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SOME NOTES ON THE CLASS OF CONTRACTIONS WITH RESPECT TO τ-DISTANCE

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Abstract

We discuss the class of contractions with respect to τ-distance. One of purposes of this paper is to understand the concept of τ-distance more deeply.

1. Introduction and preliminaries

Throughout this paper, we denote by **N**, **Z** and **R** the sets of positive integers, integers and real numbers, respectively.

Let (X,d) be a metric space. Then a mapping T on X is a *Picard operator* if T has a unique fixed point z and $\{T^n x\}$ converges to z for every $x \in X$. See [6, 9, 15]. We denote by $P(X)$ the set of all Picard operators on X. Contractions and Kannan mappings are typical examples of Picard operator.

Theorem 1 (Banach [1] and Caccioppoli [2]). *Let (X,d) be a complete metric space and let T be a contraction on X, that is, there exists $r \in [0,1)$ such that*

$$d(Tx,Ty) \le r d(x,y)$$

for all $x, y \in X$. Then $T \in P(X)$.

Theorem 2 (Kannan [5]). *Let (X,d) be a complete metric space. Let T be a Kannan mapping on X, that is, there exists $\alpha \in [0,1/2)$ such that*

$$d(Tx,Ty) \le \alpha(d(x,Tx) + d(y,Ty))$$

for all $x, y \in X$. Then $T \in P(X)$.

We denote by $C(X)$ and $K(X)$ the set of all contractions and all Kannan mappings on X, respectively. Theorems 1 and 2 tell $C(X) \subset P(X)$ and $K(X) \subset P(X)$ provided X is complete. In general, $C(X) \not\subset K(X)$ and $K(X) \not\subset C(X)$ hold.

In 2001, Suzuki introduced the concept of τ-distance in order to improve results in Tataru [18], Zhong [19, 20] and others. See also [7].

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DEFINITION 3 ([10]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

($\tau 1$) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$.

($\tau 2$) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.

($\tau 3$) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_n p(w, x_n)$ for all $w \in X$.

($\tau 4$) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.

($\tau 5$) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and propositions are stated in [4, 10–14, 16] and references therein.

We denote by $TC(X)$ the set of all mappings T on a metric space X such that there exist a τ-distance p on X and $r \in [0, 1)$ satisfying

$$p(Tx, Ty) \le r p(x, y)$$

for all $x, y \in X$. We also denote by $TK(X)$ the set of all mappings T on X such that there exist a τ-distance p and $\alpha \in [0, 1/2)$ satisfying either of the following holds:

$$p(Tx, Ty) \le \alpha(p(Tx, x) + p(Ty, y))$$

for all $x, y \in X$, or

$$p(Tx, Ty) \le \alpha(p(Tx, x) + p(y, Ty))$$

for all $x, y \in X$. We have proven $TC(X) \subset P(X)$ and $TK(X) \subset P(X)$ provided X is complete; see [10, 11]. Since the metric d is a τ-distance, $C(X) \subset TC(X)$ and $K(X) \subset TK(X)$ hold. Also we proved $TC(X) = TK(X)$ in [12]. Thus

$$C(X) \cup K(X) \subset TC(X) = TK(X) \subset P(X)$$

holds. In [17], we showed that there exists a complete metric space X such that $TC(X) \subsetneq P(X)$.

In this paper, motivated by the above facts, we continue to study $TC(X)$. One of purposes of this paper is to understand the concept of τ-distance more deeply.

2. Main results

We first give the proof of the following lemma because we did not in [17].

LEMMA 4 ([17]). *Let T be a mapping on a set X. Let A_0 be a subset of X such that $T(A_0) \subset A_0$. Define a sequence $\{A_n\}$ of subsets of X by*

$$A_1 = T^{-1}(A_0) \backslash A_0 \qquad \text{and} \qquad A_{n+1} = T^{-1}(A_n).$$

Then the following hold:

(i) *For every* $n \in \mathbf{N}$ *and* $x \in X$, $x \in A_n$ *if and only if* $T^j x \notin A_0$ *for* $j = 0, 1, \ldots, n-1$ *and* $T^n x \in A_0$, *where* T^0 *is the identity mapping on* X.

(ii) $A_m \cap A_n = \varnothing$ *for* $m, n \in \mathbf{N} \cup \{0\}$ *with* $m \neq n$.

(iii) $T(A_{n+1}) = A_n$ *for every* $n \in \mathbf{N}$.

Proof. We first show (i) by induction. It is obvious that $x \in A_1$ if and only if $x \notin A_0$ and $Tx \in A_0$. So the conclusion is true when $n = 1$. We assume the conclusion is true for some $n \in \mathbf{N}$. Let $x \in A_{n+1}$. Then $Tx \in A_n$ holds. So $T^j Tx \notin A_0$ for $j = 0, 1, \ldots, n-1$ and $T^n Tx \in A_0$. We note $x \notin A_0$ because $T(A_0) \subset A_0$. Therefore we obtain $T^j x \notin A_0$ for $j = 0, 1, \ldots, n$ and $T^{n+1} x \in A_0$. Conversely, we assume $T^j x \notin A_0$ for $j = 0, 1, \ldots, n$ and $T^{n+1} x \in A_0$. Then $T^j Tx \notin A_0$ for $j = 0, 1, \ldots, n-1$ and $T^n Tx \in A_0$. So $Tx \in A_n$, which implies $x \in A_{n+1}$. We have shown that the conclusion is true when $n := n+1$. By induction, we obtain (i). (ii) follows from (i); and (iii) is obvious. □

Let (X, d) be a metric space. Then we define a set $G(X)$ as follows: $T \in G(X)$ if and only if T is a mapping on X satisfying the following:

- There exist $z \in X$, a function f from X into $[0, \infty)$ and a continuous nondecreasing function g from $[0, \infty)$ into $[0, \infty)$ such that $f(z) = 0$, $0 < g(t) < t$ for all $t \in (0, \infty)$ and

$$d(T^n x, z) \leq g^n(f(x))$$

for all $x \in X$ and $n \in \mathbf{N}$.

Lemma 5. *Let* (X, d) *be a metric space and* $T \in G(X)$. *Let* z, f *and* g *be as in the definition of* $G(X)$. *Then the following hold:*

(i) $g(0) = 0$.

(ii) $\{g^n(t)\}$ *converges to* 0 *for every* $t \in [0, \infty)$.

(iii) z *is a unique fixed point of* T*; and* $\{T^n x\}$ *converges to* z *for every* $x \in X$.

Proof. Since $g(0) \leq g(t) < t$ for every $t \in (0, \infty)$, we have (i). We next show (ii). Since $\{g^n(t)\}$ is nonincreasing, $\{g^n(t)\}$ converges to some $\tau \in [0, \infty)$. We have

$$\tau = \lim_{n \to \infty} g^n(t) = g\Big(\lim_{n \to \infty} g^{n-1}(t)\Big) = g(\tau),$$

which implies $\tau = 0$. Let us prove (iii). Since

$$d(Tz, z) \leq g(f(z)) = g(0) = 0,$$

z is a fixed point of T. From (ii), we have $\lim_n d(T^n x, z) \leq \lim_n g^n(f(x)) = 0$ for every $x \in X$. So the fixed point z is unique. □

From Lemma 5, we have $G(X) \subset P(X)$. Then it is a natural question of which is smaller, $G(X)$ or $TC(X)$. The following theorem tells that $G(X)$ is smaller.

THEOREM 6. *Let (X,d) be a metric space. Then $G(X) \subset TC(X)$ holds. That is, for every $T \in G(X)$, there exist a τ-distance p on X and $r \in [0,1)$ such that $p(Tx,Ty) \le rp(x,y)$ for all $x,y \in X$.*

PROOF. Let $T \in G(X)$ and let z, f and g be as in the definition of $G(X)$. Fix $r \in (0,1)$. We note that there is only one periodic point, which is the unique fixed point z. Define a strictly decreasing sequence $\{t_n\}$ by $t_n = g^n(1)$. By Lemma 5, $\{t_n\}$ converges to 0. We put $t_\infty = 0$.

We shall define a function h from X into $\mathbf{Z} \cup \{\infty\}$ satisfying

- $h(x) = \infty \Leftrightarrow x = z$
- $h(Tx) \ge h(x) + 1$
- $h(x) \in \mathbf{N} \cup \{\infty\} \Rightarrow d(x,z) \le t_{h(x)}$

for all $x \in X$. We put $h(z) = \infty$. It is obvious that $h(Tz) = h(z) = \infty = h(z) + 1$ and $d(z,z) = 0 = t_{h(z)}$. Define a sequence $\{A_n\}$ of subsets of X by

$$A_0 = \{z\}, \qquad A_1 = T^{-1}(A_0) \backslash A_0 \qquad \text{and} \qquad A_{n+1} = T^{-1}(A_n).$$

Then by Lemma 4, $A_m \cap A_n = \varnothing$ for $m,n \in \mathbf{N} \cup \{0\}$ with $m \ne n$. We put $h(x) = -n$ for $x \in A_n$ with $n \in \mathbf{N}$. We have

$$h(Tx) = \begin{cases} \infty & \text{if } x \in A_1 \\ h(x) + 1 & \text{if } x \in \bigsqcup_{n=2}^{\infty} A_n \end{cases}$$

and hence $h(Tx) \ge h(x) + 1$ for $x \in \bigsqcup_{n=1}^{\infty} A_n$. Put

$$Y = X \Big\backslash \left(\bigsqcup_{n \in \mathbf{N} \cup \{0\}} A_n \right).$$

It is obvious that $T(Y) \subset Y$, $T^{-1}(Y) = Y$ and $z \notin Y$. We note

$$T^m x = T^n x \Leftrightarrow m = n$$

for $x \in Y$ and $m,n \in \mathbf{N} \cup \{0\}$. Define an equivalence relation $\sim$ on Y as follows: $x \sim y$ if and only if there exist $m,n \in \mathbf{N} \cup \{0\}$ such that $T^m x = T^n y$. By Axiom of Choice, there exists a mapping B on Y such that

$$Bx \sim x \qquad \text{and} \qquad x \sim y \Leftrightarrow Bx = By.$$

Let $u \in Y$ with $Bu = u$. Since $\lim_n g^n(f(u)) = 0$, we can choose $\nu \in \mathbf{N}$ such that $g^\nu(f(u)) \le 1$. We put $v = T^\nu u$. Then we put $h(T^n v) = n$ for $n \in \mathbf{N} \cup \{0\}$. We have

$$d(T^n v, z) = d(T^{\nu+n} u, z) \le g^{\nu+n}(f(u)) = g^n(g^\nu(f(u))) \le g^n(1) = t_n$$

for $n \in \mathbf{N}$. We also have $h(T \circ T^n v) = n+1 = f(T^n v)$ for $n \in \mathbf{N} \cup \{0\}$. Define a sequence $\{D_n\}$ of subsets of Y by

$$D_0 = \{v, Tv, T^2v, T^3v, \ldots\}, \qquad D_1 = T^{-1}(D_0) \backslash D_0 \qquad \text{and} \qquad D_{n+1} = T^{-1}(D_n).$$

Then we have $D_m \cap D_n = \varnothing$ for $m, n \in \mathbf{N} \cup \{0\}$ with $m \neq n$; and

$$\{x \in Y : x \sim u\} = \bigsqcup_{n \in \mathbf{N} \cup \{0\}} D_n.$$

We put $h(x) = -n$ for $x \in D_n$ with $n \in \mathbf{N}$. If $x \in D_1$, then $h(Tx) \geq 0 = h(x) + 1$. If $x \in D_n$ with $n \geq 2$, then $h(Tx) = -n + 1 = h(x) + 1$. We have defined h. We note that $h(x) \in \mathbf{N}$ implies $x \in Y$.

Next, we define a τ-distance p on X by

$$p(x, y) = r^{h(x)} + r^{h(y)},$$

where $r^\infty = 0$. We shall show that p is a τ-distance. For $x, y, z \in X$, we have

$$p(x, z) = r^{h(x)} + r^{h(z)} \leq r^{h(x)} + r^{h(y)} + r^{h(y)} + r^{h(z)} = p(x, y) + p(y, z).$$

These imply (τ1). Define a function η from $X \times [0, \infty)$ into $[0, \infty)$ by $\eta(x, t) = t$. (τ2) and (τ4) obviously hold. In order to show (τ3), we assume $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$. We note that the second condition is equivalent to $\lim_n h(z_n) = \lim_n h(x_n) = \infty$. So we have

$$\lim_{n \to \infty} d(x_n, z) \leq \lim_{n \to \infty} t_{h(x_n)} = 0,$$

which implies $x = z$. Thus,

$$p(w, x) = p(w, z) = r^{h(w)} = \lim_{n \to \infty} p(w, x_n)$$

holds for every $w \in X$. This implies (τ3). Let us prove (τ5). We assume $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$. Then we have $\lim_n h(x_n) = \lim_n h(y_n) = \lim_n h(z_n) = \infty$. Hence

$$\lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} (d(x_n, z) + d(y_n, z)) \leq \lim_{n \to \infty} (t_{h(x_n)} + t_{h(y_n)}) = 0,$$

which implies (τ5). Therefore we have shown that p is a τ-distance on X.

Finally, for $x, y \in X$, we have

$$p(Tx, Ty) = r^{h(Tx)} + r^{h(Ty)} \leq r^{h(x)+1} + r^{h(y)+1} = rp(x, y).$$

This complete the proof. □

Remark. We have proven that for every $T \in G(X)$ and $r \in (0, 1)$, there exist a τ-distance p satisfying $p(Tx, Ty) \leq rp(x, y)$ for all $x, y \in X$.

Next, we will show $C(X) \subset G(X)$ and $K(X) \subset G(X)$ provided X is complete.

LEMMA 7. *Let T be a mapping on a metric space (X, d). Assume there exist $z \in X$ and $r \in [0,1)$ such that $d(Tx, z) \le rd(x, z)$ for $x \in X$. Then $T \in G(X)$.*

PROOF. Put $\hat{r} := (1+r)/2 \in (0,1)$. Define a function f from X into $[0,\infty)$ and a continuous nondecreasing function g from $[0,\infty)$ into $[0,\infty)$ by

$$f(x) = d(x, z) \qquad \text{and} \qquad g(t) = \hat{r}t.$$

It is obvious that $f(z) = 0$, $0 < g(t) < t$ for $t \in (0,\infty)$ and

$$d(T^n x, z) \le r^n d(x, z) \le \hat{r}^n d(x, z) = g^n(f(x))$$

for $x \in X$ and $n \in \mathbf{N}$. □

THEOREM 8. *Let (X, d) be a complete metric space and $T \in C(X)$. Then $T \in G(X)$.*

PROOF. Since $T \in C(X)$, there exists $r \in [0,1)$ such that $d(Tx, Ty) \le rd(x, y)$ for $x, y \in X$. By Theorem 1, there exists a unique fixed point z of T. Then it is obvious that $d(Tx, z) \le rd(x, z)$. So by Lemma 7, $T \in G(X)$. □

LEMMA 9. *Let T be a mapping on a metric space (X, d). Assume there exist $z \in X$ and $r \in [0,1)$ such that*

$$\max\{d(Tx, T^2x), d(Tx, z)\} \le rd(x, Tx)$$

for $x \in X$. Then $T \in G(X)$.

PROOF. We note that z is a fixed point of T. Put $\hat{r} := (1+r)/2 \in (0,1)$. Define a function f from X into $[0,\infty)$ and a continuous nondecreasing function g from $[0,\infty)$ into $[0,\infty)$ by

$$f(x) = d(x, Tx) \qquad \text{and} \qquad g(t) = \hat{r}t.$$

It is obvious that $f(z) = 0$, $0 < g(t) < t$ for $t \in (0,\infty)$ and

$$\begin{aligned} d(T^n x, z) &\le rd(T^{n-1}x, T^n x) \le r^n d(x, Tx) \\ &\le \hat{r}^n d(x, Tx) = g^n(f(x)) \end{aligned}$$

for all $x \in X$ and $n \in \mathbf{N}$. □

THEOREM 10. *Let (X, d) be a complete metric space and $T \in K(X)$. Then $T \in G(X)$.*

PROOF. Since $T \in C(X)$, there exists $\alpha \in [0,1/2)$ such that $d(Tx, Ty) \le \alpha(d(x, Tx) + d(y, Ty))$ for $x, y \in X$. Put $r := \alpha/(1-\alpha) \in [0,1)$. We have $d(Tx, T^2x) \le rd(x, Tx)$.

By Theorem 2, there exists a unique fixed point z of T. We have

$$d(Tx, z) = d(Tx, Tz) \le \alpha(d(x, Tx) + d(z, Tz))$$
$$= \alpha d(x, Tx) \le r d(x, Tx).$$

So by Lemma 9, $T \in G(X)$. □

By Theorem 6 and so on, we obtain the following.

COROLLARY 11. *Let* (X, d) *be a complete metric space. Then*

$$C(X) \cup K(X) \subset G(X) \subset TC(X) = TK(X) \subset P(X)$$

holds.

REMARK. We do not know whether there exists a complete metric space X such that $G(X) \subsetneqq TC(X)$.

3. Additional results

Very recently, we obtained the following theorems.

THEOREM 12 ([3]). *Put* Δ *and* Δ_j $(j = 1, \dots, 4)$ *by*

$$\Delta = \{(\alpha, \beta) : \alpha \ge 0, \beta \ge 0, \alpha + \beta < 1\},$$
$$\Delta_1 = \{(\alpha, \beta) \in \Delta : \alpha \le \beta, \alpha + \beta + \alpha^2 < 1\},$$
$$\Delta_2 = \{(\alpha, \beta) \in \Delta : \alpha \ge \beta, \alpha + \beta + \beta^2 < 1\},$$
$$\Delta_3 = \{(\alpha, \beta) \in \Delta : \alpha \ge \beta, \alpha + \beta + \beta^2 \ge 1\},$$
$$\Delta_4 = \{(\alpha, \beta) \in \Delta : \alpha \le \beta, \alpha + \beta + \alpha^2 \ge 1\}.$$

Define a nonincreasing function ψ *from* Δ *into* $(1/2, 1]$ *by*

$$\psi(\alpha, \beta) = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \Delta_1 \\ 1 & \text{if } (\alpha, \beta) \in \Delta_2 \\ 1 - \beta & \text{if } (\alpha, \beta) \in \Delta_3 \\ (1 - \beta)/(1 - \beta + \alpha) & \text{if } (\alpha, \beta) \in \Delta_4. \end{cases}$$

Let T *be a mapping on a complete metric space* (X, d). *Assume that there exists* $(\alpha, \beta) \in \Delta$ *such that*

$$\psi(\alpha, \beta) d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le \alpha d(x, Tx) + \beta d(y, Ty)$$

for all $x, y \in X$. *Then* $T \in P(X)$.

THEOREM 13 ([8]). *Put* Δ *and* Δ_j $(j = 1, \dots, 6)$ *by*

$$\Delta = [0,1)^2,$$

$$\Delta_1 = \{(\alpha,\beta) \in \Delta : \alpha + \alpha^2 < 1 \ \ or \ \ \beta + \beta^2 < 1\},$$

$$\Delta_2 = \{(\alpha,\beta) \in \Delta : \alpha \geq \beta, (\sqrt{5}-1)/2 \leq \beta \leq 1/\sqrt{2}\},$$

$$\Delta_3 = \{(\alpha,\beta) \in \Delta : \alpha \geq \beta, 1/\sqrt{2} \leq \beta < 1\},$$

$$\Delta_4 = \{(\alpha,\beta) \in \Delta : \alpha \leq \beta, (\sqrt{5}-1)/2 \leq \alpha \leq 1/\sqrt{2}, \beta \leq \alpha^2 - \alpha + 1\},$$

$$\Delta_5 = \{(\alpha,\beta) \in \Delta : (\sqrt{5}-1)/2 \leq \alpha \leq 1/\sqrt{2}, \alpha^2 - \alpha + 1 \leq \beta \leq 1 - \alpha^3/(1+\alpha)\},$$

$$\Delta_6^* = \{(\alpha,\beta) \in \Delta : (\sqrt{5}-1)/2 \leq \alpha \leq 1/\sqrt{2}, 1 - \alpha^3/(1+\alpha) \leq \beta\},$$

$$\Delta_6^{**} = \{(\alpha,\beta) \in \Delta : \alpha \leq \beta, 1/\sqrt{2} \leq \alpha < 1\},$$

$$\Delta_6 = \Delta_6^* \cup \Delta_6^{**}.$$

Define a function φ *from* Δ *into* $(1/2, 1]$ *by*

$$\varphi(\alpha,\beta) = \begin{cases} 1 & if \ \ (\alpha,\beta) \in \Delta_1 \\ (1-\beta)/\beta^2 & if \ \ (\alpha,\beta) \in \Delta_2 \\ 1/(1+\beta) & if \ \ (\alpha,\beta) \in \Delta_3 \\ (1-\alpha)/\alpha^2 & if \ \ (\alpha,\beta) \in \Delta_4 \\ (1-\beta)/\alpha^3 & if \ \ (\alpha,\beta) \in \Delta_5 \\ 1/(1+\alpha) & if \ \ (\alpha,\beta) \in \Delta_6. \end{cases}$$

Let T *be a mapping on a complete metric space* (X,d). *Assume that there exists* $(\alpha,\beta) \in \Delta$ *such that*

$$\varphi(\alpha,\beta) d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) \leq \max\{\alpha d(x,Tx), \beta d(y,Ty)\}$$

for all $x, y \in X$. *Then* $T \in P(X)$.

The conclusion of both theorems is $T \in P(X)$. We will prove stronger results.

THEOREM 14. *Let* T *be a mapping on a complete metric space* (X,d). *Assume that* T *satisfies the assumption in Theorem* 12. *Then* $T \in G(X)$.

PROOF. By Theorem 12, there exists a unique fixed point z of T. We put $r := \alpha/(1-\beta) \in [0,1)$. In [3], we have shown

- $d(Tx, T^2x) \leq r d(x,Tx)$ for $x \in X$
- $d(z,Tx) \leq \beta d(x,Tx)$ for $x \in X \setminus \{z\}$.

Since z is a fixed point of T, $d(z,Tx) \leq \beta d(x,Tx)$ holds for all $x \in X$. By Lemma 9, we obtain $T \in G(X)$. □

THEOREM 15. *Let T be a mapping on a complete metric space (X, d). Assume that T satisfies the assumption in Theorem* 13. *Then $T \in G(X)$.*

PROOF. By Theorem 13, there exists a unique fixed point z of T. We put $r := \min\{\alpha, \beta\} \in [0, 1)$. In [8], we have shown

- $d(Tx, T^2x) \leq rd(x, Tx)$ for $x \in X$
- $d(z, Tx) \leq \beta d(x, Tx)$ for $x \in X \backslash \{z\}$.

As in the proof of Theorem 14, we can prove the desired result. □

By Theorem 6, we obtain the following.

COROLLARY 16. *Let T be a mapping on a complete metric space (X, d). Assume that T satisfies the assumption in either Theorem* 12 *or* 13. *Then $T \in TC(X)$.*

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