat{c}_1}{\underline{\tau}_1}-1\right)\right\}\tag{70}$$

$$\psi(s)=e^{G(s)},\quad G(s)=\int_{L/2}^{s}Q(t)dt,\ s\in[0,L)\tag{71}$$

The function ψ is continuous, increasing and bounded on $[0, L)$. It is verified that ψ satisfies

$$\begin{aligned}&\left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0 \sigma_0(s)\right] \\ &\quad \leq \frac{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)}{\hat{c}_1} \left[\psi \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\end{aligned} \tag{72}$$

Replace (55), (56), (59) and (60) by

$$\hat{\zeta}_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)} \sqrt{\frac{\hat{c}_1}{\underline{\tau}_1}} \left[\hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \tag{73}$$

$$\hat{\zeta}_1(s) = \left[\alpha_0 \circ \bar{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \hat{\sigma}_1^{-1} \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]\left[\psi \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \tag{74}$$

$$\lambda_0(s) = \frac{\hat{c}_0}{(\hat{c}_0 - 1)} \sqrt{\frac{\hat{c}_1}{\underline{\tau}_1}} \left[\hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1}(s)\right] \tag{75}$$

$$\begin{aligned}\lambda_1(s) = &\left[\alpha_0 \circ \bar{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \hat{\sigma}_1^{-1} \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \\ &\cdot \left[\nu \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right] \left[\psi \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]\end{aligned} \tag{76}$$

respectively. Redefine U_1 and $\lambda_{M.1}$ by replacing τ_1 and $1/\tau_1$ with $\tau_1 \circ$ and $\tau_1^{-1} \circ$, respectively. Then, using (46) and (72), we obtain (64) for some $\alpha_{cl} \in \mathcal{K}$. Therefore, we arrive at (67) for $r_i(t) \equiv 0$, $i = 1, 2$, in the case of *(a)*. For *(b)* and *(c)*, it is directly verified that (43) does not hold if one of *(H1)*, *(H2)* and *(H3)* holds.

Finally, one of the easiest choices of $F_i(s)$ fulfilling the requirement is shown explicitly. Given an interval $U_k \in \mathbb{R}_+$, where $\hat{\lambda}_i(s)$ for $s \in U_k$ is locally minimum, let the non-negative number p_k denote the left endpoint of U_k. Let $\{p_k\}$ be the monotonically increasing sequence of such endpoints. In the same manner, let $\{b_k\}$ be the monotonically increasing sequence of the left endpoints of the intervals where $\hat{\lambda}_i(s)$ is locally maximum. Suppose that the sequence $\{p_k\}$ has no accumulation points. Pick $d \in (0, \infty)$ and set $F_i(d) = 1$. Using $p_k < b_k < p_{k+1} < b_{k+1}$, we can determine $F_i(s)$ from $s = d$ to ∞ by using

$$F_i(s) = \begin{cases} F_i(p_k) & s \in (p_k, b_k] \\ \dfrac{\hat{\lambda}_i(b_k) F_i(b_k)}{\hat{\lambda}_i(s)} & s \in (b_k, p_{k+1}] \end{cases} \tag{77}$$

and $F_i(s)$ from $s = d$ to 0 by using

$$F_i(s) = \begin{cases} F_i(b_l) & s \in [p_l, b_l) \\ \dfrac{\hat{\lambda}_i(p_l) F_i(p_l)}{\hat{\lambda}_i(s)} & s \in [b_{l-1}, p_l) \end{cases} \tag{78}$$

Then, $F_i(s)$ becomes non-decreasing. If $\{p_k\}$ has accumulation points, the function $F_i(s)$ can be constructed in a piecewise manner by means of (77) and (78) for $\lim_{k \to \infty} p_{k+1} - p_k = 0$ and $\lim_{l \to -\infty} p_l - p_{l-1} = 0$, respectively. Since the local absolute continuity of $\hat{\lambda}_i(s)$ on $(0, \infty)$ and (6) guarantees that $1/\hat{\lambda}_i(s)$ is absolutely continuous on any finite interval in $(0, \infty)$, we have $0 < F_i(s) < \infty$ for $s \in (0, \infty)$. The property (58) is ensured by (42).

5.2 Proof of Theorem 2

Using (21) we can decompose $\hat{\lambda}_0$ as

$$\begin{aligned}&\hat{\lambda}_0(v_0) = \hat{\lambda}_{A0}(|v_0|) \hat{\lambda}_{B0}(v_0), \quad \hat{\lambda}_{B0}(v_0) > 0, \quad \forall v_0 \in \mathbb{R}^{n_0} \\ &\limsup_{s \to \infty} \hat{\lambda}_{A0}(s) < \infty\end{aligned}$$

By virtue of (30), the inequality (27) ensures that

$$0 \le \hat{\lambda}_{A0}(|v_0|)\left[-\alpha_0(|v_0|)+\sigma_0(|x_1|)\right]+\hat{\lambda}_{Ar0}(|v_0|)\sigma_{r0}(|r_0|)$$

holds for some $\mathbf{C}^0$ function $\hat{\lambda}_{Ar0}:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\limsup_{s\to\infty}\hat{\lambda}_{Ar0}(s)/\hat{\lambda}_{A0}(s)<\infty$. The rest is the same as the proof of Theorem 1. The iISS and ISS Lyapunov functions are given by

$$V_{cl}(t,x)=\int_0^{V_1(t,x_1)}\lambda_{M.1}(s)ds \tag{79}$$

5.3 Proof of Theorem 3

From (21), (37) and (39) we obtain

$$-\alpha_1(|x_1|)+\sigma_1(|v_0|)\le-\delta_1\alpha_1(|x_1|)$$

where $\delta_1=(1-1/c_1)>0$. Let $\lambda_{M.1}:\mathbb{R}_+\to\mathbb{R}_+$ be any function fulfilling

$$\lambda_{M.1}(s)>0,\ \forall s\in(0,\infty),\quad 0<\liminf_{s\to\infty}s^k\lambda_{M.1}(s)<\infty$$

The assumptions (40) and (41) guarantee

$$\lim_{s\to\infty}\sup_{s\le|x_1|,t\in\mathbb{R}_+}\lambda_{M.1}(V_1(t,x_1))\hat{\lambda}_{r1}(t,x_1)<\infty$$
$$\lim_{s\to\infty}\inf_{s\le|x_1|,t\in\mathbb{R}_+}\lambda_{M.1}(V_1(t,x_1))\hat{\lambda}_1(t,x_1)\alpha_1(|x_1|)=\infty$$

respectively. It is verified that (79) is a Lyapunov function proving UGAS, iISS and ISS.

6 Examples

Consider the following:

$$\Sigma_0:\ \dot{x}_0=12x_0^2\left(\frac{x_1}{x_1+1}\right)^2+x_0v_0+r_0,\ x_0(0)\in\mathbb{R}_+ \tag{80}$$
$$\Sigma_1:\ \dot{x}_1=-\frac{3x_1}{x_1+1}+x_0+r_1^2,\quad x_1(0)\in\mathbb{R}_+ \tag{81}$$

The state $x=[x_0,x_1]^T$ evolves on $\mathbb{R}_+^2$ for the disturbance $r=[r_0,r_1]^T\in\mathbb{R}_+^2$. In other words, $\mathbb{R}_+^2$ is invariant. For the purpose of illustrating the small-gain-type criterion developed in this paper, we now design a scalar input $v_0(t)$ in the form of local feedback $v_0(x_0)$ to make the whole interconnected system iISS with respect to input r to state x. Since Σ_1 is not ISS, the control $v_0(t)$ should render Σ_0 stable strongly enough to compensate the shortage of stability. Let

$$V_1=x_1,\ \alpha_1(s)=\frac{3s}{s+1},\ \sigma_1(s)=s,\ \sigma_{r1}(s)=s^2$$
$$\hat{\lambda}_1(s)=1,\ \hat{\lambda}_{r1}(s)=1$$

For the choice of $V_0=x_0$, we obtain

$$\dot{V}_0=\frac{4x_0^2}{3}\left\{\frac{3v_0}{4x_0}+\left(\frac{3x_1}{x_1+1}\right)^2\right\}+r_0$$

Define

$$\alpha_0(s) = \frac{3v_0(s)}{4s},\ \sigma_0(s) = \left(\frac{3s}{s+1}\right)^2,\ \sigma_{r2}(s) = s$$
$$\hat{\lambda}_0(s) = \frac{4}{3}s^2,\ \hat{\lambda}_{r0}(s) = 1$$

where $\alpha_0 \in \mathcal{K}_\infty$ has yet to be determined. These functions satisfy (4)-(7), (12), (13), (14) and (15), *(H1)*. Using $\bar{\alpha}_i = \underline{\alpha}_i = s$, $i = 1, 2$, we obtain (11) as

$$\alpha_0^{-1}\left(c_0\left(\frac{3s}{s+1}\right)^2\right) \leq \frac{3s}{c_1(s+1)},\ \forall s \in \mathbb{R}_+$$

This inequality holds for some $c_0, c_1 > 1$ if and only if $\alpha_0(s) > s^2$ is satisfied for all $s \in (0, \infty)$. Hence, Theorem 1 guarantees that $v_0(x_0) = -kx_0^3$ with any $k > 4/3$ renders the interconnected system iISS. An important feature of this paper is that we obtain an iISS Lyapunov function explicitly as $V_{cl}(x)$ in (66), where

$$\lambda_{M.0}(s) = \frac{k+3}{k-1}\sqrt{\frac{\hat{c}_1}{\tau_1}}\, s^{\phi+1},\ \lambda_{M.1}(s) = \frac{4k}{3}\left(\frac{3s}{\tau_1(s+1)}\right)^{\phi+4}$$
$$\hat{c}_1 = 2\sqrt{\frac{k}{k+3}},\quad \tau_1 = 2\sqrt{\frac{k}{2(k+3)} + \frac{1}{8}}$$
$$L = \sqrt{\frac{1}{2} + \frac{1}{8}\sqrt{\frac{k+3}{k}}},\quad \phi = \max\left\{\frac{\log\frac{(\tau_1-1)(k-1)}{4}}{\log L}, 0\right\}$$

The techniques in [13, 14, 21, 8, 9, 10, 11] do not lead to the stability and the Lyapunov function of this example. It is mentioned that the small-gain-type evaluation is not the only approach to the stabilization of (80)-(81). For instance, a cancellation approach can yield a full-state feedback easily.

We next shall show a simple practical example. Consider the following bioreactor model with the monod kinetics [15, 12]:

$$\dot{X} = -bX + \mu\frac{S}{K+S}X + D(X_{in} - X),\quad X(0) \geq 0 \tag{82}$$
$$\dot{S} = -c\frac{S}{K+S}X + D(S_{in} - S),\quad S(0) \geq 0 \tag{83}$$

where $X(t)$ and $S(t)$ denote the concentration of biomass and organic substrate, respectively, which are non-negative real numbers. The kinetic parameters μ, c, K, b and the dilution rate D are positive constants and satisfy $c \geq \mu$. The symbols X_{in} and S_{in} indicate the biomass and the organic substrate concentrations in the inflow, which are non-negative. It is verified that the set $\mathbb{R}_+^2$ in which the bioreactor evolves is positively invariant for the solution $(X(t), S(t))$ of (82)-(83). Suppose that $X_{in} = 0$ and S_{in} is constant (stationary inflow containing no useful bacteria). Then, the system (82)-(83) has an equilibrium at $(X, S) = (0, S_{in})$. For the state variables $(X(t), S(t) - S_{in}) \in \mathbb{R}_+ \times [-S_{in}, \infty)$, take the simplest choice

$$V_X(X) = X,\quad V_S(S - S_{in}) = \frac{1}{2}(S - S_{in})^2$$

Then, we obtain $\dot{V}_X = \dot{X}$ and

$$\dot{V}_S = -D(S - S_{in})^2 - c(S - S_{in})\frac{S}{K+S}X$$

along the trajectories of (82)-(83). Rewrite these equations as follows:

$$\dot{V}_X \leq (K+X)\left(-(D+B)\frac{X}{K+X}+\mu\frac{|S-S_{in}|}{K+|S-S_{in}|}\right) \tag{84}$$

$$\dot{V}_S \leq |S-S_{in}|\left(-D|S-S_{in}|+cX\right) \tag{85}$$

$$B = b-\mu \sup_{S\in\mathbb{R}_+}\left(\frac{S}{K+S}-\frac{|S-S_{in}|}{K+|S-S_{in}|}\right)$$

Assume that $D+B>0$. Let

$$\hat{\lambda}_0(s) = K+s,\ \alpha_0(s) = \frac{(D+B)s}{K+s},\ \sigma_0(s) = \frac{\mu s}{K+s}$$
$$\hat{\lambda}_1(s) = \sqrt{2s},\ \alpha_1(s) = Ds,\ \sigma_1(s) = cs$$
$$\bar{\alpha}_1(s) = \underline{\alpha}_1(s) = s,\ \bar{\alpha}_2(s) = \underline{\alpha}_2(s) = \frac{1}{2}s^2,$$

The tool proposed in [9] cannot be applied to the pair (84)-(85). This paper, however, enables us to apply (11) to the pair and we obtain the condition

$$\frac{c_1cc_0\mu Ks}{(D+B-c_0\mu)s+(D+B)K} \leq Ds,\ \forall s\in\mathbb{R}_+$$

There exist $c_1, c_0 > 1$ such that this inequality holds if and only if $\mu < D+B$ and $c\mu < D(D+B)$. Hence, Theorem 1 establishes the global asymptotic stability of $(X,S)=(0,S_{in})$ for $\mu\max\{1,\frac{c}{D}\} < D+B$. A Lyapunov function $V_{cl}(x)$ of the system (82)-(83) is computed as in (66). For example, we obtain

$$\lambda_{M.0}(s) = \frac{4\sqrt{6}c^4s^4}{3D}, \quad \lambda_{M.1}(s) = \frac{27(D+B)D^3s\sqrt{2s}}{32}$$

for $c_0 = c_1 = 2$. It is mentioned that, for merely dealing with (82)-(83), there are other ways. Although this brief example does not illustrate all features of the proposed approach, it shows compactly how well the formulation of the flexible inequality fits to a practical system in invoking a small-gain type argument and constructing a Lyapunov function.

7 Concluding remarks

In this paper, a flexible Lyapunov formulation has been introduced into the small-gain methodology for stability analysis of interconnected systems. Namely, the technique of bilateral flexibility in Lyapunov inequalities is proposed, which can be considered as a thoroughly nonlinear counterpart of the popular scaling technique in linear robust control[3, 4]. The bilaterality and treating iISS and ISS systems equally are new in the literature. Examples have shown that the flexibility is useful in exploiting and coping with nonlinearity in stability analysis and feedback design. The Lyapunov function of the interconnected system is expressed explicitly in terms of a $\mathbf{C}^1$ nonlinear combination of given Lyapunov functions of the individual subsystems. This paper has focused on the construction a continuously differentiable Lyapunov function since such a Lyapunov function is directly amenable to a large variety of techniques for further analysis and design of control systems. This contrasts with the max-type construction leading only to a Lipschitz continuous function, which requires methods of non-smooth analysis or additional mathematical process of smoothing(see [13]). Finally, it is worth mentioning that $c_i\sigma_i$ in (11), (29) and (39) can be replaced by $(\mathbf{Id}+\rho_i)\circ\sigma_i$ where ρ_i is of class $\mathcal{K}_\infty$ in the ISS case. It can be relaxed further when iISS or UGAS is targeted. This paper employs the simplest choice $s+\rho_i(s)=c_is$ for brevity of the presentation of Lyapunov functions. For the generalization, we can combine the result in this paper with [11] devoted to the iISS small-gain theorem without flexible parameters $\{\hat{\lambda}_i,\ \hat{\lambda}_{ri}\}$.

Appendix

A Derivation of Eq. (65) and Eq. (64)

Since (49) ensures $\lim_{s\to\infty}\hat{\sigma}_1(s)\geq\lim_{s\to\infty}(1/\tau_1)\hat{\alpha}_1(s)$ or $\lim_{s\to\infty}\hat{\sigma}_1(s)=\infty$, the functions λ_0 and λ_1 defined in (59) and (60) are of class $\mathcal{K}$ and satisfy

$$\lambda_i(s)=\lambda_{M.i}(s)\hat{\lambda}_i(s),\quad s\in\mathbb{R}_+,\ i=0,1 \tag{86}$$

Case (c): Pick a constant δ satisfying $\sqrt{\tau_1/\hat{c}_1}<\delta<1$. Let $\tau_{r0},\tau_{r1}>1$ be defined with

$$(1-\delta)\left(1-\frac{1}{\tau_i}\right)=\frac{1}{\tau_{ri}},\quad i=0,1$$

where $\tau_0=\hat{c}_0$. Define

$$\begin{aligned}
&\theta_0(s)=\overline{\alpha}_0\circ\alpha_0^{-1}\circ\tau_0\sigma_0(s),\quad \theta_1(s)=\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}\circ\tau_1\hat{\sigma}_1(s)\\
&\theta_{r0}(s)=\overline{\alpha}_0\circ\alpha_0^{-1}\circ\tau_{r0}\tilde{\sigma}_{r0}(s),\quad \theta_{r1}(s)=\overline{\alpha}_1\circ\hat{\alpha}_1^{-1}\circ\tau_{r1}\tilde{\sigma}_{r1}(s)\\
&\lambda_{\theta i}(s)=\lambda_i\circ\theta_i(s),\quad i=0,1
\end{aligned}\tag{87}$$

By the hypothesis *(H3)*, we have $\theta_i,\ \theta_{ri}\in\mathcal{K}$. Combining the individual cases separated by $\alpha_0(|x_0|)\geq\tau_0\sigma_0(|x_1|)$, $\alpha_0(|x_0|)<\tau_0\sigma_0(|x_1|)$, $\alpha_0(|x_0|)\geq\tau_{r0}\tilde{\sigma}_{r0}(|r_0|)$ and $\alpha_0(|x_0|)<\tau_{r0}\tilde{\sigma}_{r0}(|r_0|)$, we obtain

$$\begin{aligned}
&\lambda_0(V_0(t,x_0))\{-\alpha_0(|x_0|)+\sigma_0(|x_1|)+\tilde{\sigma}_{r0}(|r_0|)\}\\
&\leq\delta\left(-1+\frac{1}{\tau_0}\right)\lambda_0(\underline{\alpha}_0(|x_0|))\alpha_0(|x_0|)\\
&\quad+\lambda_{\theta0}(|x_1|)\sigma_0(|x_1|)+\lambda_0(\theta_{r0}(|r_0|))\tilde{\sigma}_{r0}(|r_0|)
\end{aligned}\tag{88}$$

We also obtain a similar upper bound of $\lambda_1(V_1)\{-\hat{\alpha}_1(|x_1|)+\hat{\sigma}_1(|x_0|)+\tilde{\sigma}_{r1}(|r_1|)\}$. Thus, the desired inequality (65) is fulfilled with

$$\begin{aligned}
\alpha_{cl}(s)=\min_{s=|x|}&\left\{\left(\delta-\sqrt{\frac{\tau_1}{\hat{c}_1}}\right)\frac{\tau_1-1}{\tau_1}\lambda_1(\underline{\alpha}_1(|x_1|))\hat{\alpha}_1(|x_1|)\right.\\
&\left.+\left(\delta-\sqrt{\frac{\tau_1}{\hat{c}_1}}\right)\frac{\tau_0-1}{\tau_0}\lambda_0(\underline{\alpha}_0(|x_0|))\alpha_0(|x_0|)\right\}\\
\sigma_{cl}(s)=\max_{s=|r|}&\{\lambda_1(\theta_{r1}(|r_1|))\tilde{\sigma}_{r1}(|r_1|)+\lambda_0(\theta_{r0}(|r_0|))\tilde{\sigma}_{r0}(|r_0|)\}
\end{aligned}$$

if λ_1 and λ_0 satisfy

$$\lambda_{\theta1}(s)\hat{\sigma}_1(s)\leq\sqrt{\frac{\tau_1}{\hat{c}_1}}\cdot\frac{\tau_0-1}{\tau_0}\lambda_0(\underline{\alpha}_0(s))\alpha_0(s) \tag{89}$$

$$\lambda_{\theta0}(s)\sigma_0(s)\leq\sqrt{\frac{\tau_1}{\hat{c}_1}}\cdot\frac{\tau_1-1}{\tau_1}\lambda_1(\underline{\alpha}_1(s))\hat{\alpha}_1(s) \tag{90}$$

for all $s \in \mathbb{R}_+$. The function σ_{cl} is of class $\mathcal{K}$, while $\alpha_{cl} \in \mathcal{K}_\infty$ since *(H3)*. By virtue of $\overline{\alpha}_i^{-1} \circ \underline{\alpha}_i(s) \le s$, $\theta_0 \in \mathcal{K}$ and $\hat{c}_0\sigma_0(s) = \alpha_0(\overline{\alpha}_0^{-1}(\theta_0(s)))$, we have (89) and (90) if

$$\begin{aligned} &\hat{\sigma}_1(\underline{\alpha}_0^{-1}(\theta_0(s)))\lambda_{\theta 1}(\underline{\alpha}_0^{-1}(\theta_0(s))) \\ &\quad \le \frac{(\tau_1-1)(\hat{c}_0-1)}{\hat{c}_1}\hat{\alpha}_1(\overline{\alpha}_1^{-1}(\underline{\alpha}_1(s)))\lambda_1(\underline{\alpha}_1(s)) \end{aligned} \tag{91}$$

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) = \sqrt{\frac{\tau_1}{\hat{c}_1}} \cdot \frac{\hat{c}_0-1}{\hat{c}_0}\lambda_0(\underline{\alpha}_0(s))\alpha_0(\overline{\alpha}_0^{-1}(\underline{\alpha}_0(s))) \tag{92}$$

hold for all $s \in \mathbb{R}_+$. The property (92) is directly verified from (59) and (60). By the non-decreasing property of ν and (46), we have

$$\begin{aligned} &[\hat{c}_0\sigma_0(s)]\left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \theta_0(s)\right] \le \\ &\quad \left[\alpha_0 \circ \overline{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \theta_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\nu \circ \frac{1}{\tau_1}\hat{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] \end{aligned}$$

Combining the above inequality with (61) yields

$$\begin{aligned} &[\hat{c}_0\sigma_0(s)]\left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \theta_0(s)\right]\left[\hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0\sigma_0(s)\right]^{\phi+1} \\ &\le \frac{1}{\hat{c}_1\tau_1^\phi}(\tau_1-1)(\hat{c}_0-1)\left[\alpha_0 \circ \overline{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \theta_1^{-1} \circ \underline{\alpha}_1(s)\right] \\ &\quad \cdot \left[\nu \circ \frac{1}{\tau_1}\hat{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\left[\hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]^{\phi+1} \end{aligned}$$

Then, the property (91) follows from (60).

Case (b): The claim (c) proves (65) for $\alpha_{cl} \in \mathcal{K}_\infty \subset \mathcal{K}$ and $\sigma_{cl} \in \mathcal{K}$ in the case of *(H3)*. Thus, suppose that *(H3)* is not satisfied. The logical sum of *(H1)* and *(H2)* implies that one of

$$\lim_{s\to\infty} \alpha_0(s) < \infty \text{ and } \lim_{s\to\infty} \hat{\sigma}_1(s) < \infty \tag{93}$$

and

$$\lim_{s\to\infty} \alpha_0(s) = \infty \text{ and } \lim_{s\to\infty} \hat{\alpha}_1(s) < \infty \tag{94}$$

holds true. First, we consider the case of (93). Using (62), we have

$$\lambda_i(V_i(t,x_i))\tilde{\sigma}_{ri}(|r_i|) \le E_i\tilde{\sigma}_{ri}(|r_i|), \quad \forall x_i \in \mathbb{R}^{n_i},\ r_i \in \mathbb{R}^{b_i},\ t \in \mathbb{R}_+ \tag{95}$$

with $E_i = \lim_{s\to\infty}\lambda_i(s) < \infty$ for $i = 0,1$, so that we can use E_i instead of $\lambda_i(\theta_{ri}(|r_i|))$ for $i = 0,1$. The property $\lim_{s\to\infty}\alpha_0(s) \ge \tau_0 \lim_{s\to\infty}\sigma_0(s)$ implied by (46) guarantees that $\lambda_{\theta 0}$ in (87) is well-defined on $\mathbb{R}_+$. Redefine $\lambda_{\theta 1}$ as

$$\lambda_{\theta 1}(s) = \begin{cases} \lambda_1 \circ \theta_1(s) & , s \in [0, Y_1) \\ \lim_{s\to\infty}\lambda_1(s) & , s \in [Y_1, \infty) \end{cases} \tag{96}$$

$$Y_1 = \lim_{s\to\infty} \hat{\sigma}_1^{-1} \circ \tau_1^{-1}\hat{\alpha}_1(s) < \infty \tag{97}$$

Here, $Y_1 < \infty$ is due to $\tau_1 > 1$ and (49). Replace (92) required on $s \in \mathbb{R}_+$ by

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) = \sqrt{\frac{\tau_1}{\hat{c}_1}} \cdot \frac{\hat{c}_0-1}{\hat{c}_0}\lambda_0(\underline{\alpha}_0(s))\alpha_0(\overline{\alpha}_0^{-1}(\underline{\alpha}_0(s)),\ s \in [0, Y_1) \tag{98}$$

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) \le \sqrt{\frac{\tau_1}{\hat{c}_1}} \cdot \frac{\hat{c}_0-1}{\hat{c}_0}\lambda_0(\underline{\alpha}_0(s))\alpha_0(\overline{\alpha}_0^{-1}(\underline{\alpha}_0(s)),\ s \in [Y_1, \infty) \tag{99}$$

Since (46) assures $\underline{\alpha}_0^{-1} \circ \theta_0(s) \le Y_1$, the inequality (65) is satisfied for some $\alpha_{cl}, \sigma_{cl} \in \mathcal{K}$ if (91), (98) and (99) are fulfilled. The property (99) is immediate from (96), (59) and the non-decreasing property of ν. Using the arguments employed in (c), we can verify (91) and (98). We next consider the case of (94). Due to (12) and (58), ν is bounded on $[0, \hat{\sigma}_1(\infty))$. The property (49) ensures (95) for $i = 1$. We do not need (95) for $i = 0$ since $\lim_{s\to\infty} \alpha_0(s) = \infty$ allows us to use θ_{r0}. Define $\lambda_{\theta 1}$ as in (96). The rest is the same as the proof in the (93) case.

Case (a): Recall that $r_i = 0$ for $i = 0, 1$. Suppose that (43) is violated. Deleting $\tilde{\sigma}_{r0}$ and $\tilde{\sigma}_{r1}$, the arguments used in (b) yield (64) with some $\alpha_{cl} \in \mathcal{K}$. Note that the (a) case does not need the technique of (95). The properties (12) and (58) guarantees that U_1 is bounded on $[0, \hat{\sigma}_1(\infty))$. Next consider the case where (43) holds. The choice of τ_1 given in (69) implies $\lim_{s\to\infty} \tau_1 \circ \hat{\sigma}_1(s) = \lim_{s\to\infty} \hat{\alpha}_1(s)$, and the property $\mathbf{Id} - \tau_1^{-1} \in \mathcal{K}_\infty$ follows from $\tau_1 \in \mathcal{K}_\infty$ and $(\mathbf{Id} - \tau_1^{-1}) \circ \tau_1(s) = \tau_1 - s \in \mathcal{K}_\infty$. The function U_1 is guaranteed to be bounded on $[0, \hat{\sigma}_1(\infty))$. Defining $\theta_1 \in \mathcal{K}_\infty$ by

$$\theta_1(s) = \overline{\alpha}_1 \circ \hat{\alpha}_1^{-1} \circ \tau_1 \circ \hat{\sigma}_1(s)$$

we obtain

$$\begin{aligned}&\lambda_1(V_1(t, x_1))\{-\hat{\alpha}_1(|x_1|) + \hat{\sigma}_1(|x_0|)\} \\ &\quad \le -\lambda_1(\underline{\alpha}_1(|x_1|))[(\mathbf{Id} - \tau_1^{-1}) \circ \hat{\alpha}_1(|x_1|)] + \lambda_{\theta 1}(|x_0|)\hat{\sigma}_1(|x_0|)\end{aligned} \tag{100}$$

Using $\underline{\tau}_1 s \le \tau_1(s)$ and arguments similar to the (c) case, we can verify that (64) holds with the class $\mathcal{K}$ function

$$\begin{aligned}\alpha_{cl}(s) = \min_{s=|x|} \Bigg\{ &\left(1 - \sqrt{\frac{\underline{\tau}_1}{\hat{c}_1}}\right) \lambda_1(\underline{\alpha}_1(|x_1|))[(\mathbf{Id} - \tau_1^{-1}) \circ \hat{\alpha}_1(|x_1|)] \\ &+ \left(1 - \sqrt{\frac{\underline{\tau}_1}{\hat{c}_1}}\right) \frac{\tau_0 - 1}{\tau_0} \lambda_0(\underline{\alpha}_0(|x_0|))\alpha_0(|x_0|) \Bigg\}\end{aligned} \tag{101}$$

if

$$\begin{aligned}&\hat{\sigma}_1(\underline{\alpha}_0^{-1}(\theta_0(s)))\lambda_{\theta 1}(\underline{\alpha}_0^{-1}(\theta_0(s))) \\ &\quad \le \frac{(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)}{\hat{c}_1} \hat{\alpha}_1(\overline{\alpha}_1^{-1}(\underline{\alpha}_1(s)))\lambda_1(\underline{\alpha}_1(s))\end{aligned} \tag{102}$$

$$\lambda_{\theta 1}(s)\hat{\sigma}_1(s) = \sqrt{\frac{\underline{\tau}_1}{\hat{c}_1}} \cdot \frac{\hat{c}_0 - 1}{\hat{c}_0} \lambda_0(\underline{\alpha}_0(s))\alpha_0(\overline{\alpha}_0^{-1}(\underline{\alpha}_0(s))) \tag{103}$$

are satisfied for all $s \in \mathbb{R}_+$. The property (103) is immediate from (75) and (76). From (46), (69), the non-decreasing property of ν and $\lim_{s\to\infty} c_1\hat{\sigma}_1(s) = \lim_{s\to\infty} \hat{\alpha}_1(s)$, we obtain

$$\begin{aligned}&[\hat{c}_0\sigma_0(s)] \left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \theta_0(s)\right] \\ &\quad \le \left[\alpha_0 \circ \overline{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \theta_1^{-1} \circ \underline{\alpha}_1(s)\right] \left[\nu \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\end{aligned}$$

Combining this inequality with (72), we obtain

$$\begin{aligned}&[\hat{c}_0\sigma_0(s)] \left[\nu \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \theta_0(s)\right] \left[\psi \circ \hat{\sigma}_1 \circ \underline{\alpha}_0^{-1} \circ \bar{\alpha}_0 \circ \alpha_0^{-1} \circ \hat{c}_0\sigma_0(s)\right] \\ &\quad \le \frac{\underline{\tau}_1}{\hat{c}_1}(\underline{\tau}_1 - 1)(\hat{c}_0 - 1) \left[\alpha_0 \circ \overline{\alpha}_0^{-1} \circ \underline{\alpha}_0 \circ \theta_1^{-1} \circ \underline{\alpha}_1(s)\right] \\ &\qquad \cdot \left[\nu \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right] \left[\psi \circ \tau_1^{-1} \circ \hat{\alpha}_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)\right]\end{aligned}$$

Using (76), (46) and $1 < \underline{\tau}_1 < \hat{c}_1$ in both sides of the above inequality, we arrive at (102). Therefore, the property (64) has been verified.

B Derivation of Eq. (72)

Recall that (43) is assumed for (72). By definition, we can verify

$$\frac{d}{ds}\psi(s) = \psi(s)Q(s) \tag{104}$$

$$\frac{d^2}{ds^2}\psi(s) = \psi(s)\left(Q(s)^2 + \frac{d}{ds}Q(s)\right) \tag{105}$$

$$\psi(s) \geq 0 \tag{106}$$

$$Q(s)^2 + \frac{d}{ds}Q(s) \geq 0 \tag{107}$$

for $s \in (0, L)$. Here, the property (107) is derived from

$$Q(s)^2 + \frac{d}{ds}Q(s) = \frac{k}{p(s)^2 w(s)}\left(\frac{2(k+1)(\hat{c}_1 - \underline{\tau}_1)\tau_1^{-1}(\hat{c}_1 s)}{L} + k\underline{\tau}_1 + \underline{\tau}_1 - \hat{c}_1\right)$$

where

$$\begin{aligned}
k &= \max\left\{\left(\frac{\hat{c}_1}{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)} - 1\right), \left(\frac{\hat{c}_1}{\underline{\tau}_1} - 1\right)\right\} > 0 \\
p(s) &= \tau_1^{-1}(\hat{c}_1 s) - s > 0, \quad \forall s \in (0, L) \\
w(s) &= \underline{\tau}_1 + \frac{2(\hat{c}_1 - \underline{\tau}_1)\tau_1^{-1}(\hat{c}_1 s)}{L} > 0, \ \forall s \in (0, L)
\end{aligned}$$

From (104) and (70) we obtain

$$\psi(s) \leq \frac{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)}{\hat{c}_1}\left(\psi(s) + (\tau_1^{-1}(\hat{c}_1 s) - s)\frac{d}{ds}\psi(s)\right)$$

for $s \in (0, L)$. Then, the non-decreasing property of $d\psi/ds$ ensured by (105), (106) and (107) implies

$$\psi(s) \leq \frac{\underline{\tau}_1(\underline{\tau}_1 - 1)(\hat{c}_0 - 1)}{\hat{c}_1}\psi(\tau_1^{-1}(\hat{c}_1 s)), \ \forall s \in (0, L) \tag{108}$$

Hence, the property (72) follows from (46).

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