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Global Robust Control Design of Nonlinearly-Bounded Interconnected Systems ¶§

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Abstract: This paper presents a novel approach to a new problem of global stabilization with $\mathcal{L}_2$ disturbance attenuation and local matching for a class of uncertain nonlinear systems. An innovative concept of state-dependent scaling is developed to deal with interconnection of nonlinear-gain bounded dynamic systems which unnecessarily have finite linear-gain. The state-dependent scaling approach is not only a natural extension of robust linear $\mathcal{H}^{\infty}$ control, but also the approach offers flexibility to render the nonlinear controller identical with the linear control at the equilibrium. The effect of the disturbance on the controlled output can be attenuated to an arbitrarily small level with global asymptotic stability by partial-state feedback control if the nonlinear system belongs to a generalized class of interconnected strict-feedback systems. A procedure of designing such partial-state feedback controllers is described in the form of recursive selection of state-dependent scaling factors.

Keywords: State-dependent scaling, Nonlinear gain, Robust global stabilization, Almost disturbance decoupling, Robust control Lyapunov function

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1 Introduction

The use of the small gain theorem is one of popular theoretical tools for assessing stability of interconnected systems. Indeed, the small gain technique forms the basis of most popular linear robust control methods such as [2, 6, 31, 3] and references therein. Roughly speaking, if an appropriate composition of the size of two systems is smaller than unit, the feedback interconnection of the two systems is guaranteed to be stable. Scaled $\mathcal{L}_p$-gain has been found reasonable and popular in measuring the size of linear systems. However, it has been widely known that boundedness of such linear gains is far too strong a requirement for a nonlinear system. Thus, different points of view such as notions of nonlinear gain [27, 17, 30] have been needed to deal with essential nonlinearities.

Recently, an idea of 'nonlinear scaling' has been introduced into the $\mathcal{L}_2$-gain of nonlinear systems by [10, 11]. These papers employ scaling factors which are functions of state variables so that the gain is not a linear gain any more. The notion of state-dependent scaling allows one system of a feedback interconnection comprising two systems to have infinite $\mathcal{L}_2$-gain. The other system in the loop is still required to be $\mathcal{L}_2$-gain bounded in primitive results presented in the previous papers [10, 11]. In spite of this limitation, the setting has fitted robust control of uncertain nonlinear systems successfully[11, 12]. Indeed, the $\mathcal{L}_2$-gain ball is one of popular models to describe uncertainties. The $\mathcal{L}_2$-gain is the ratio between the average power of output and input of a system, which is the direct extension of the peak magnitude of the bode frequency plot to nonlinear systems. However, if we have more knowledge on the uncertainty such as nonlinear gain other than linear gain, the result based on [10, 11] might be too conservative. The previous papers [10, 11] could employ state-dependent scaling only for static uncertainties so that its capability of global stabilization is limited in the presence of dynamic uncertainties. The problem of disturbance attenuation was not addressed either.

In the last decade, global stabilization of nonlinear systems with unknown parameters and uncertain static nonlinearities has been extensively studied in the literature of nonlinear control. For instance, backstepping with parameter uncertainty in strict-feedback systems has been considered in [25, 23, 20, 5] and references therein. Unknown unmodeled dynamics has attracted less attention than static uncertainties. The effect of unmodeled dynamics is essentially different from static uncertainty in that inadequate high gain domination leads to the loss of robustness to unmodeled dynamics[26], and dynamic uncertainties often cause drastic shrinking of the region of attraction[21]. At the present time of writing, there are several tools concentrating on such troublesome uncertainties. A gain margin for a fixed linear unmodeled dynamics at control input was investigated in [21] and dynamic nonlinear damping was introduced. Linear unmodeled dynamics was also considered in [16]. These results have been extended to nonlinear unmodeled dynamics by [24]. Optimal control posses a certain margin of stability in the sense of [26]. Inverse optimal control is, however, not sufficient to secure global robustness to general classes of dynamic uncertainties although it may establish robustness to static uncertainties. The concept of robust control Lyapunov functions defined in [5] is not applicable to dynamic uncertainties either. Usually, redesign of control Lyapunov functions are required to robustify control against dynamic uncertainties[19, 7, 13, 1]. One has yet to develop a tool which unifies treatment of static and dynamic uncertainties in global nonlinear stabilization.

In this paper, the concept of state-dependent(SD) scaling is extensively developed further in a fruitful from to deal with global control of nonlinear systems described by interconnection of nonlinear-gain bounded dynamic systems. Advanced state-dependent scaling factors are proposed for the first time to achieve this goal. Thereby, the use of linear small gain and nonlinear small gain is unified successfully within a single framework which locally falls in with popular linear robust control. Thus, the SD scaling approach becomes a natural extension of popular scaling techniques in linear $\mathcal{H}^{\infty}$ control to nonlinear systems. The SD scaling concept of this paper not only allows nonlinear gains, but also brings in a unique way of constructing non-quadratic (control) Lyapunov functions directly. The development is considered as a global robustification of the previous results[5, 11] against dynamic uncertainties and an extension of them to $\mathcal{L}_2$ disturbance attenuation problems. This paper allows scaling factors to depend on state variables even in the presence of dynamic uncertainties and exogenous disturbances. Dynamic and static uncertainties are treated in a unified way so that design formulas for the two types of uncertainties are identical. A difference only appears in classes from which scaling factors are chosen. This paper does not limit dynamic uncertainties to input unmodeled dynamics either. They are allowed to enter a system in a general manner.

The main purpose of this paper is largely to present a unique solution to a new problem for interconnected uncertain nonlinear dynamic systems. In brief, the primary goal is to design nonlinear controllers which achieve three control objectives simultaneously. One is global asymptotic stabilization and another is disturbance attenuation in a $\mathcal{L}_2$-gain sense. These two properties are required to be robust against all admissible uncertainties. The other one is to secure desirable local properties of the nonlinear control in the way that nonlinear controllers are identical with robust linear $\mathcal{H}^{\infty}$ control at the equilibrium. The SD scaling characterization developed in this paper leads to an explicit construction of partial-state nonlinear feedback laws which are natural extensions of linear robust $\mathcal{H}^{\infty}$ controllers. This point is quite unique and useful in practice. For a class of interconnected systems, the control laws are systematically generated by selecting SD scaling factors and parameters of the coordinate change recursively. The design equations are obtained as affine algebraic inequalities with respect to the design parameters, so that the SD scaling approach is advantageous to systematic numerical computation as well as analytical computation. This feature contrasts with the past literature in constructive nonlinear control. The layout of interconnected uncertain systems for which this paper guarantees the existence of solutions is broader than setups considered in previous papers [8, 22, 29, 9] which deal with $\mathcal{L}_2$ disturbance attenuation. Thanks to SD scaling and non-quadratic Lyapunov functions generated by the scaling, this paper allows uncertainties to be dynamic and linearly unbounded as well. For much more general systems, nonlinear weighting in the level of $\mathcal{L}_2$ disturbance attenuation and a nonlinear gain margin are addressed in this paper. This paper also introduce a new class of nonlinear uncertainties including systems which are not input-to-state stable.

This paper is organized as follows. Section 2 starts with a general description of nonlinear systems subject to dynamic and static structured nonlinear uncertainties. In Section 3, a new class of nonlinearly bounded uncertainties is defined. Section 4 develops a novel concept of state-dependent scaling for dynamic and static uncertainties. Global stability and the level of $\mathcal{L}_2$ disturbance

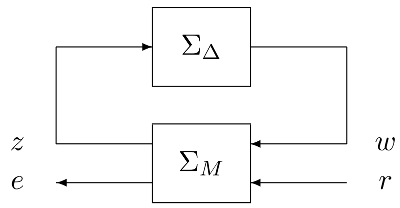

Figure 1: Interconnected nonlinear system Σ

attenuation of the uncertain nonlinear systems are recasted as a state-dependent scaling problem, which forms the basis of this paper. After Section 3, the topic is shifted to a partial-state feedback controller design problem which is formulated in Section 5. The design goal consists of global stabilization, $\mathcal{L}_2$ disturbance attenuation and local matching with scaled $\mathcal{H}^\infty$ linear controllers. Section 6 introduces a new class of uncertain systems, which is called the generalized robust strict-feedback form. For this wide class of systems, the existence of controllers is demonstrated in Section 7. The section also proposes a recursive procedure to construct partial-state feedback controllers. In addition, a result of nonlinear gain margin and nonlinear weighting in $\mathcal{L}_2$ disturbance attenuation is presented. An example is provided to illustrate the proposed methodology. Section 8 gathers some important modifications. Another set of new state-dependent scaling which guarantees solutions to another class of systems is also presented briefly. Some conclusions are drawn in Section 9. Proofs can be found in Appendix.

2 General layout of interconnected system

Consider the nonlinear uncertain system Σ comprising a time-invariant system Σ_M and a time-varying uncertain system Σ_Δ shown in Fig.1. The lower part Σ_M is described by

$$\Sigma_M : \left\{ \begin{array}{l} \dot{x} = A(x)x + B(x)\bar{w} \\ \bar{z} = C(x)x + D(x)\bar{w} \end{array} \right. , \begin{array}{c} x(t) \in \mathcal{R}^n \\ \bar{w}(t),\ \bar{z}(t) \in \mathcal{R}^{p+q} \end{array} \tag{1}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \quad \begin{array}{c} w_i(t), z_i(t) \in \mathcal{R}^{p_i} \\ p_i \geq 0 \\ p = \sum_{i=1}^m p_i \end{array}$$

$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}, \quad \begin{array}{c} r_i(t), e_i(t) \in \mathcal{R}^{q_i} \\ q_i \geq 0 \\ q = \sum_{i=1}^m q_i \end{array}$$

$$\bar{w}_i = \begin{bmatrix} w_i \\ r_i \end{bmatrix}, \ \bar{z}_i = \begin{bmatrix} z_i \\ e_i \end{bmatrix} \in \mathcal{R}^{p_i+q_i}, \ \bar{w} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_m \end{bmatrix}, \ \bar{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_m \end{bmatrix}$$

The matrices A, B, C, and D are assumed to be $\mathcal{C}^0$ functions of x. The part Σ_Δ which is a nonlinear mapping from z to w consists of the following mappings.

$$\Delta_i : z_i = \begin{bmatrix} z_{is} \\ z_{id} \end{bmatrix} \mapsto w_i = \begin{bmatrix} w_{is} \\ w_{id} \end{bmatrix}, \quad w_i = \begin{bmatrix} \Delta_{is} & 0 \\ 0 & \Delta_{id} \end{bmatrix} z_i \ . \tag{2}$$

Here, Δ_{is} and Δ_{id} represent a time-varying static system and a time-varying dynamic system, respectively. It is unnecessary for Δ_i to have the both types. The static part Δ_{is} and the dynamic part Δ_{id} are defined by

$$\Delta_{is} \ : w_{is} = h_{\Delta_{is}}(z_{is}, t) \tag{3}$$

$$\Delta_{id} \ : \begin{cases} \dot{x}_{\Delta_i} = f_{\Delta_{id}}(x_{\Delta_i}, z_{id}, t) \\ w_{id} = h_{\Delta_{id}}(x_{\Delta_i}, t) \end{cases} , \tag{4}$$

We assume $f_{\Delta_{id}}(0,0,t) = 0$, $h_{\Delta_{id}}(0,t) = 0$ and $h_{\Delta_{is}}(0,t) = 0$ for all $t \geq 0$. Functions $f_{\Delta_{id}}$, $h_{\Delta_{i*}}$ ($*$ stands for s or d) are locally Lipschitz in (x_{Δ_i}, z_{i*}) on $\mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{i*}}$, uniformly in $t \in \mathcal{R}$. In order to assure the well-posedness of the interconnection $(\Sigma_M, \Sigma_\Delta)$, the static mapping between $\bar{w}$ and z seen as a part of the direct-feedthrough matrix $D(x)$ is assumed to be zero. The system Σ_Δ represents uncertainty arising from the system Σ so that knowledge of the functions $f_{\Delta_{id}}$, $h_{\Delta_{id}}$ and $h_{\Delta_{is}}$ is unnecessary. We only assume that information about nonlinear or linear gain is available in a sense described in Section 3. The state of the whole interconnected system Σ is $x_{cl} = [x^T, x_\Delta^T]^T = [x^T, x_{\Delta_1}^T, x_{\Delta_2}^T, \cdots, x_{\Delta_m}^T]^T \in \mathcal{R}^{n+n_\Delta}$. The system Σ is said to be globally uniformly asymptotically stable if the equilibrium $x_{cl} = 0$ is globally uniformly asymptotically stable. In this paper, the system Σ is said to have $\mathcal{L}_2$-gain less than or equal to τ if there exists a storage function $V(x_{cl})$ which is positive definite and radially unbounded in x_{cl} such that for all initial states $x_{cl}(0) \in \mathcal{R}^{n+n_\Delta}$, and all $r \in \mathcal{L}_2[0,T]$, the inequality

$$V(x_{cl}(T)) \leq V(x_{cl}(0)) + \int_0^T (\tau^2 \|r\|^2 - \|e\|^2) dt$$

holds for all $T \geq 0$.

3 Nonlinearly bounded uncertainty

This section defines a class of nonlinearly bounded systems for describing the uncertainty Σ_Δ. It will be shown that input-to-state stable systems fall into that class under a mild assumption.

In this paper, the uncertainty Σ_Δ is supposed to belong to the following class of nonlinearly bounded systems.

Assumption 1 *The uncertain system Σ_Δ satisfies* **(i)** *and* **(ii)** *for each $i = 1, 2, \ldots, m$.*

(i) *There exists a $\mathcal{C}^0$ function $\psi_{is} : [0,\infty) \to [0,\infty)$ such that*

$$\|w_{is}\|^2 \leq \psi_{is}(\|z_{is}\|)\|z_{is}\|^2 \tag{5}$$

holds for all $t \in [0,\infty)$.

(ii) *There exists a $\mathcal{C}^0$ function $\psi_{id} : [0,\infty) \to [0,\infty)$ and a $\mathcal{C}^1$ function $W_{\Delta i} : [0,\infty) \times \mathcal{R}^{n_{\Delta_i}} \to \mathcal{R}$ such that*

$$\underline{\beta}_i(\|x_{\Delta_i}\|) \leq W_{\Delta i}(t, x_{\Delta_i}) \leq \bar{\beta}_i(\|x_{\Delta_i}\|) \tag{6}$$

$$\frac{\partial W_{\Delta i}}{\partial t} + \frac{\partial W_{\Delta i}}{\partial x_{\Delta_i}} f_{\Delta_{id}} \leq -\beta_i(x_{\Delta_i}) - \|w_{id}\|^2 + \psi_{id}(\|z_{id}\|)\|z_{id}\|^2 \tag{7}$$

hold for all $(t, x_{\Delta_i}, z_{id}) \in [0,\infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$, where $\underline{\beta}_i$ and $\bar{\beta}_i$ are class $\mathcal{K}_\infty$ functions, and β_i is a positive definite $\mathcal{C}^0$ function of x_{Δ_i}.

A system Σ_Δ is said to be admissible if Assumption 1 is true. The assumption does not require uncertain systems to have finite $\mathcal{L}_2$-gain. Instead, they are supposed to have finite nonlinear-gain. When Δ_{is} (Δ_{id}) exhibits finite $\mathcal{L}_2$-gain, the parameter ψ_{is} (ψ_{id}, respectively) in Assumption 1 reduces to a positive constant. In such a case, we obtain $\psi_{is} = \gamma_{is}^2$ and $\psi_{id} = \gamma_{id}^2$, where γ_{is} and γ_{id} are $\mathcal{L}_2$-gain. The new class of uncertain systems is broad and it includes input-to-state stable(ISS) systems in the following sense.

Lemma 1 **(i)** *Suppose that a static system Δ_{is} admits class $\mathcal{K}_\infty$ functions α_i and σ_i such that*

$$\alpha_i(\|w_{is}\|) \leq \sigma_i(\|z_{is}\|) \tag{8}$$

holds for all $t \in [0, \infty)$ and

$$\lim_{s \to 0^+} \frac{\sigma_i(s)}{\alpha_i(s)} < +\infty \tag{9}$$

holds. Then, there exists a $\mathcal{C}^0$ function ψ_{is} such that (5) holds for all $t \in [0, \infty)$.

(ii) *Suppose that a dynamic system Δ_{id} admits a $\mathcal{C}^1$ function $V_{\Delta i} : [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \to \mathcal{R}$ such that*

$$\underline{\alpha}_i(\|x_{\Delta_i}\|) \leq V_{\Delta i}(t, x_{\Delta_i}) \leq \bar{\alpha}_i(\|x_{\Delta_i}\|) \tag{10}$$

$$\frac{\partial V_{\Delta i}}{\partial t} + \frac{\partial V_{\Delta i}}{\partial x_{\Delta_i}} f_{\Delta_{id}} \leq -\alpha_i(\|x_{\Delta_i}\|) + \sigma_i(\|z_{id}\|) \tag{11}$$

are satisfied for all $(t, x_{\Delta_i}, z_{id}) \in [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$ where $\underline{\alpha}_i$, $\bar{\alpha}_i$ and α_i are class $\mathcal{K}_\infty$ functions and σ_i is a class $\mathcal{K}$ function and they satisfy

$$\lim_{\|x_{\Delta_i}\| \to 0} \frac{\|w_{id}\|^2}{\alpha_i(\|x_{\Delta_i}\|)} < +\infty, \quad \lim_{\|z_{id}\| \to 0} \frac{\sigma_i(\|z_{id}\|)}{\|z_{id}\|^2} < +\infty \tag{12}$$

uniformly in t. Then, there exists a $\mathcal{C}^0$ function ψ_{id}, a $\mathcal{C}^1$ function $W_{\Delta i}$, class $\mathcal{K}_\infty$ functions $\underline{\beta}_i$ and $\bar{\beta}_i$ and a positive definite $\mathcal{C}^0$ function β_i such that (6) and (7) hold for all $(t, x_{\Delta_i}, z_{id}) \in [0, \infty) \times \mathcal{R}^{n_{\Delta_i}} \times \mathcal{R}^{p_{id}}$.

The proof exploits a technique introduced in [28] and appropriate modifications. According to the proof, a function ψ_{is} can be obtained from

$$(\alpha_i^{-1} \circ \sigma_i(s))^2 \leq \psi_{is}(s)s^2 \quad \forall s \in [0, \infty)$$

A choice of ψ_{id} is a function satisfying

$$\left[q \circ \bar{\alpha}_i \circ \alpha_i^{-1} \circ \pi_i \sigma_i(s)\right] \sigma_i(s) \leq \psi_{id}(s)s^2$$

for some $\tau_i > 1$ and $\pi_i > \tau_i/(\tau_i - 1)$. Here, $q : [0, \infty) \to [0, \infty)$ is a non-decreasing $\mathcal{C}^0$ function such that $q(s) > 0, \forall s > 0$ and

$$q \circ \underline{\alpha}_i(\|x_{\Delta_i}\|) \geq \frac{\tau_i \|w_{id}\|^2}{\alpha_i(\|x_{\Delta_i}\|)}, \quad \forall (t, x_{\Delta_i}) \in \mathcal{R} \times \mathcal{R}^{n_{\Delta_i}}$$

Then, β_i is obtained as

$$\beta_i(x_{\Delta_i}) = \left(1 - \frac{1}{\pi_i} - \frac{1}{\tau_i}\right) [q \circ \underline{\alpha}_i(\|x_{\Delta_i}\|)] \, \alpha_i(\|x_{\Delta_i}\|)$$

Note that $\beta_i(x_{\Delta_i})$ can be always chosen as a class $\mathcal{K}_\infty$ function of $\|x_{\Delta_i}\|$ for ISS systems defined in Lemma 1(ii). The existence of all functions appearing in the above are guaranteed by Lemma 1. It is emphasized that Assumption 1 admits systems which are not ISS. To bring out a better interpretation of Lemma 1, we define

$$\hat{z}_{is} = \upsilon_{is}(z_{is}) = \psi_{is}^{1/2}(\|z_{is}\|)z_{is}$$
$$\hat{z}_{id} = \upsilon_{id}(z_{id}) = \psi_{id}^{1/2}(\|z_{id}\|)z_{id}$$

Lemma 1 demonstrates that the mappings Δ_{is} and Δ_{id} can be decomposed into $\Delta_{is} = \hat{\Delta}_{is}\circ\upsilon_{is}(z_{is})$ and $\Delta_{id} = \hat{\Delta}_{id} \circ \upsilon_{id}(z_{id})$. Here, new mappings $\hat{\Delta}_{is} : \hat{z}_{is} \mapsto w_{is}$ and $\hat{\Delta}_{id} : \hat{z}_{id} \mapsto w_{id}$ are globally $\mathcal{L}_2$-gain bounded due to (5) and (7), respectively. An example of nonlinearly bounded static mappings which violate (9) is $w_{is} = h_{\Delta_{is}}(z_{is}) = \sqrt{\|z_{is}\|}$ which is not Lipschitz at $z_{is} = 0$. Indeed, if $h_{\Delta_{is}}(z_{is}, t)$ is Lipschitz at $z_{is} = 0$ uniformly in t as assumed in Section 2, there always exists a class $\mathcal{K}_\infty$ pair of $\{\alpha(s), \sigma(s)\}$ satisfying (9) and (8). Therefore, a static nonlinear mapping $\Delta_{is} : z_{is} \mapsto w_{is}$ can be always decomposed in such a way. A dynamic nonlinear mapping $\Delta_{id} : z_{id} \mapsto w_{id}$ is guaranteed to be decomposable under the condition (12). The condition together with (11) is common in asymptotic analysis based on the nonlinear small-gain technique[14, 15, 17]. It is known that (12) is always satisfied for appropriate functions $\alpha_i \in \mathcal{K}_\infty$ and $\sigma_i \in \mathcal{K}$ if the Jacobian linearization of Δ_{id} at $x_{\Delta_i} = 0$ is uniformly asymptotically stable(e.g. see [18]).

4 SD scaling characterization

This section derives a characterization of global robustness properties of the interconnected system Σ via a generalized concept of state-dependent (SD) scaling which incorporates the nonlinear gain.

In order to characterize the robustness of the interconnected system Σ, this paper introduces several new classes of SD scaling factors associated with uncertain components. First, a set of scaling factors associated with dynamic components Δ_{id} is defined as

$$\mathbf{\Phi}_{id} = \left\{ \Phi_{id}(x) = \phi_d(V_0(x)) \begin{bmatrix} \check{\phi}_{id} I & 0 \\ 0 & I \end{bmatrix} : \begin{array}{l} \phi_d(\cdot) \in \mathcal{C}^0, \quad \phi_d(s) > 0,\ \forall s \in [0,\infty), \quad \check{\phi}_{id} > 0 \\ \exists \mu_\phi(\cdot) \in \mathcal{K} \text{ s.t. } \dfrac{s}{\phi_d(s)} \geq \mu_\phi(s),\ \forall s \in [0,\infty) \end{array} \right\} \tag{13}$$

for $i = 1, 2, \ldots, m$. Here, the block partition of Φ_{id} is compatible in size with that of $[z_{id}^T, e_i^T]^T$. All sets $\mathbf{\Phi}_{id}$, $i = 1, 2, \ldots, m$ are defined with a common function $\phi_d(\cdot)$. The parameters $\check{\phi}_{id}$ for $i = 1, 2, \ldots, m$ are constant. The function $V_0(\cdot)$ is a $\mathcal{C}^0$ function $\mathcal{R}^n \to [0, \infty)$ of the state x, which has yet to be defined. For static components Δ_{is}, a set of scaling factors is defined by

$$\mathbf{\Phi}_{is} = \{\Phi_{is}(x) = \phi_{is}(x)I : \phi_{is}(\cdot) \in \mathcal{C}^0,\ \phi_{is}(x) > 0 \quad \forall x \in \mathcal{R}^n\} \tag{14}$$

The identity matrix I is compatible in size with z_{is}. The definition of Φ_{is} is similar to state-dependent scaling factors employed in [11, 10]. The new scaling factor Φ_{id} is different from those SD scaling factors in that the growth rate of the state dependence is constrained. Note that constant scaling factors are members of the set $\mathbf{\Phi}_{id}$, i.e., a matrix Φ_{id} with a positive constant ϕ_d

belongs to $\mathbf{\Phi}_{id}$. The following lemma will play an important role in constructing non-quadratic Lyapunov functions with the new SD scaling.

Lemma 2 *Suppose that $\eta(x)$ is a $\mathcal{C}^1$ function of $x \in \mathcal{R}^n$ which is positive definite and radially unbounded. Let $\phi(\cdot)$ be a $\mathcal{C}^0$ function which fulfills the following properties.*

$$\begin{aligned}
&\phi(s) > 0,\ \ \forall s \in [0,\infty) \\
&\exists \mu(\cdot) \in \mathcal{K} \text{ s.t. } \frac{s}{\phi(s)} \geq \mu(s),\ \ \forall s \in [0,\infty)
\end{aligned}$$

Then, the function

$$\zeta(x) = \int_0^{\eta(x)} \frac{1}{\phi(s)} ds$$

is $\mathcal{C}^1$, positive definite and radially unbounded.

For $i = 1, 2, \ldots, m$, define $\Phi_i(x)$ as

$$\mathbf{\Phi}_i = \left\{ \Phi_i(x) = \begin{bmatrix} \Phi_{is}(x) & 0 \\ 0 & \Phi_{id}(x) \end{bmatrix} : \begin{array}{c} \Phi_{is} \in \mathbf{\Phi}_{is} \\ \Phi_{id} \in \mathbf{\Phi}_{id} \end{array} \right\} \tag{15}$$

Using $\mathcal{C}^0$ functions $\psi_{id}, \psi_{is} : [0,\infty) \to [0,\infty)$ in Assumption 1, define $\bar{\Psi}(x)$ as

$$\bar{\Psi}(x) = \operatorname*{block\text{-}diag}_{i=1}^{m} \bar{\Psi}_i(x) \tag{16}$$

$$\bar{\Psi}_i(x) = \begin{bmatrix} \psi_{is}(\|z_{is}\|)^{1/2} I & 0 & 0 \\ 0 & \psi_{id}(\|z_{id}\|)^{1/2} I & 0 \\ 0 & 0 & \tau^{-1} I \end{bmatrix} \tag{17}$$

The block diagonal structure of $\bar{\Psi}_i$ is conformable in size to the partition $\bar{z}_i = [z_{is}^T, z_{id}^T, e_i^T]^T$. The scalar τ is a positive number to describe the level of disturbance attenuation. We are now ready to define three sets of SD scaling matrices by

$$\mathbf{\Phi} = \left\{ \Phi(x) = \operatorname*{block\text{-}diag}_{i=1}^{m} \Phi_i(x),\ \ \Phi_i \in \mathbf{\Phi}_i \right\} \tag{18}$$

$$\mathbf{\Theta} = \left\{ \Theta(x) : \mathcal{R}^n \to \mathcal{R}^{(p+q)\times(p+q)},\ \ \Theta(\cdot) \in \mathcal{C}^0,\ \ \Theta(x) > 0\ \forall x \in \mathcal{R}^n \right\} \tag{19}$$

$$\mathbf{\Psi} = \left\{ \Psi(x) : \mathcal{R}^n \to \mathcal{R}^{(p+q)\times(p+q)},\ \ \Psi(\cdot) \in \mathcal{C}^0,\ \ \Psi(x) \geq 0\ \forall x \in \mathcal{R}^n \right\} \tag{20}$$

All scaling matrices Φ, Θ and Ψ are 'state-dependent'.

Based on the triplet of new scaling matrices defined in the above, we shall characterize stability and $\mathcal{L}_2$ disturbance attenuation of Σ shown in Fig.1. Consider a diffeomorphism between $x \in \mathcal{R}^n$ and $\chi \in \mathcal{R}^n$ as follows:

$$\chi = S(x)x\ . \tag{21}$$

The time-derivative of χ is obtained as

$$\dot{\chi} = \left[\frac{\partial S}{\partial x_1} x, \frac{\partial S}{\partial x_2} x, \cdots, \frac{\partial S}{\partial x_n} x \right] \dot{x} + S(x)\dot{x} = T(x)\dot{x}\ .$$

Let $\chi_{[\kappa]}$ and $\chi_{\langle\kappa\rangle}$ denote

$$\chi_{[\kappa]} = [\chi_1, \chi_2, \cdots, \chi_\kappa]^T, \qquad \chi_{\langle\kappa+1\rangle} = [\chi_{\kappa+1}, \chi_{\kappa+2}, \cdots, \chi_n]^T$$

respectively. Note that $\chi = \chi_{[n]} = \chi_{\langle 1\rangle}$. The following is the main result of this section.

Theorem 1 *Suppose that there exist an integer $\kappa \in [0, n]$, constant symmetric matrices $P_{[\kappa]} \in \mathcal{R}^{\kappa\times\kappa}$, $P_{\langle\kappa+1\rangle} \in \mathcal{R}^{(n-\kappa)\times(n-\kappa)}$, and SD scaling matrices $\Phi \in \boldsymbol{\Phi}$, $\Theta \in \boldsymbol{\Theta}$, $\Psi \in \boldsymbol{\Psi}$ such that*

$$\begin{bmatrix} S^{-T}A^T T^T \Xi + \Xi T A S^{-1} & \Xi T B & S^{-T} C^T \Psi\Phi \\ B^T T^T \Xi & -\Theta & D^T \Psi \Phi \\ \Phi\Psi C S^{-1} & \Phi\Psi D & -\Phi \end{bmatrix} < 0 \tag{22}$$

$$P = \begin{bmatrix} P_{[\kappa]} & 0 \\ 0 & P_{\langle\kappa+1\rangle} \end{bmatrix} > 0 \tag{23}$$

$$\Theta \le \Phi \tag{24}$$

$$\bar{\Psi} \le \Psi \tag{25}$$

are satisfied for all $x \in \mathcal{R}^n$ with

$$V_0(x) = \begin{cases} \chi_{[\kappa]}^T P_{[\kappa]} \chi_{[\kappa]} & , \ \kappa \ge 1 \\ 0 & , \ \kappa = 0 \end{cases} \tag{26}$$

$$\Xi(x) = \begin{bmatrix} I_\kappa & 0 \\ 0 & \phi_d(V_0(x)) I_{n-\kappa} \end{bmatrix} P \tag{27}$$

Then, the system Σ is globally uniformly asymptotically stable, and it has $\mathcal{L}_2$-gain less than or equal to τ.

In the case of $\kappa = 0$, scaling factors Φ_{id}, $i = 1, 2, \dots, n$ defined in (13) become constant matrices. For systems Σ_M with $q = 0$ (equivalently, $\bar{w} = w$ and $\bar{z} = z$), Theorem 1 only addresses stability. Note that $\Xi = P$ holds if $\kappa = n$ or if $\kappa = 0$ and $\phi_d(0) = 1$. In the case of $\{q = 0,\ \kappa = 0,\ \phi_d = 1,\ \Theta = \Phi,\ \Psi = I\}$, Theorem 1 reduces to the primitive result of [11]. In Section 7, it will become clear that the choice $\kappa = 0$ is not sufficient for ensuring the global properties if the system Σ involves dynamic components in Σ_Δ or signals for $\mathcal{L}_2$ disturbance attenuation.

The Lyapunov function employed in the above theorem is

$$V(t, x_{cl}) = \int_0^{V_{0[\kappa]}(\chi_{[\kappa]})} \frac{1}{\phi_d(s)} ds + \chi_{\langle\kappa+1\rangle}^T P_{\langle\kappa+1\rangle} \chi_{\langle\kappa+1\rangle} + \sum_{i=1}^m \check{\phi}_{id} W_{\Delta i}(t, x_{\Delta i}) \tag{28}$$

The first term of the Lyapunov function is in a form which is similar to recent Lyapunov techniques(e.g. [27, 17, 19, 26]). It, however, rises from different purposes and its role and mechanism are distinct from them. In this paper, the integrand $1/\phi_d(s)$ in (28) is given a character of scaling and the function is determined directly by the matrix inequality (22). Theorem 1 demonstrates how to construct the integrand for guaranteeing stability and disturbance attenuation with respect to the uncertain dynamics Σ_Δ explicitly. An explicit procedure for selecting $\phi_d(s)$ for a wide class of nonlinear systems will be also explained in this paper.

It is worth mentioning that the SD scaling characterization presented in Theorem 1 does not require the system to fit in some geometric structure, such as strict-feedback form, which is necessary in backstepping and nested controller designs[20, 5, 29, 15].

In the cases of $\kappa = 0$ and $\kappa = n$, the inequality (22) implies that the 'scaled' system

$$\begin{aligned} \dot{x} &= A(x)x + B(x)\Theta^{-1/2}\bar{w} \\ \bar{z} &= \Phi^{1/2}\Psi C(x)x + \Phi^{1/2}\Psi D(x)\Theta^{-1/2}\bar{w} \end{aligned}$$

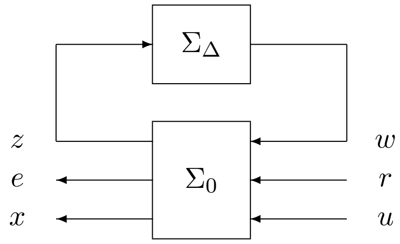

Figure 2: Interconnected plant Σ_P

has $\mathcal{L}_2$-gain less than one with a storage function $x^T S^T P S x$. Here, $\Theta^{-1/2}$ is scaling at the disturbance input, and $\Phi^{1/2}\Psi$ is scaling at the regulated output. The original system (1) unnecessarily has $\mathcal{L}_2$-gain less than one since Φ, Θ and Ψ are functions of the state x. Indeed, it is unnecessary for the original system (1) to have finite $\mathcal{L}_2$-gain. In addition, the uncertain system Σ_Δ is allowed to be linearly unbounded. In this sense, Theorem 1 is much more general and less conservative than the classical input-output small-gain theorem.

5 Design problem setup

From this section, we consider feedback controller design for the nonlinear plant Σ_P shown in Fig.2. Let x denote the state variable of Σ_0. Several control objectives are posed to the feedback design at the same time. The design goal is to find a partial-state feedback controller of the form

$$\Sigma_K \ : \ u = K(x)x \tag{29}$$

which

- globally uniformly asymptotically stabilizes Σ_P
- makes the mapping between r and e have $\mathcal{L}_2$-gain less than or equal to τ
- is identical to a scaled $\mathcal{H}^\infty$ linear controller given arbitrarily at the equilibrium point.

The state x_Δ of the uncertain system Σ_Δ is not measured for the feedback. We assume that Σ_0 in Fig.2 is described by

$$\Sigma_0 : \begin{cases} \dot{x} = A(x)x + B(x)\bar{w} + G(x)u & x(t) \in \mathcal{R}^n, u(t) \in \mathcal{R}^\rho \\ \bar{z} = C(x)x + D(x)\bar{w} + H(x)u\text{'} & \bar{w}(t), \bar{z}(t) \in \mathcal{R}^{p+q} \end{cases} \tag{30}$$

We also assume that the uncertain system Σ_Δ is admissible as defined in Section 3.

The feedback connection consisting of Σ_0 and Σ_K corresponds to Σ_M in Fig.1, and the whole system comprising Σ_0, Σ_K and Σ_Δ is regarded as Σ. According to Theorem 1, the system Σ_P is globally uniformly asymptotically stabilized by a partial-state feedback controller Σ_K and achieves $\mathcal{L}_2$-gain less than or equal to τ if there exist $P > 0$, $\Phi \in \mathbf{\Phi}$, $\Theta \in \mathbf{\Theta}$ such that (23) and

$$M = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T \Xi + \Xi T \hat{A} \hat{S} & \Xi T B & \hat{S}^T \hat{C}^T \Psi \Phi \\ B^T T^T \Xi & -\Theta & D^T \Psi \Phi \\ \Phi \Psi \hat{C} \hat{S} & \Phi \Psi D & -\Phi \end{bmatrix} < 0 \tag{31}$$

$$\Theta \leq \Phi \tag{32}$$

are satisfied for all $x \in \mathcal{R}^n$ with (26-27) and a given matrix $\Psi \in \boldsymbol{\Psi}$ meeting the constraint (25), where $\hat{A}$ and $\hat{S}$ are given by

$$\hat{S} = \begin{bmatrix} S^{-1} \\ KS^{-1} \end{bmatrix}, \quad \hat{A} = [\, A \quad G \,], \quad \hat{C} = [\, C \quad H \,]$$

In addition to the above global properties, this paper takes into account local properties of the feedback system. Define constant matrices $\underline{A}$, $\underline{B}$, $\underline{G}$, $\underline{C}$, $\underline{D}$ and $\underline{H}$ as

$$\begin{aligned}
\underline{A} &= \left.\frac{\partial f(x,\bar{w},u)}{\partial x}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = A(0), \ \underline{B} = \left.\frac{\partial f(x,\bar{w},u)}{\partial \bar{w}}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = B(0)\\
\underline{G} &= \left.\frac{\partial f(x,\bar{w},u)}{\partial u}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = G(0), \ \underline{C} = \left.\frac{\partial h(x,\bar{w},u)}{\partial x}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = C(0)\\
\underline{D} &= \left.\frac{\partial h(x,\bar{w},u)}{\partial \bar{w}}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = D(0), \ \underline{H} = \left.\frac{\partial h(x,\bar{w},u)}{\partial u}\right|_{\substack{x=0\\ \bar{w}=0\\ u=0}} = H(0)\\
& f(x,\bar{w},u) = A(x)x + B(x)\bar{w} + G(x)u\\
& h(x,\bar{w},u) = C(x)x + D(x)\bar{w} + H(x)u
\end{aligned}$$

which define a Jacobian linearized system of Σ_0. The control law $u = \underline{K}x$ with a constant gain matrix $\underline{K}$ is said to be a scaled $\mathcal{H}^\infty$ linear controller if there exist a constant matrix $X > 0$ and constant scalars $\underline{\phi}_{js} > 0$ and $\underline{\phi}_{jd} > 0$, $j = 1, 2, \ldots, m$ such that

$$\underline{M} = \begin{bmatrix} (\underline{A}+\underline{G}\underline{K})^T X + X(\underline{A}+\underline{G}\underline{K}) & X\underline{B} & (\underline{C}+\underline{H}\underline{K})^T\bar{\Psi}(0)\underline{\Phi} \\ \underline{B}^T X & -\underline{\Phi} & \underline{D}^T\bar{\Psi}(0)\underline{\Phi} \\ \underline{\Phi}\bar{\Psi}(0)(\underline{C}+\underline{H}\underline{K}) & \underline{\Phi}\bar{\Psi}(0)\underline{D} & -\underline{\Phi} \end{bmatrix} < 0 \tag{33}$$

$$\underline{\Phi} = \operatorname*{block-diag}_{j=1}^{m} \begin{bmatrix} \underline{\phi}_{js} I & 0 & 0 \\ 0 & \underline{\phi}_{jd} I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{34}$$

are satisfied. Here, the block-diagonal partition of $\underline{\Phi}$ in (34) is compatible with the partition of $\bar{z}$. According to linear robust control theory(e.g.[6, 3]), such parameters X, $\underline{\phi}_{js}$ and $\underline{\phi}_{jd}$ exist if and only if the linear control gain $\underline{K}$ achieves robust stability and robust $\mathcal{H}^\infty$ performance of the disturbance attenuation level τ of the linear system ($\underline{A}$, $\underline{B}$, $\underline{G}$, $\underline{C}$, $\underline{D}$, $\underline{H}$) for all time-varying mappings Δ_i whose constituents have $\mathcal{L}_2$-gain less than or equal to diagonal elements of $\bar{\Psi}(0)$. Recall that $\psi_{is}(0)^{1/2}$ and $\psi_{id}(0)^{1/2}$ represent local $\mathcal{L}_2$-gain of each uncertain components. Thus, it is practically desirable that the nonlinear controller Σ_K agrees locally with the scaled $\mathcal{H}^\infty$ linear controller, i.e.

$$\underline{K} = K(0) = \left.\frac{\partial K(x)x}{\partial x}\right|_{x=0}$$

In this way, this paper seeks a nonlinear controller which is a natural extension of a linear controller.

6 A class of uncertain systems

This section defines a new class of uncertain nonlinear plants Σ_P shown in Fig.2. For that class, this paper will give control laws solving the partial-state feedback problem formulated in Section

5. Consider the system Σ_0 described by (30) with a scalar input $\rho = 1$. Let $m = 2n$. Signals are partitioned as follows:

$$\bar{w}_i = \begin{bmatrix} w_i \\ r_i \end{bmatrix}, \ \bar{z}_i = \begin{bmatrix} z_i \\ e_i \end{bmatrix}, \ \begin{array}{l} w_i(t), z_i(t) \in \mathcal{R}^{p_i}, \ u(t) \in \mathcal{R} \\ r_i(t), e_i(t) \in \mathcal{R}^{q_i} \end{array} \tag{35}$$

$$\bar{w} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_{2n-1} \\ \bar{w}_{2n} \end{bmatrix}, \ \bar{z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{2n-1} \\ \bar{z}_{2n} \end{bmatrix}, \quad \begin{array}{ll} p_i \geq 0, & q_i \geq 0 \\ p = \sum_{i=1}^{2n} p_i, & q = \sum_{i=1}^{2n} q_i \end{array}$$

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{2n-1} \\ w_{2n} \end{bmatrix}, \ z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{2n-1} \\ z_{2n} \end{bmatrix}, \ r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{2n-1} \\ r_{2n} \end{bmatrix}, \ e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{2n-1} \\ e_{2n} \end{bmatrix}$$

Then, the uncertain system Σ_Δ is a mapping in the block-diagonal form of

$$w = \Delta z = \left[\operatorname*{block-diag}_{j=1}^{2n} \Delta_j \right] z \tag{36}$$

where each component is represented by

$$w_i = \begin{bmatrix} w_{is} \\ w_{id} \end{bmatrix} = \begin{bmatrix} \Delta_{is} & 0 \\ 0 & \Delta_{id} \end{bmatrix} \begin{bmatrix} z_{is} \\ z_{id} \end{bmatrix} = \Delta_i z_i \tag{37}$$

Here, Δ_{is} and Δ_{id} are mappings of a static system and a dynamic system, respectively, which are defined in (3-4). Dimension of each vector w_{is}, w_{id}, z_{is}, z_{id} can be zero. Mappings Δ_i with even i represent uncertain components situated at virtual and actual inputs, which will be seen clearly from the structure of matrices B and C defined below(See also [11]). Suppose that all Δ_i, $i = 1, 2, \ldots, 2n$ fulfill Assumption 1. We assume that Σ_0 has the following structure.

$$A(x) = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots\cdots & 0 \\ a_{21} & a_{22} & a_{23} \ 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots \ \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n-1,1} & a_{n-1,2} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, G(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{n,n+1} \end{bmatrix} \tag{38}$$

$$a_{ij}(x) = a_{ij}(x_1, x_2, \cdots, x_i), \quad 1 \leq i \leq n, \ 1 \leq j \leq i+1 \tag{39}$$

$$B(x) = \begin{bmatrix} B_{11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{21} & U_{21} & B_{22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{n1} & U_{n1} & B_{n2} & U_{n2} & \cdots & B_{nn} & U_{Ln} \end{bmatrix}, \ H(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ U_{Rn} \end{bmatrix} \tag{40}$$

$$D(x) = \begin{bmatrix} D_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \ D_i(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{R,i} \end{bmatrix} \tag{41}$$

$$C(x) = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{21} & C_{22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ C_{n-1,1} & C_{n-1,2} & \cdots & \cdots & C_{n-1,n-1} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,n-1} \\ C_{n1} & C_{n2} & \cdots & \cdots & C_{n,n-1} & C_{nn} \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{42}$$

where $B_{ij} \in \mathcal{R}^{1\times(p_{2i-1}+q_{2i-1})}$, $C_{ij} \in \mathcal{R}^{(p_{2i-1}+q_{2i-1})\times 1}$, $D_i \in \mathcal{R}^{(p_{2i-1}+q_{2i-1})\times(p_{2i-1}+q_{2i-1})}$, $U_{L,i} \in \mathcal{R}^{1\times(p_{2i}+q_{2i})}$, $U_{R,i} \in \mathcal{R}^{(p_{2i}+q_{2i})\times 1}$ and $U_{li} \in \mathcal{R}^{1\times(p_{2i}+q_{2i})}$ are consistent with

$$B_{ij}(x)=B_{ij}(x_1,x_2,\cdots,x_i),\ C_{ij}(x)=C_{ij}(x_1,x_2,\cdots,x_i),\ D_i(x)=D_i(x_1,x_2,\cdots,x_i) \tag{43}$$

$$U_{Li}(x)=U_{Li}(x_1,x_2,\cdots,x_i),\ U_{Ri}(x)=U_{Ri}(x_1,x_2,\cdots,x_i) \tag{44}$$

$$U_{li}(x)=U_{li}(x_1,x_2,\cdots,x_l),\quad i+1\le l\le n \tag{45}$$

for $1 \le i \le n$ and $1 \le j \le i$. The block partition of D_i is consistent with that of $[z_{2i-1,s}^T, z_{2i-1,d}^T, e_{2i-1,}^T]^T$. We also assume

$$I - \tau^{-2} D_{R,i} D_{R,i}^T > 0 \qquad \text{if } q_{2i-1} \neq 0 \tag{46}$$

$$a_{i,i+1} \neq 0 \qquad \text{if } p_{2i} + q_{2i} = 0 \tag{47}$$

for all $x \in \mathcal{R}^n$ and $i = 1, 2, \ldots, n$. Finally, it is assumed that there exist positive numbers $\hat{\phi}_{js}$ and $\hat{\phi}_{jd}$ such that

$$a_{i,i+1}^2 > U_{Ri}^T \bar{\Psi}_{2i}^2 \hat{\Phi}_{2i} U_{Ri} U_{Li} \hat{\Phi}_{2i}^{-1} U_{Li}^T \qquad \text{if } p_{2i} + q_{2i} \neq 0 \tag{48}$$

holds for all $x \in \mathcal{R}^n$ and $i = 1, 2, \ldots, n$, where $\hat{\Phi}_j$ is defined by

$$\hat{\Phi}_j = \begin{bmatrix} \hat{\phi}_{js} I & 0 & 0 \\ 0 & \hat{\phi}_{jd} I & 0 \\ 0 & 0 & I \end{bmatrix} > 0 \quad j = 1, 2, \ldots, 2n \tag{49}$$

The block partition of $\hat{\Phi}_j$ is compatible in size with that of $[z_{js}^T, z_{jd}^T, e_j^T]^T$. The inequalities (47) and (48) prevent coefficients of virtual and actual inputs from being zero[5]. These conditions are for simplification of solution formulas(see Subsection 8.1). In either case of $U_{Ri}(x) \equiv 0$ and $U_{Li}(x) \equiv 0$, the condition $a_{i,i+1}(x) \neq 0$ is sufficient for (48). The inequality (46) is necessary for achieving the prescribed level τ of $\mathcal{L}_2$-gain between r and e. When we consider Σ_0 with $q = 0$ (namely, $\bar{w} = w$ and $\bar{z} = z$), our concern is only stabilization. In this paper, a system Σ_P consisting of Σ_0 and Σ_Δ satisfying (35-48) and Assumption 1 is said to be in the generalized robust strict-feedback form. This class of Σ_P extends several types of strict-feedback forms [20, 5, 11] to general interconnected systems with uncertain static and dynamic systems which are nonlinearly bounded, and involving signals to define disturbance attenuation. The uncertain components, disturbances and regulated outputs are allowed to be situated at any locations in the system equations of the plant.

7 Recursive design

This section shows how to solve the two characteristic inequalities (31) and (33) simultaneously. Solutions are shown to exist for systems in the generalized robust strict-feedback form.

7.1 Recursive parameterization

Let $x_{[k]}$ denote $x_{[k]} = [x_1, x_2, \cdots, x_k]^T$. Consider parameterization of the nonsingular matrix $S(x)$, the partial-state feedback and the constant matrix P as follows:

$$S(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ s_1(x) & 1 & 0 & 0 & \cdots & 0 \\ d_{21} & s_2(x) & 1 & 0 & \ddots & 0 \\ d_{31} & d_{32} & s_3(x) & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{n-1,1} & \cdots & \cdots & d_{n-1,n-2} & s_{n-1}(x) & 1 \end{bmatrix}^{-1} \tag{50}$$

$$u = [\, d_{n,1} \ \cdots \ d_{n,n-1} \ \ s_n(x) \,]\, \chi \tag{51}$$

$$P = \mathrm{diag}[P_1, P_2, \cdots, P_n] \tag{52}$$

where $s_1(x_{[1]})$, $s_2(x_{[2]})$, $\cdots$, $s_n(x_{[n]})$ are smooth functions, and parameters d_{ij}, $2 \le i \le n$, $1 \le j \le i-1$, are real constants. The scalars P_i, $i = 1, \ldots, n$ are positive real constants. Candidates of state-dependent scaling matrices are also parameterized as follows:

$$\Phi = \operatorname*{block-diag}_{j=1}^{2n} \Phi_j, \quad \Phi_j = \begin{cases} \phi_j(x_{[(j+1)/2]})\hat{\Phi}_j & \text{for odd } j \\ \phi_j(x_{[j/2]})\hat{\Phi}_j & \text{for even } j \end{cases} \tag{53}$$

$$\Theta = \operatorname*{block-diag}_{j=1}^{2n} \Theta_j, \quad \Theta_j = \begin{cases} \theta_j(x_{[(j+1)/2]})\hat{\Phi}_j & \text{for odd } j \\ \theta_j(x_{[j/2]})\hat{\Phi}_j & \text{for even } j \end{cases} \tag{54}$$

$$\phi_j(x) > 0, \quad \theta_j(x) > 0, \quad \forall x \in \mathcal{R}^n$$

Scalar functions $\phi_j(x)$ and $\theta_j(x)$ have yet to be determined. The block partition of Φ and Θ is compatible in size with that of $\bar{z}$. Pick a SD scaling matrix $\Psi \in \mathbf{\Psi}$ so that (25) and

$$\Psi(x) = \operatorname*{block-diag}_{j=1}^{2n} \Psi_j(x) \tag{55}$$

$$\Psi_j(x) = \begin{cases} \Psi_j(x_{[(j+1)/2]}) & \text{for odd } j \\ \Psi_j(x_{[j/2]}) & \text{for even } j \end{cases} \tag{56}$$

$$\bar{\Psi}(0) = \Psi(0) \tag{57}$$

$$I - \Psi_{2i-1} D_i D_i^T \Psi_{2i-1} > 0, \ \forall i \in [1, n] \setminus \{q_{2i-1} = 0\} \tag{58}$$

$$a_{i,i+1}^2 > U_{Ri}^T \Psi_{2i}^2 \hat{\Phi}_{2i} U_{Ri} U_{Li} \hat{\Phi}_{2i}^{-1} U_{Li}^T, \ \forall i \in [1, n] \setminus \{p_{2i} + q_{2i} = 0\} \tag{59}$$

are fulfilled. Such a SD scaling matrix Ψ exists due to the structure of $[C,\, H]$ and D, and (46) and (48). A simple choice is $\bar{\Psi} = \Psi$. The matrix Ξ in (31) is given by

$$\Xi(x) = \begin{bmatrix} I_\kappa & 0 \\ 0 & \phi_d(\chi_{[\kappa]}^T P_{[\kappa]} \chi_{[\kappa]}) I_{n-\kappa} \end{bmatrix} P \tag{60}$$

with an integer $k \in [0, n]$ and a $\mathcal{C}^0$ function $\phi_d(\cdot)$ to be selected. Let $M_{[k]}(x_k)$ be defined as

$$M_{[k]} = \begin{bmatrix} \left\{ \begin{matrix} \hat{S}_{[k]}^T \hat{A}_{[k]}^T T_{[k]}^T \Xi_{[k]} + \\ \Xi_{[k]} T_{[k]} \hat{A}_{[k]} \hat{S}_{[k]} \end{matrix} \right\} & \Xi_{[k]} T_{[k]} B_{[k]} & \hat{S}_{[k]}^T \hat{C}_{[k]}^T \Psi_{[k]} \Phi_{[k]} \\ B_{[k]}^T T_{[k]}^T \Xi_{[k]} & -\Theta_{[k]} & D_{[k]}^T \Psi_{[k]} \Phi_{[k]} \\ \Phi_{[k]} \Psi_{[k]} \hat{C}_{[k]} \hat{S}_{[k]} & \Phi_{[k]} \Psi_{[k]} D_{[k]} & -\Phi_{[k]} \end{bmatrix}$$

for $k = 1, 2, \ldots, n$, where individual matrices are given by

$$\hat{A}_{[k]} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & \cdots & a_{k-1,k} & 0 \\ a_{k1} & a_{k2} & \cdots & \cdots & a_{kk} & a_{k,k+1} \end{bmatrix}$$

$$B_{[k]} = \begin{bmatrix} B_{11} & U_{L1} & 0 & 0 & \cdots & 0 & 0 \\ B_{21} & U_{21} & B_{22} & U_{L2} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ B_{k1} & U_{k1} & B_{k2} & U_{k2} & \cdots & B_{kk} & U_{Lk} \end{bmatrix}$$

$$\hat{C}_{[k]} = \begin{bmatrix} C_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & U_{R1} & 0 & \cdots & 0 & 0 \\ C_{21} & C_{22} & 0 & \ddots & 0 & 0 \\ 0 & 0 & U_{R2} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{k,1} & C_{k,2} & \cdots & \cdots & C_{k,k} & 0 \\ 0 & 0 & \cdots & \cdots & 0 & U_{R,k} \end{bmatrix}, \quad D_{[k]} = \begin{bmatrix} D_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & D_k & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{S}_{[k]} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ s_1 & 1 & 0 & \cdots & 0 \\ d_{21} & s_2 & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & s_{k-1} & 1 \\ d_{k,1} & \cdots & & d_{k,k-1} & s_k \end{bmatrix}$$

$$\Xi_{[k]} = \operatorname*{diag}_{j=1}^{k} \Xi_j, \quad \Phi_{[k]} = \operatorname*{block\text{-}diag}_{j=1}^{2k} \Phi_j \quad \Theta_{[k]} = \operatorname*{block\text{-}diag}_{j=1}^{2k} \Theta_j, \quad \Psi_{[k]} = \operatorname*{block\text{-}diag}_{j=1}^{2k} \Psi_j$$

In a similar manner, $T_{[k]}$ is the $k \times k$ upper left part of T. Then, the next properties follow immediately.

Proposition 1 *For $k = 1, 2 \ldots, n$, $M_{[k]}$ satisfies the following.*

(i) *$M_{[k]}$ is independent of $\{x_{k+1}, x_{k+2}, \cdots, x_n\}$.*

(ii) *$M_{[k]}$ does not include $\{s_{k+1}, \cdots, s_{n-1}, s_n\}$, $\{\phi_{2k+1}, \phi_{2k+2}, \cdots, \phi_{2n}\}$ and $\{\theta_{2k+1}, \theta_{2k+2}, \cdots, \theta_{2n}\}$.*

(iii) *$M_{[k]}(x_{[k]}) < 0$ implies $M_{[k-1]}(x_{[k-1]}) < 0$.*

(iv) *$M_{[n]}(x_{[n]}) = M(x)$*

The following lemma is verified from $S(0) = T(0)$, $\bar{\Psi}(0) = \Psi(0)$ and comparison between (31) and (33).

Proposition 2 *Suppose that $\underline{K}$ is the constant feedback gain of a scaled $\mathcal{H}^\infty$ linear controller given arbitrarily. Let $X > 0$ and $\underline{\Phi}$ be matrices satisfying (33). If the set of parameters $\{P_i$, s_i, d_{ik}, ϕ_j, θ_j, $\hat{\Phi}_j$, $\phi_d\}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, 2n$, $k = 1, 2, \ldots, i-1$, solving (31) fulfills*

$$X = S(0)^T P S(0) \tag{61}$$

$$\underline{K} = [\, d_{n,1} \; \cdots \; d_{n,n-1} \;\; s_n(0) \,]\, S(0) \tag{62}$$

$$1 = \phi_j(0) = \theta_j(0), \;\; j = 1, 2, \ldots, 2n \tag{63}$$

$$\underline{\Phi} = \operatorname*{block\text{-}diag}_{j=1}^{2n} \hat{\Phi}_j \tag{64}$$

$$1 = \phi_d(0) \tag{65}$$

then, the nonlinear controller Σ_K agrees locally with the scaled $\mathcal{H}^\infty$ linear controller.

Let $X > 0$ and $\underline{\Phi}$ be matrices satisfying (33). Using the Cholesky factorization[4], the matrix $X > 0$ is decomposed into

$$\begin{aligned} X &= L^{-T}\Lambda L^{-1} \\ \Lambda &= [\,\lambda_1 \ \lambda_2 \ \cdots \ \lambda_n\,] > 0 \\ L &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{11} & 1 & 0 & & \vdots \\ l_{21} & l_{22} & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ l_{n-1,1} & \cdots & l_{n-1,n-2} & l_{n-1,n-1} & 1 \end{bmatrix} \end{aligned}$$

Define scalars $\{l_{n,1}, \ldots, l_{n,n}\}$ by

$$\underline{K}L = [\,l_{n,1} \ \cdots \ l_{n,n-1} \ \ l_{n,n}\,]$$

On the basis of Proposition 1 and 2, a new recursive method of solving the SD scaling problem posed by (31-32) and (33) is now proposed as follows.

Recursive procedure with local matching: *Pick $d_{k,j}$, P_k, $\hat{\phi}_{2k-1,s}$, $\hat{\phi}_{2k-1,d}$ $\hat{\phi}_{2k,s}$ and $\hat{\phi}_{2k,d}$ as*

$$d_{k,j} = l_{k,j}, \ 1 \le j \le k-1 \tag{66}$$

$$P_k = \lambda_k \tag{67}$$

$$\hat{\phi}_{2k-1,s} = \underline{\phi}_{2k-1,s}, \ \hat{\phi}_{2k-1,d} = \underline{\phi}_{2k-1,d}, \ \hat{\phi}_{2k,s} = \underline{\phi}_{2k,s}, \ \hat{\phi}_{2k,d} = \underline{\phi}_{2k,d} \tag{68}$$

and solve $M_{[k]}(x_{[k]}) < 0$ for $\{s_k(x_{[k]}),\ \phi_{2k-1}(x_{[k]}),\ \phi_{2k}(x_{[k]}),\ \theta_{2k-1}(x_{[k]}),\ \theta_{2k}(x_{[k]})\}$ subject to

$$s_k(0) = l_{k,k} \tag{69}$$

$$\phi_{2k-1}(0) = \theta_{2k-1}(0) = 1 \tag{70}$$

$$\phi_{2k}(0) = \theta_{2k}(0) = 1 \tag{71}$$

from $k = 1$ through $k = n$.

Requirements of (61), (62) and (64) for local matching properties are implied by (69), (66), (67) and (68). Parameters $\{P_i,\ s_i,\ d_{ik},\ \phi_j,\ \theta_j,\ \hat{\Phi}_j\}$ are free from the local matching conditions (69), (66), (67) (70), (71) and (68) as long as we do not care about local properties of the controller.

Recursive procedure without local matching: *Solve $M_{[k]}(x_{[k]}) < 0$ for $\{s_k(x_{[k]}),\ \phi_{2k-1}(x_{[k]}),\ \phi_{2k}(x_{[k]}),\ \theta_{2k-1}(x_{[k]}),\ \theta_{2k}(x_{[k]})\}$ from $k = 1$ through $k = n$.*

The latter procedure does not force nonlinear controllers to agree with scaled $\mathcal{H}^\infty$ linear controllers.

7.2 Existence of solution

Consider a nonsingular matrix

$$Q_k = \begin{bmatrix} I_{k-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & I_{\rho_a} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\rho_b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\rho_c} & 0 \\ 0 & 0 & I_{\rho_a} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\rho_b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\rho_c} \end{bmatrix}$$

where I_k denotes a $k \times k$ identity matrix and, $\rho_a = \sum_{i=1}^{2(k-1)}(p_i + q_i)$, $\rho_b = p_{2k-1} + q_{2k-1}$ and $\rho_c = p_{2k} + q_{2k}$ are used. Then, we have

$$Q_k^T M_{[k]}(x_{[k]}) Q_k = \begin{bmatrix} M_{[k-1]}(x_{[k-1]}) & M_{[k-1]k}(x_{[k]}) \\ M_{[k-1]k}^T(x_{[k]}) & M_{kk}(x_{[k]}) \end{bmatrix} \tag{72}$$

The right hand side of (72) becomes $M_{11}(x_{[1]})$ in the case of $k = 1$. Let $J_k \in \mathcal{R}$, $E_k \in \mathcal{R}^{1\times 2(\rho_b+\rho_c)}$ and $F_k \in \mathcal{R}^{2(\rho_b+\rho_c)\times 2(\rho_b+\rho_c)}$ be defined with

$$M_{kk} - M_{[k-1]k}^T M_{[k-1]}^{-1} M_{[k-1]k} = \begin{bmatrix} J_k & E_k \\ E_k^T & F_k \end{bmatrix} \tag{73}$$

The left hand side of (73) is M_{11} in the case of $k = 1$. Since F_k is in the form of

$$F_k = \begin{bmatrix} F_{ak} & 0 \\ 0 & F_{bk} \end{bmatrix}$$

we arrive at the following properties by applying Schur complements formula to $M_{[k]} < 0$.

Proposition 3 *Assume that $M_{[k-1]}(x_{[k-1]}) < 0$ is satisfied for all $x_{[k-1]} \in \mathcal{R}^{k-1}$ unless $k = 1$. Then, $M_{[k]}(x_{[k]}) < 0$ holds for all $x_{[k]} \in \mathcal{R}^k$ if and only if the following is satisfied for all $x_{[k]} \in \mathcal{R}^k$.*

(i) $p_{2k-1} + q_{2k-1} \neq 0$ *&* $p_{2k} + q_{2k} \neq 0$ *Case:* $F_{ak} < 0, F_{bk} < 0, J_k - E_k F_k^{-1} E_k^T < 0$

(ii) $p_{2k-1} + q_{2k-1} \neq 0$ *&* $p_{2k} + q_{2k} = 0$ *Case:* $F_{ak} < 0, J_k - E_k F_k^{-1} E_k^T < 0$

(iii) $p_{2k-1} + q_{2k-1} = 0$ *&* $p_{2k} + q_{2k} \neq 0$ *Case:* $F_{bk} < 0, J_k - E_k F_k^{-1} E_k^T < 0$

(iv) $p_{2k-1} + q_{2k-1} = 0$ *&* $p_{2k} + q_{2k} = 0$ *Case:* $J_k < 0$

The following two lemmas establish the existence of decision variables solving the set of global inequalities and local equations posed in each step of the recursive procedure.

Lemma 3 *Assume $p_{2k-1} + q_{2k-1} \neq 0$ and that $M_{[k-1]}(x_{[k-1]}) < 0$ holds for all $x_{[k-1]} \in \mathcal{R}^{k-1}$ unless $k = 1$. Then, the following are true.*

(i) *There exists a $\mathcal{C}^0$ function $\nu_{2k-1}(x_{[k]})$ such that each $\mathcal{C}^0$ function $\phi_{2k-1}(x_{[k]})$ satisfying*

$$\nu_{2k-1}(x_{[k]}) \geq \phi_{2k-1}(x_{[k]}) > 0, \quad \forall x_{[k]} \in \mathcal{R}^k \tag{74}$$

admits the existence of a $\mathcal{C}^0$ function $\theta_{2k-1}(x_{[k]})$ achieving

$$\phi_{2k-1}(x_{[k]}) \geq \theta_{2k-1}(x_{[k]}) > 0, \quad F_{ak}(x_{[k]}) < 0, \quad \forall x_{[k]} \in \mathcal{R}^k \tag{75}$$

In the case of $k = 1$, any positive constant ν_{2k-1} fulfills the property.

(ii) *For any constant $\bar{\phi}_{2k-1} > 0$, there exists a $\mathcal{C}^0$ function $\bar{\nu}_{2k-1}(\chi_{[k]}^T P_{[k]} \chi_{[k]})$ such that each $\mathcal{C}^0$ function $\phi_{2k-1}(x_{[k]})$ satisfying*

$$\bar{\phi}_{2k-1} \geq \bar{\nu}_{2k-1}(\chi_{[k]}^T P_{[k]} \chi_{[k]}) \geq \phi_{2k-1}(x_{[k]}) > 0, \quad \forall x_{[k]} \in \mathcal{R}^k \tag{76}$$

admits the existence of a $\mathcal{C}^0$ function $\theta_{2k-1}(x_{[k]})$ achieving (75).

(iii) *The properties (i) and (ii) are true even if* $\phi_{2k-1} \geq \theta_{2k-1}$ *in (75) is replaced by* $\phi_{2k-1} = \theta_{2k-1}$.

(iv) *Suppose that* $\underline{M} < 0$ *and*

$$d_{i,j} = l_{i,j}, \quad 1 \leq j \leq i-1 \tag{77}$$

$$P_i = \lambda_i \tag{78}$$

$$\hat{\phi}_{2i-1,s} = \underline{\phi}_{2i-1,s}, \ \hat{\phi}_{2i-1,d} = \underline{\phi}_{2i-1,d}, \ \hat{\phi}_{2i,s} = \underline{\phi}_{2i,s}, \ \hat{\phi}_{2i,d} = \underline{\phi}_{2i,d} \tag{79}$$

hold for all $i = 1, 2, \ldots, k$, *and* ϕ_d *in (60) satisfies* $\phi_d(0) = 1$. *If*

$$s_i(0) = l_{i,i} \tag{80}$$

$$\phi_{2i-1}(0) = \theta_{2i-1}(0) = 1 \tag{81}$$

$$\phi_{2i}(0) = \theta_{2i}(0) = 1 \tag{82}$$

are satisfied for all $i = 1, 2, \ldots, k-1$, *there exist* $\mathcal{C}^0$ *functions* $\nu_{2k-1}(\cdot)$ *and* $\bar{\nu}_{2k-1}(\cdot)$ *which fulfill* $\nu_{2k-1}(0) = \bar{\nu}_{2k-1}(0) = 1$ *in addition to properties of (i), (ii) and (iii).*

Lemma 4 *Assume that* $M_{[k-1]}(x_{[k-1]}) < 0$ *holds for all* $x_{[k-1]} \in \mathcal{R}^{k-1}$ *unless* $k = 1$.

(i) $p_{2k} + q_{2k} \neq 0$ *Case: There exist a* $\mathcal{C}^0$ *function* $\nu_{2k}(x_{[k]})$ *such that each* $\mathcal{C}^0$ *function* $\phi_{2k}(x_{[k]})$ *satisfying*

$$\nu_{2k}(x_{[k]}) \geq \phi_{2k}(x_{[k]}) > 0, \quad x_{[k]} \in \mathcal{R}^k \tag{83}$$

admits the existence of a $\mathcal{C}^0$ *function* $\theta_{2k}(x_{[k]})$ *and a smooth function* $s_k(x_{[k]})$ *for which*

$$\phi_{2k}(x_{[k]}) \geq \theta_{2k}(x_{[k]}) > 0, \quad F_{bk}(x_{[k]}) < 0 \tag{84}$$

$$J_k(x_{[k]}) - E_k(x_{[k]})F_k^{-1}(x_{[k]})E_k^T(x_{[k]}) < 0 \tag{85}$$

hold for all $x_{[k]} \in \mathcal{R}^k$. *Furthermore, there exist a* $\mathcal{C}^0$ *function* $\bar{\nu}_{2k}(\chi_{[k]}^T P_{[k]} \chi_{[k]})$ *and a finite constant* $\bar{\phi}_{2k}$ *such that each* $\mathcal{C}^0$ *function* $\phi_{2k}(x_{[k]})$ *satisfying*

$$\bar{\phi}_{2k} \geq \bar{\nu}_{2k}(\chi_{[k]}^T P_{[k]} \chi_{[k]}) \geq \phi_{2k}(x_{[k]}) > 0, \quad x_{[k]} \in \mathcal{R}^k \tag{86}$$

admits the existence of a $\mathcal{C}^0$ *function* $\theta_{2k}(x_{[k]})$ *and a smooth function* $s_k(x_{[k]})$ *for which (84-85) hold for all* $x_{[k]} \in \mathcal{R}^k$. *In addition,* $\phi_{2k} \geq \theta_{2k}$ *in (84) can be replaced by* $\phi_{2k} = \theta_{2k}$.

(ii) $p_{2k} + q_{2k} = 0$ *Case: There exists a smooth function* $s_k(x_{[k]})$ *such that*

$$J_k(x_{[k]}) - E_k(x_{[k]})F_k^{-1}(x_{[k]})E_k^T(x_{[k]}) < 0 \quad \text{if } p_{2k-1} + q_{2k-1} \neq 0 \tag{87}$$

$$J_k(x_{[k]}) < 0 \quad \text{if } p_{2k-1} + q_{2k-1} = 0 \tag{88}$$

is satisfied for all $x_{[k]} \in \mathcal{R}^k$.

(iii) *Suppose* $\underline{M} < 0$ *holds and (77), (78), (79) and (81) hold for all* $i = 1, 2, \ldots, k$, *and* ϕ_d *in (60) satisfies* $\phi_d(0) = 1$. *If (80) and (82) are satisfied for all* $i = 1, 2, \ldots, k-1$, *there exist* $\mathcal{C}^0$ *functions* $\nu_{2k}(\cdot)$, $\bar{\nu}_{2k}(\cdot)$ *and a smooth function* $s_k(\cdot)$ *which fulfill* $\nu_{2k}(0) = \bar{\nu}_{2k}(0) = 1$ *and* $s_k(0) = l_{k,k}$ *in addition to properties of (i) or (ii).*

We are now in position to state the main results of this section. First, the next lemma guarantees that local controllers exist for generalized robust strict-feedback systems.

Lemma 5 *If the system Σ_P is in the generalized robust strict-feedback form with $r = r_j$ for an integer $j \in [1, 2n]$ or $q = 0$, then there exist scaled $\mathcal{H}^\infty$ linear controllers.*

The existence of solutions to the partial-state feedback nonlinear design problem is addressed by the following theorem. The constructive proof demonstrates how to obtain a desired solution.

Theorem 2 *Suppose that the system Σ_P is in the generalized robust strict-feedback form and satisfies either:*

(i) *dynamic components are not involved in Σ_Δ, and there are no exogenous disturbances(i.e., $q = 0$).*

(ii) *dynamic components in Σ_Δ are situated only at Δ_j, and the exogenous disturbance is $r = r_j$ for an integer $j \in [1, 2n]$.*

(iii) *dynamic components in Σ_Δ are situated only at $\{\Delta_i : i = 1, 2, \ldots, 2\rho\}$, the exogenous disturbance is $r = r_j$, and $\hat{A}_{[\rho]}$, $B_{[\rho]}$, $\hat{C}_{[\rho]}$ and $D_{[\rho]}$ are independent of x for integers $\rho \in [1, n]$ and $j \in [1, 2\rho]$.*

Then,

(a) *the system Σ_P can be globally uniformly asymptotically stabilized, and the $\mathcal{L}_2$-gain from r to e can be rendered less than or equal to τ by the smooth partial-state feedback control law (51).*

Furthermore,

(b) *the control law (51) can be made to agree localy with any scaled $\mathcal{H}^\infty$ linear controller.*

Desired solutions $\{s_k(x_{[k]}), \phi_{2k-1}(x_{[k]}), \phi_{2k}(x_{[k]}), \theta_{2k-1}(x_{[k]}), \theta_{2k}(x_{[k]})\}$ can be constructed sequentially from $k = 1$ through $k = n$. In each step of the recursive procedure given in Subsection 7.1, existence of the solutions is guaranteed by Lemma 3 and Lemma 4, and proofs of those lemmas have described a way to obtain the solutions. Computation proceeds in the k-th step as follows:

(1) If $p_{2k-1} + q_{2k-1} \neq 0$, solve

$$\nu_{2k-1} > 0, \quad -\nu_{2k-1}\lambda_{min}\left(F_{dk}^{-1}\hat{\Phi}_{2k-1}^{1/2}F_{ck}\hat{\Phi}_{2k-1}^{1/2}F_{dk}^{-1}\right) < 1 \tag{89}$$

for $\nu_{2k-1}(x_{[k]})$. If $2k-1 = j$, calculate $\bar{\nu}_{2k-1}(\chi_{[k]}^T P_{[k]}\chi_{[k]})$ satisfying $0 < \bar{\nu}_{2k-1} \leq \min\{\nu_{2k-1}, \bar{\phi}_{2k-1}\}$ for some finite constant $\bar{\phi}_{2k-1} > 0$.

(2) If $p_{2k} + q_{2k} \neq 0$, solve

$$\nu_{2k} > 0, \quad \alpha U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk}\nu_{2k} < a_{k,k+1}^2 - U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} U_{Lk} \hat{\Phi}_{2k}^{-1} U_{Lk}^T \tag{90}$$

for $\nu_{2k}(x_{[k]})$. If $2k = j$, calculate $\bar{\nu}_{2k}(\chi_{[k]}^T P_{[k]}\chi_{[k]})$ satisfying $0 < \bar{\nu}_{2k} \leq \min\{\nu_{2k}, \bar{\phi}_{2k}\}$ for some finite constant $\bar{\phi}_{2k} > 0$.

(3) Solve

$$U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} \phi_{2k} s_k^2 + 2\Xi_k a_{k,k+1} s_k + U_{Lk} \hat{\Phi}_{2k}^{-1} U_{Lk}^T \Xi_k^2 \theta_{2k}^{-1} + \Xi_k^2 \alpha < 0 \quad \text{when } p_{2k} + q_{2k} \neq 0 \tag{91}$$

$$2\Xi_k a_{k,k+1} s_k + \Xi_k^2 \alpha < 0 \quad \text{when } p_{2k} + q_{2k} = 0 \tag{92}$$

for $s_k(x_{[k]})$.

Here, $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue of a matrix, and $F_{dk}(x_{[k]})$ is defined as

$$F_{dk}^2 = \begin{bmatrix} I & 0 \\ 0 & I - \tau^{-2} D_{R,k} D_{R,k}^T \end{bmatrix}$$

The functions $F_{ck}(x_{[k]})$ and $\alpha(x_{[k]})$ are given in the proof of Lemma 3 and Lemma 4, respectively. A simpler choice of $\{\theta_*, \phi_*\}$ is $\nu_* = \theta_* = \phi_*$. In the cases of $2k-1 = j$ and $2k = j$, it is replaced with $\bar{\nu}_* = \theta_* = \phi_*$. In the case of (iii) of Theorem 2, parameters $\{\nu_{2k-1}, \nu_{2k}, \phi_{2k-1}, \phi_{2k}, \theta_{2k-1}, \theta_{2k}\}$ are chosen as constants for all $k = 1, 2, \ldots, \rho$. The parameters κ and ϕ_d in (60) are selected as follows:

Case (i) $\kappa = 0$; $\phi_d = 1$.

Case (ii) $\kappa = j/2$ for even j, and $\kappa = (j+1)/2$ for odd j; $\phi_d(\chi_{[\kappa]}^T P_{[\kappa]} \chi_{[\kappa]}) = \phi_j$.

Case (iii) $\kappa = 0$; $\phi_d = \phi_j$.

Then, $\Phi \in \boldsymbol{\Phi}$ and $\Theta \in \boldsymbol{\Theta}$ are fulfilled. Accoding to Proposition 1 and Proposition 3, the parameters obtained achieve $0 < \Theta(x) \leq \Phi(x)$ and $M(x) < 0$ for all $x \in \mathcal{R}^n$. Thus, Theorem 1 proves that the control law (51) globally uniformly asymptotically stabilizes Σ_P and achieves $\mathcal{L}_2$-gain less than or equal to τ. This completes the proof of the part (a). To achieve the property of the part (b), pick $d_{k,*}$, P_k, $\hat{\phi}_{2k-1,s}$, $\hat{\phi}_{2k-1,d}$ $\hat{\phi}_{2k,s}$ and $\hat{\phi}_{2k,d}$ as (66), (67) and (68) from $k = 1$ and $k = n$. Then, Lemma 3 and 4 imply that parameters $\{s_k, \phi_{2k-1}, \phi_{2k}, \theta_{2k-1}, \theta_{2k}\}$ of each k-th step can be constructed so that the additional conditions of local matching (69), (70) and (71) are satisfied. According to Proposition 2, the nonlinear control law (51) with such parameters agrees with a given scaled $\mathcal{H}^\infty$ linear controller. Thereby, the proof of Theorem 2 has been completed.

All scalar inequalities (89), (90) and (92) are affine in decision variables. The inequality (91) which is quadratic in s_k also reduces to a pair of affine inequalities since (90) guarantees it. All inequalities are functions of $x_{[k]}$. Coefficients appearing in the inequalities are obtained easily both analytically and numerically by application of (73) to appropriate minors of M. Solutions to the affine inequalities are also calculated easily both analytically and numerically. It should be noted that instead of the set of inequalities (89), (90), (92) and (91), we can solve an equivalent single matrix inequality $M_{[k]} < 0$ directly for decision variables. The computation is amenable to numerical calculation since $M_{[k]}$ is jointly affine in $\{s_k, \phi_{2k-1}, \theta_{2k-1}\}$, and jointly affine in $\{\phi_{2k-1}, \phi_{2k}, \theta_{2k-1}, \theta_{2k}\}$.

Although Theorem 2 addresses $\mathcal{L}_2$-gain disturbance attenuation of a fixed level τ , the partial-state feedback law can be designed for arbitrarily small τ unless (46) and (48) are violated. Thus, the almost $\mathcal{L}_2$ disturbance decoupling with the local matching can be solved by the smooth partial-state feedback (51).

The new class of SD scaling (13) employed in this paper enables us to solve the global stabilization in the presence of dynamic components in Σ_Δ and $\mathcal{L}_2$-gain disturbance attenuation. Indeed, constant scaling employed in [11] could only guarantee solutions to the global stabilization problem only for $j = 1$ in (ii) of Theorem 2.

When Σ_0 is linear, no restriction of locations of dynamic uncertain components is necessary, according to Theorem 2(iii). Indeed, linear systems in the generalized robust strict-feedback form are stabilizable for all admissible dynamic nonlinear uncertainties by a linear control law. However, in the case of nonlinear Σ_0, Theorem 2 does not allow dynamic components to be involved in all Δ_i, $i = 1, 2, \ldots, 2n$ simultaneously. A benefit from this restriction is that the designed controller achieves robust $\mathcal{L}_2$ disturbance attenuation globally in addition to robust asymptotic stability. The disturbance r can enter the system Σ_0 at any single location j as $r = r_j$. Furthermore, the nonlinear control can be made identical to robust $\mathcal{H}^\infty$ linear control, thanks to the concept of SD scaling design. These are unique features of this paper.

When dynamic components are allowed to be included in all Δ_i, $i = 1, 2, \ldots, 2n$ simultaneously, it is possible to obtain a nonlinear stability margin. The disturbance attenuation is also achievable if one admits nonlinear weighting in the level of attenuation, similar to [14], instead of requiring the standard $\mathcal{L}_2$-gain.

Theorem 3 *Suppose that the pair of Σ_0 and Σ_Δ defines an uncertain system Σ_P in the generalized robust strict-feedback form. Then, there exist uniformly bounded $\mathcal{C}^0$ diagonal matrices $\bar{W}(x)$ and $\hat{W}(x)$ satisfying*

$$\bar{W}(0) = I, \quad \bar{W}(x) > 0, \quad \hat{W}(x) > 0, \quad \forall x \in \mathcal{R}^n$$

such that the system $\hat{\Sigma}_P$ shown in Fig.3 can be globally uniformly asymptotically stabilized, and the $\mathcal{L}_2$-gain from $\hat{r}$ to $\hat{e}$ can be rendered less than or equal to τ by the smooth partial-state feedback control law (51) for all diagonal matrices $W(x)$ satisfying $0 < W(x) \leq \bar{W}(x)$ for all $x \in \mathcal{R}^n$. Furthermore, if there exist a scaled $\mathcal{H}^\infty$ linear controller associated with the original system Σ_P consisting of Σ_0 and Σ_Δ, the control law (51) can be made to agree locally with the linear controller, and $\hat{W}(x)$ satisfies $\hat{W}(0) = I$.

Since $\hat{W}(x) \leq \alpha$ holds with a finite constant α for all x, it is possible to replace $\hat{W}(x)$ and $\hat{W}^{-1}(x)$ in Fig.3 with $\alpha\hat{W}(x)$ and I, respectively.

7.3 An example

This subsection presents an example briefly just to illustrate an achievement in Subsection 7.2. Consider the system Σ_0 given by

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^2 w_1 \\
\dot{x}_2 &= u + x_1 w_3 + r_3 \\
z_1 = x_1 \ , \quad z_3 &= x_2 \ , \quad e_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

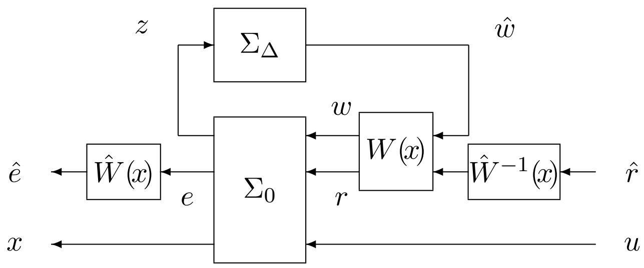

Figure 3: Nonlinearly weighted plant $\hat{\Sigma}_P$

and the uncertain system Σ_Δ between z_* and w_* in the form of

$$
\begin{aligned}
&w_1 = h_{\Delta_{1s}}(z_1, t) \\
&\dot{x}_{\Delta_3} = f_{\Delta_{3d}}(x_{\Delta_3}, z_3, t), \quad w_3 = h_{\Delta_{3d}}(x_{\Delta_3}, t)
\end{aligned}
$$

These uncertain components are supposed to be admissible in the sense of Assumption 1 with

$$
\psi_{1s} = 1, \quad \psi_{3d} = 2.2 z_3^2
$$

Note that the dynamic system between z_3 and w_3 has zero $\mathcal{L}_2$-gain locally although it is globally bounded only in nonlinear gain. The objective is to find a partial-state feedback controller which globally uniformly asymptotically stabilizes Σ_P shown in Fig.2 and achieves the level $\tau = 0.5$ of disturbance attenuation between r_3 and e_3. A feedback gain of scaled $\mathcal{H}^\infty$ linear controllers associated with this design problem is computed as

$$
\underline{K} = [\, -14.980 \;\; -11.411 \,] \tag{93}
$$

which solves (33) with

$$
\begin{aligned}
X &= \begin{bmatrix} 21.995 & 9.806 \\ 9.806 & 8.082 \end{bmatrix} \\
\underline{\phi} &= \begin{bmatrix} \underline{\phi}_{1s} & 0 & 0 & 0 \\ 0 & \underline{\phi}_{3d} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.7764 & 0 & 0 & 0 \\ 0 & 0.7810 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

One of simple solutions we can obtain using formulas of the recursive procedure in Subsection 7.2 is

$$
u = [\, -1.1352 \;\; s_2 \,]\, \chi = K(x) x \tag{94}
$$

where calculated parameters are

$$
\begin{aligned}
&P = \begin{bmatrix} 10.097 & 0 \\ 0 & 8.0821 \end{bmatrix} \\
&\phi_1 = \theta_1 = x_1^2 + 1, \quad s_1 = -7x_1^2 - 1.2133 \\
&\phi_3 = \theta_3 = \frac{1}{(\chi^T P \chi)^2 + 1}, \quad s_2 = -4(\chi^T P \chi + 0.4)^3 - 0.4^3) - 11.411
\end{aligned}
$$

This nonlinear controller satisfies $K(0) = \underline{K}$. One of admissible uncertain components $z_3 \mapsto w_3$ is

$$\begin{cases} \dot{x}_{\Delta_3} = -x_{\Delta_3}(1 - z_3^4) \\ w_3 = \mathrm{sat}(x_{\Delta_3}) \end{cases} \tag{95}$$

It satisfies (7) for $\psi_{3d} = 2.2 z_3^2$ and

$$W_{\Delta_3} = \int_0^{x_{\Delta_3}^2} \frac{1.1}{s} ds, \quad \beta_3 = \frac{x_{\Delta_3}^2}{5(x_{\Delta_3}^2 + 1)}$$

Note that the system (95) is not input-to-state stable although it is globally asymptotically stable when $z_3 \equiv 0$. Figure 4 shows state transition of Σ_0 in the presence of (95), $w_1 = 0.8 z_1$ and the disturbance

$$r_3(t) = \begin{cases} 1 \ , \ 1 \leq t < 2 \\ 0 \ , \ \text{otherwise} \end{cases}$$

for the initial condition $x(0) = [1.1, 1.2]^T$ and $x_{\Delta_3}(0) = 1$. The solid lines are x_1 and x_2 in the case of the nonlinear control (94). The dashed lines are of the linear control (93). The state response of the linear control approaches infinity before the disturbance comes in the system. The nonlinear case converges to zero and the effect of the disturbance is attenuated substantially. Figure 5 shows phase trajectories around the origin for $r_3 \equiv 0$. The behavior of the nonlinear control(solid lines) near the equilibrium is almost the same as that of the linear control(dashed lines).

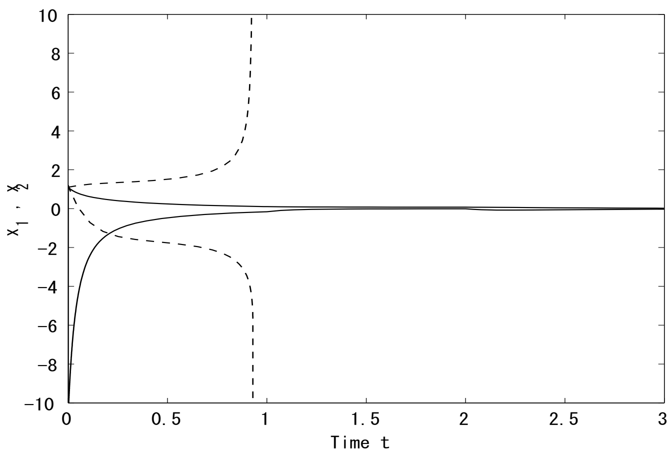

Figure 4: State transition of feedback control systems.

8 Further results

8.1 Some useful modifications

The strict inequalities (22) and (31) require Σ_P to be locally exponentially stabilizable. Systems which are only asymptotically stabilizable can be dealt with in the same manner by replacing the strict inequalities with non-strict ones. For instance, Theorem 1 can be modified as follows.

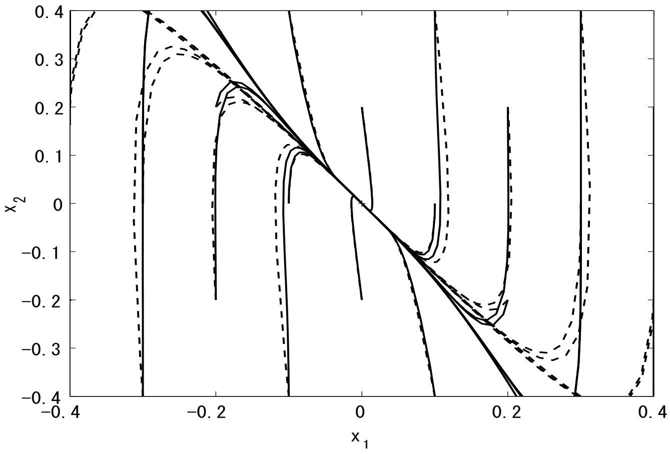

Figure 5: Phase portraits of feedback control systems.

Corollary 1 *Suppose that there exist* $\kappa \in [0, n]$, $P_{[\kappa]} \in \mathcal{R}^{\kappa\times\kappa}$, $P_{\langle\kappa+1\rangle} \in \mathcal{R}^{(n-\kappa)\times(n-\kappa)}$, $\Phi \in \mathbf{\Phi}$, $\Theta \in \mathbf{\Theta}$, $\Psi \in \mathbf{\Psi}$ *and a positive definite function* $v(x)$ *such that*

$$\begin{bmatrix} S^{-T}A^TT^T\Xi+\Xi TAS^{-1}\Xi+\Upsilon & \Xi TB & S^{-T}C^T\Psi\Phi \\ B^TT^T\Xi & -\Theta & D^T\Psi\Phi \\ \Phi\Psi CS^{-1} & \Phi\Psi D & -\Phi \end{bmatrix} \leq 0 \tag{96}$$

and (23), (24), (25) are satisfied for all $x \in \mathcal{R}^n$ *and a* $\mathcal{C}^0$ *function* $\Upsilon(x)$ *with* $v(x) = \chi^T\Upsilon\chi$ *and (26) and (27). Then, the system* Σ *is globally uniformly asymptotically stable, and it has* $\mathcal{L}_2$*-gain less than or equal to* τ.

The existence of parameters in the above corollary is mathematically equivalent to the existence of parameters which satisfies the strict inequality version of (96) for almost all $x \in \mathcal{R}^n$. This fact may be useful in numerical computation. Note that there are no scaled $\mathcal{H}^\infty$ linear controllers for systems which are not locally exponentially stabilizable. In such a case, as a matter of course, a nonlinear controller obtained in Section 7 is not able to agree locally with any linear controller.

Regarding the inequalities (47) and (48) in the definition of generalized robust strict-feedback form, they are often unnecessary assumptions. The condition (47) can be removed when $M_{[i]}(x_{[i]}) < 0$ (or $M_{[i]}(x_{[i]}) \leq 0$ for Corollary 1) holds with $s_i = 0$ at all points of $x_{[i]}$ where (47) is violated. The condition (48) can be also removed when $M_{[i]}(x_{[i]}) < 0$ (or ≤ 0) holds for some $\phi_{2i}(x_{[i]})$ and $\theta_{2i}(x_{[i]})$ with $s_i = 0$ at $x_{[i]}$ where (48) are violated. The existence of $\phi_{2i-1}(x_{[i]})$ and $\theta_{2i-1}(x_{[i]})$ is independent of (47) and (48).

8.2 Another class of scaling

This subsection presents another control Lyapunov function based on another set of scaling functions. That new class of scaling allows dynamic uncertain components to enter all channels $\bar{w}_i$ simultaneously under some assumption. Let the definition of $\mathbf{\Phi}_{id}$ in (13) be replaced by the following.

$$\tilde{\mathbf{\Phi}}_{id} = \left\{ \tilde{\Phi}_{id}(x) = \begin{bmatrix} \check{\phi}_{id}I & 0 \\ 0 & I \end{bmatrix} \prod_{j=1}^{k} \phi_j(\eta_j(x)) : \begin{array}{ll} k = (i+1)/2 & \text{for odd } i \\ k = i/2 & \text{for even } i \end{array} \right.$$

$$\left.\begin{array}{l}\phi_j(\cdot) \in \mathcal{C}^0, \quad \phi_j(s) > 0, \ \forall s \in [0,\infty), \quad \check{\phi}_{id} > 0 \\ \phi_j(s) \geq \phi_j(s+\epsilon), \ \forall \epsilon > 0\end{array}\right\} \tag{97}$$

The $\mathcal{C}^0$ function $\eta_j(x) : \mathcal{R}^n \to [0,\infty)$ will be determined later. Functions ϕ_j and η_j are common among all $\tilde{\Phi}_{id}$, $i = 1, 2, \ldots, n$. Define the set $\tilde{\mathbf{\Phi}}$ as (18) by replacing $\mathbf{\Phi}_{id}$ by $\tilde{\mathbf{\Phi}}_{id}$.

Theorem 4 *Suppose that the uncertain system Σ_P is in the generalized robust strict-feedback form. Let the state-feedback control and the matrix $S(x)$ are in the form of (50) and (51). If there exist a diagonal matrix P and diagonal matrices of SD scaling $\tilde{\Phi} \in \tilde{\mathbf{\Phi}}$, $\Theta \in \mathbf{\Theta}$, $\Psi \in \mathbf{\Psi}$ such that*

$$M = \begin{bmatrix} \hat{S}^T A^T T^T \Xi + \Xi T A \hat{S} & \Xi T B & \hat{S}^T C^T \Psi \tilde{\Phi} \\ B^T T^T \Xi & -\Theta & D^T \Psi \tilde{\Phi} \\ \tilde{\Phi} \Psi C \hat{S} & \tilde{\Phi} \Psi D & -\tilde{\Phi} \end{bmatrix} < 0 \tag{98}$$

$$P = \operatorname*{block-diag}_{i=1}^{n} P_i > 0 \tag{99}$$

$$\Theta \leq \tilde{\Phi} \tag{100}$$

$$\bar{\Psi} \leq \Psi \tag{101}$$

are satisfied for all $x \in \mathcal{R}^n$ with

$$\eta_j(x) = \hat{V}_{[j-1]}(\chi_{[j-1]}) \tag{102}$$

$$\Xi(x) = \operatorname*{diag}_{i=1}^{n} \xi_i(x), \quad \xi_i(x) = P_i \prod_{j=1}^{i} \phi_j(\eta_j(x)) \tag{103}$$

$$\hat{V}_{[0]} = 0 \tag{104}$$

$$\hat{V}_{[k]}(\chi_{[k]}) = \int_0^{\hat{V}_{[k-1]}(\chi_{[k-1]})} \frac{1}{\phi_k(s)} ds + \chi_k P_k \chi_k, \ k = 1, 2, \ldots, n \tag{105}$$

Then, the system Σ is globally uniformly asymptotically stable, and the $\mathcal{L}_2$-gain between $[r_{2n-1}^T, r_{2n}^T]^T$ and $[e_{2n-1}^T, e_{2n}^T]^T$ is less than or equal to τ. Furthermore, there exist a $\mathcal{C}^0$ diagonal matrix $W(x, x_\Delta) > 0$ for which the $\mathcal{L}_2$-gain between the disturbance r and the weighted output $W(x, x_\Delta)e$ is less than or equal to τ.

This theorem is based on a Lyapunov function $V(x) = V_{[n]}(t, \chi_{[n]}, x_{\Delta[n]})$, where

$$V_{[0]} = 0 \tag{106}$$

$$\begin{aligned} &V_{[k]}(t, \chi_{[k]}, x_{\Delta[k]}) = \\ &\int_0^{V_{[k-1]}(\chi_{[k-1]}, x_{\Delta[k-1]})} \frac{1}{\phi_k(s)} ds + \chi_k P_k \chi_k + \check{\phi}_{2k-1,d} W_{\Delta 2k-1}(t, x_{\Delta 2k-1}) + \check{\phi}_{2k,d} W_{\Delta 2k}(t, x_{\Delta 2k}) \end{aligned} \tag{107}$$

In contrast with Theorem 1, it is emphasized that, Theorem 4 is only applicable to the system Σ_P in the generalized robust strict-feedback form. We again apply the recursive procedure in Subsection 7.1 to the new matrix M defined in (98). It is verified that F_k defined as in (73) is independent of ξ_i for (98). Therefore, the following is obtained straightforwardly from Lemma 3 and Lemma 4.

Theorem 5 *Suppose that the system Σ_P is in the generalized robust strict-feedback form and it does not have dynamic uncertain components at virtual control inputs, i.e. $p_i = 0$, $i = 2, 4, \ldots, 2n$. Assume that all components of the matrix $\bar{\Psi}_{[k]}(x_{[k]}) C_{[k]}(x_{[k]})$ are uniformly bounded in x_k for each $k = 2, 3, \ldots, n$. Then, the system Σ_P can be globally uniformly asymptotically stabilized, and*

(i) $r = r_{2n-1}$ *case : the* $\mathcal{L}_2$*-gain from* r *to* e *can be rendered less than or equal to* τ *by the smooth partial-state feedback control law (51).*

(ii) $r = [r_1^T, r_3^T, \cdots, r_{2n-1}]^T$ *case : there exist a* $\mathcal{C}^0$ *diagonal matrix* $W(x, x_\Delta) > 0$ *for which the* $\mathcal{L}_2$*-gain between the disturbance* r *and the weighted output* $W(x, x_\Delta)e$ *can be rendered less than or equal to* τ *by the smooth partial-state feedback control law (51).*

Furthermore, the control law (51) can be made to agree locally with any scaled $\mathcal{H}^\infty$ *linear controller.*

9 Conclusion

The state-dependent scaling approach proposed in this paper enables us to deal with a broader class of nonlinear systems involving uncertain nonlinearities, unmodeled dynamics and linearly-unbounded systems, and standard $\mathcal{L}_2$-gain and nonlinear $\mathcal{L}_2$-gain to characterize the disturbance rejection level. It should be emphasized that a single design procedure provides us with solutions in such various situations. The state-dependent scaling is not only the first avenue to this accomplishment, but also naturally extends popular linear control frameworks such as scaled $\mathcal{H}^\infty$ control, LPV control and gain scheduling[2, 6] to global control of nonlinear systems with significant nonlinearities and nonlinear uncertainties. Thanks to the extension, nonlinear controller constructed in this paper can be always made identical to such linear controllers at the equilibrium. It is also worth mentioning that the SD scaling characterization does not require systems to fit in some geometric structures. The SD scaling approach has unified treatment of static and dynamic uncertainties, and provides us with a unique way of constructing new Lyapunov functions directly securing robustness against static and dynamic uncertainties.

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A Appendix

Proof of Lemma 1:
(i) From $\alpha_i(\|w_{is}\|) \le \sigma_i(\|z_{is}\|)$ it follows that

$$\|w_{is}\|^2 \le (\alpha_i^{-1} \circ \sigma_i(\|z_{is}\|))^2 = f(\|z_{is}\|), \quad t \in [0, \infty)$$

The assumption (9) implies that the class $\mathcal{K}_\infty$ function $f(\cdot)$ can be decomposed as $f(\|z_{is}\|) = \psi_{is}(\|z_{is}\|)\|z_{is}\|^2$ with a $\mathcal{C}^0$ non-negative function ψ_{is}.
(ii) Consider a non-decreasing continuous function $q : [0, \infty) \to [0, \infty)$ satisfying $q(s) > 0$ for all $s > 0$. Define a $\mathcal{C}^1$ function by

$$W_{\Delta_i}(t, x_{\Delta_i}) = \int_0^{V_{\Delta_i}(t, x_{\Delta_i})} q(s) ds$$

Here, W_{Δ_i} satisfies (6). From (11) we obtain,

$$\frac{dW_{\Delta_i}}{dt} \le q(V_{\Delta_i}(t, x_{\Delta_i}))\left\{-\alpha_i(\|x_{\Delta_i}\|) + \sigma_i(\|z_{id}\|)\right\}$$

Let π_i be any scalar satisfying $\pi_i > 1$. If $\alpha_i(\|x_{\Delta_i}\|) \ge \pi_i \sigma_i(\|z_{id}\|)$ holds, we have

$$\frac{dW_{\Delta_i}}{dt} \le -\left(1 - \frac{1}{\pi_i}\right) q(V_{\Delta_i}(t, x_{\Delta_i}))\alpha_i(\|x_{\Delta_i}\|) \tag{108}$$

In the case of $\alpha_i(\|x_{\Delta_i}\|) \leq \pi_i\sigma_i(\|z_{id}\|)$,

$$\frac{dW_{\Delta_i}}{dt} \leq -q(V_{\Delta_i}(t,x_{\Delta_i}))\alpha_i(\|x_{\Delta_i}\|) + q(\theta(\|z_{id}\|))\sigma_i(\|z_{id}\|) \tag{109}$$

is satisfied, where $\theta(\cdot)$ is

$$\theta(\|z_{id}\|) = \bar{\alpha}_i \circ \alpha_i^{-1} \circ \pi_i\sigma_i(\|z_{id}\|)$$

Here, note that $\bar{\alpha}_i(\|x_{\Delta_i}\|) \leq \theta(\|z_{id}\|)$. Combining (108) and (109) together, we have

$$\frac{dW_{\Delta_i}}{dt} \leq -\left(1-\frac{1}{\pi_i}\right) q(V_{\Delta_i}(t,x_{\Delta_i}))\alpha_i(\|x_{\Delta_i}\|) + q(\theta(\|z_{id}\|))\sigma_i(\|z_{id}\|)$$

The first condition in (12) implies that $\|w_{id}\|^2/\alpha_i(\|x_{\Delta_i}\|)$ is bounded for all $x_{\Delta_i} \in \mathcal{R}^{n_{\Delta_i}}$ uniformly in t. Thus, the non-decreasing continuous function $q(x)$ can be chosen such that

$$q(\underline{\alpha}_i(\|x_{\Delta_i}\|)) \geq \frac{\tau_i\|w_{id}\|^2}{\alpha_i(\|x_{\Delta_i}\|)} \geq 0, \quad \forall (t,x_{\Delta_i}) \in \mathcal{R}\times\mathcal{R}^{n_{\Delta_i}}$$

for some τ_i satisfying $\tau_i > \pi_i/(\pi_i - 1)$. Using (10) and the non-decreasing property of q, we obtain

$$\frac{dW_{\Delta_i}}{dt} \leq -\left(1-\frac{1}{\pi_i}-\frac{1}{\tau_i}\right)\frac{1}{4}q(\underline{\alpha}_i(\|x_{\Delta_i}\|))\alpha_i(\|x_{\Delta_i}\|) - \|w_{id}\|^2 + q(\theta(\|z_{id}\|))\sigma_i(\|z_{id}\|) \tag{110}$$

The second condition in (12) guarantees existence of a $\mathcal{C}^0$ function $\psi_{id} : [0,\infty) \to [0,\infty)$ satisfying

$$q(\theta(\|z_{id}\|))\sigma_i(\|z_{id}\|) \leq \psi_{id}(\|z_{id}\|)\|z_{id}\|^2$$

Finally, substituting this inequality into (110), we arrive at (7) with

$$\beta_i(x_{\Delta_i}) = \left(1-\frac{1}{\pi_i}-\frac{1}{\tau_i}\right)\left[q\circ\underline{\alpha}_i(\|x_{\Delta_i}\|)\right]\alpha_i(\|x_{\Delta_i}\|)$$

which is a class $\mathcal{K}_\infty$ function of $\|x_{\Delta_i}\|$.

Proof of Lemma 2:
By definition, the function $\zeta(x)$ is $\mathcal{C}^1$ and satisfies $\zeta(x) \geq 0$. It is zero only if $\eta(x) = 0$. Positive definiteness of $\eta(x)$ implies that $\zeta(x)$ is also positive definite. Since $\mu(\cdot)$ is a class $\mathcal{K}$ function, there exist finite numbers $k, \alpha > 0$ such that $\mu(s) \geq ks/(s+1)$ holds for all $s \geq \alpha$. Thus, we obtain

$$\zeta(x) \geq \int_0^\alpha \frac{1}{\phi(s)}ds + \int_\alpha^{\eta(x)} \frac{k}{s+1}ds = \int_0^\alpha \frac{1}{\phi(s)}ds + k\log\left(\frac{\eta(x)+1}{\alpha+1}\right)$$

Since $\eta(x)$ is radially unbounded, so is $\zeta(x)$.

Proof of Theorem 1:
Suppose that (22-27) are satisfied for all $x \in \mathcal{R}^n$. Let $V_{0[\kappa]}(\chi_{[\kappa]}) = V_0(x)$ which is positive definite radially unbounded $\mathcal{C}^1$ function of $\chi_{[\kappa]}$. Define a function $V(t,x_{cl})$ as (28). Since (21) defines a diffeomorphism from $x \in \mathcal{R}^n$ to $\chi \in \mathcal{R}^n$, Lemma 2 assures that $V(\cdot,\cdot)$ is a $\mathcal{C}^1$ function, and there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$ and $\bar{\alpha}_{cl}$ such that

$$\underline{\alpha}_{cl}(\|x_{cl}\|) \leq V(t,x_{cl}) \leq \bar{\alpha}_{cl}(\|x_{cl}\|)$$

The time-derivative of V along the trajectory x_{cl} of Σ satisfies

$$\begin{aligned}
\frac{d}{dt}V(t,x_{cl}) &\le \frac{1}{\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))}\left[\frac{d}{dt}V_{0[\kappa]}(\chi_{[\kappa]})+\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))\frac{d}{dt}\chi_{\langle\kappa+1\rangle}^TP_{\langle\kappa+1\rangle}\chi_{\langle\kappa+1\rangle}\right.\\
&\qquad\left.+\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))\sum_{i=1}^{m}\check{\phi}_{id}\frac{d}{dt}W_{\Delta i}(t,x_{\Delta i})\right]\\
&\le \frac{1}{\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))}\left\{\frac{d}{dt}V_{0[\kappa]}(\chi_{[\kappa]})+\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))\frac{d}{dt}\chi_{\langle\kappa+1\rangle}^TP_{\langle\kappa+1\rangle}\chi_{\langle\kappa+1\rangle}\right.\\
&\qquad\left.+\sum_{i=1}^{m}\begin{bmatrix}w_{id}\\ z_{id}\end{bmatrix}^T\begin{bmatrix}-\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))\check{\phi}_{id}I & 0\\ 0 & \psi_{id}\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))\check{\phi}_{id}I\end{bmatrix}\begin{bmatrix}w_{id}\\ z_{id}\end{bmatrix}\right\}-\sum_{i=1}^{m}\check{\phi}_{id}\beta_i(x_{\Delta i})
\end{aligned}$$

Making use of (24), (25) and

$$0\le\begin{bmatrix}w_{is}\\ z_{is}\end{bmatrix}^T\begin{bmatrix}-\Phi_{is} & 0\\ 0 & \psi_{is}\Phi_{is}\end{bmatrix}\begin{bmatrix}w_{is}\\ z_{is}\end{bmatrix},\quad \forall x\in\mathcal{R}^n$$

we arrive at

$$\begin{aligned}
\frac{d}{dt}V(t,x_{cl}) &\le \frac{q(x,w)}{\phi_d(V_{0[\kappa]}(\chi_{[\kappa]}))}+r^Tr-\tau^{-2}e^Te-\sum_{i=1}^{m}\check{\phi}_{id}\beta_i(x_{\Delta i}) \qquad (111)\\
q(x,w) &= \begin{bmatrix}\chi\\ w\end{bmatrix}^T\begin{bmatrix}S^{-T}A^TT^T\Xi+\Xi TAS^{-1} & \Xi TB\\ B^TT^T\Xi & 0\end{bmatrix}\begin{bmatrix}\chi\\ w\end{bmatrix}+\begin{bmatrix}w\\ z\end{bmatrix}^T\begin{bmatrix}-\Theta & 0\\ 0 & \Psi^2\Phi\end{bmatrix}\begin{bmatrix}w\\ z\end{bmatrix}
\end{aligned}$$

The function $q(x,w)$ satisfies $q(x,w)<0$ for all $x\in\mathcal{R}^n\setminus\{0\}$ if

$$\begin{bmatrix}S^{-T}A^TT^T\Xi+\Xi TAS^{-1} & \Xi TB\\ B^TT^T\Xi & 0\end{bmatrix}+\begin{bmatrix}0 & S^{-T}C^T\Psi\\ I & D^T\Psi\end{bmatrix}\begin{bmatrix}-\Theta & 0\\ 0 & \Phi\end{bmatrix}\begin{bmatrix}0 & I\\ \Psi CS^{-1} & \Psi D\end{bmatrix}<0 \qquad (112)$$

holds for all $x\in\mathcal{R}^n$. According to Schur complements formula, the inequality (22) is equivalent to a pair of (112) and $\Phi>0$. Thus, under the condition (22), the global uniform asymptotic stability of Σ follows from (111) with $r\equiv 0$. Finally, integrating (111) from $t=0$ to $t=T>0$, we obtain

$$V(t,x_{cl}(T))-V(t,x_{cl}(0))\le\int_0^T\left(r^Tr-\tau^{-2}e^Te\right)dt$$

This proves that Σ has $\mathcal{L}_2$-gain less than or equal to τ.

Proof of Lemma 3:

Proof : For $k=1,2,\dots,n$ the matrix F_{ak} is obtained as

$$\begin{aligned}
F_{ak} &= \begin{bmatrix}-\Theta_{2k-1} & D_k^T\Psi_{2k-1}\Phi_{2k-1}\\ \Phi_{2k-1}\Psi_{2k-1}D_k & -\Phi_{2k-1}-\Phi_{2k-1}F_{ck}\Phi_{2k-1}\end{bmatrix}\\
F_{c1} &= 0,\quad F_{cj}=\begin{bmatrix}\star_{j,j-1}\\ 0\end{bmatrix}^TM_{[j-1]}^{-1}\begin{bmatrix}\star_{j,j-1}\\ 0\end{bmatrix},\quad j=2,3,\dots,n
\end{aligned}$$

where $\star_{i,j}$ denotes any function depending only on $x_{[i]}$, and the functions s_1 through s_j and their partial derivatives.

(i) The condition (75) is obtained as

$$\phi_{2k-1}\ge\theta_{2k-1}>0,\quad\begin{bmatrix}0 & 0\\ 0 & \phi_{2k-1}\tau^{-2}D_{R,k}D_{R,k}^T\end{bmatrix}<\theta_{2k-1}(I+\phi_{2k-1}\hat{\Phi}_{2k-1}^{1/2}F_{ck}\hat{\Phi}_{2k-1}^{1/2}) \qquad (113)$$

There exists $\mathcal{C}^0$ functions $\phi_{2k-1}(x_{[k]})$ and $\theta_{2k-1}(x_{[k]})$ satisfying (113) if and only if there exists a $\mathcal{C}^0$ function $\nu_{2k-1}(x_{[k]})$ satisfying

$$\nu_{2k-1} > 0, \quad -\nu_{2k-1}\hat{\Phi}_{2k-1}^{1/2}F_{ck}\hat{\Phi}_{2k-1}^{1/2} < \begin{bmatrix} I & 0 \\ 0 & I - \tau^{-2}D_{R,k}D_{R,k}^T \end{bmatrix} \tag{114}$$

for all $x_{[k]} \in \mathcal{R}^k$. Note that $F_{ck} \leq 0$ follows from $M_{[k-1]} < 0$. Since D_k and F_{ck} are $\mathcal{C}^0$ functions defined on $\mathcal{R}^k$, the assumption (58) implies that such a $\mathcal{C}^0$ function $\nu_{2k-1}(x_{[k]})$ exists. Write θ_{2k-1} as

$$\theta_{2k-1} = \phi_{2k-1}(1-\epsilon)$$

Then, (113) is satisfied if and only if ϵ satisfies

$$0 \leq \epsilon < 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & \tau^{-2}D_{R,k}D_{R,k}^T \end{bmatrix} < (1-\epsilon)(I + \phi_{2k-1}\hat{\Phi}_{2k-1}^{1/2}F_{ck}\hat{\Phi}_{2k-1}^{1/2}) \tag{115}$$

For any $\mathcal{C}^0$ function $\phi_{2k-1}(x_{[k]})$ satisfying (74), there exists a $\mathcal{C}^0$ function $\epsilon(x_{[k]})$ satisfying (115) for all $x_{[k]} \in \mathcal{R}^k$ since we have $\nu_{2k-1}F_{ck} \leq \phi_{2k-1}F_{ck}$ and (114). Finally, if $k=1$, the assumption (46) and $F_{c1} = 0$ imply (114) for any constant $\nu_{2k-1} > 0$.

(ii) Let $\nu_{2k-1}(x_{[k]})$ be a $\mathcal{C}^0$ function satisfying (114). There exists a $\mathcal{C}^0$ function $\bar{\nu}_{2k-1}(s)$ such that

$$0 < \bar{\nu}_{2k-1}(s) \leq \left\{ \bar{\phi}_{2k-1}, \min_{x_{[k]} \in \{x_{[k]} \,:\, s = \chi_{[k]}^T P_{[k]}\chi_{[k]}\}} \nu_{2k-1}(x_{[k]}) \right\}$$

for all $s \in [0, \infty)$ since $\chi_{[k]} = S_{[k]}x_{[k]}$ is a diffeomorphism and $P_{[k]} > 0$. The rest of the proof is the same as (i).

(iii) Choose $\epsilon = 0$ in the proof of (i) and (ii).

(iv) The inequality $\underline{M} < 0$ implies $M_{[k-1]}(0) < 0$ with $\phi_d(0) = 1$. The assumptions also guarantee that $M_{[k]}(0) < 0$ with $\phi_d(0) = 1$ is achievable by $\phi_{2k-1}(0) = \theta_{2k-1}(0) = 1$. Proposition 3 implies that $F_{ak}(0) < 0$ can be also achieved for $\phi_{2k-1}(0) = \theta_{2k-1}(0) = 1$. Thus, $\nu_{2k-1}(0) = 1$ solves (114) at $x_{[k]} = 0$. Finally, $\nu_{2k-1}(0) = 1$ implies

$$1 = \bar{\nu}_{2k-1}(0) = \min_{x_{[k]} \in \{x_{[k]} \,:\, 0 = \chi_{[k]}^T P_{[k]}\chi_{[k]}\}} \nu_{2k-1}(x_{[k]})$$

Proof of Lemma 4:

(i) For $k = 1, 2, \ldots, n$, the matrix F_{bk} is obtained as

$$F_{bk} = \begin{bmatrix} -\Theta_{2k} & 0 \\ 0 & -\Phi_{2k} \end{bmatrix}$$

The inequality $F_{bk} < 0$ is satisfied if and only if $\phi_{2k} > 0$ and $\theta_{2k} > 0$ hold. The inequality (85) at $x_{[k]}$ is obtained as

$$U_{Rk}^T\Psi_{2k}^2\hat{\Phi}_{2k}U_{Rk}\phi_{2k}s_k^2 + 2\Xi_k a_{k,k+1}s_k + U_{Lk}\hat{\Phi}_{2k}^{-1}U_{Lk}^T\Xi_k^2\theta_{2k}^{-1} + \Xi_k^2\alpha < 0 \tag{116}$$

where the function $\alpha(x_{[k]})$ is defined so that the left hand sides of (116) and (85) are identical. The function α is independent of s_k, ϕ_{2k} and θ_{2k}. Note that α, $a_{k,k+1}$, U_{Rk}, U_{Lk}, Ξ_k and Ψ_{2k} are $\mathcal{C}^0$ functions of $x_{[k]}$. There exists a scalar s_k satisfying (116) at $x_{[k]}$ if and only if

$$\alpha U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} \phi_{2k} < a_{k,k+1}^2 - U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} U_{Lk} \hat{\Phi}_{2k}^{-1} U_{Lk}^T \phi_{2k} \theta_{2k}^{-1} \tag{117}$$

holds. There exist ϕ_{2k} and θ_{2k} satisfying (117) and $\phi_{2k} \geq \theta_{2k}$ at $x_{[k]}$ if and only if there exists a scalar ν_{2k} satisfying

$$\alpha U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} \nu_{2k} < a_{k,k+1}^2 - U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} U_{Lk} \hat{\Phi}_{2k}^{-1} U_{Lk}^T \quad 0 < \nu_{2k} \tag{118}$$

Since we have (59), there exists a $\mathcal{C}^0$ function $\nu_{2k}(x_{[k]})$ satisfying (118) for all $x_{[k]} \in \mathcal{R}^k$. Write θ_{2k} as

$$\theta_{2k} = \frac{\phi_{2k}}{1+\epsilon}$$

Then, (117) is satisfied for all $x_{[k]} \in \mathcal{R}^k$ if and only if ϵ satisfies

$$\alpha U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} \phi_{2k} < a_{k,k+1}^2 - U_{Rk}^T \Psi_{2k}^2 \hat{\Phi}_{2k} U_{Rk} U_{Lk} \hat{\Phi}_{2k}^{-1} U_{Lk}^T (1+\epsilon) \tag{119}$$

for all $x_{[k]} \in \mathcal{R}^k$. For any $\mathcal{C}^0$ function $\phi_{2k}(x_{[k]})$ satisfying $0 < \phi_{2k}(x_{[k]}) \leq \nu_{2k}(x_{[k]})$, there exist a $\mathcal{C}^0$ function $\epsilon(x_{[k]})$ satisfying (119) and $\epsilon(x_{[k]}) \geq 0$ for all $x_{[k]} \in \mathcal{R}^k$ since we have (118) and (59). Note that $\epsilon \equiv 0$ is a solution. Furthermore, the function $\nu_{2k}(x_{[k]})$ can be replaced by another $\mathcal{C}^0$ function $\bar{\nu}_{2k}(\chi_{[k]}^T P_{[k]} \chi_{[k]})$ obtained from

$$0 < \bar{\nu}_{2k}(s) \leq \min \left\{ \bar{\phi}_{2k}, \min_{x_{[k]} \in \{x_{[k]} \,:\, s = \chi_{[k]}^T P_{[k]} \chi_{[k]}\}} \nu_{2k}(x_{[k]}) \right\}$$

Finally, the inequality (117) implies that the inequality (116) has a smooth solution $s_k(x_{[k]})$ since U_{Rk}, U_{Lk}, $a_{k,k+1}$, α, Ξ_k and Ψ_{2k} are $\mathcal{C}^0$ functions of $x_{[k]}$.
(ii) If $p_{2k} + q_{2k} = 0$, the inequalities (87) and (88) are obtained as

$$2\Xi_k a_{k,k+1} s_k + \Xi_k^2 \alpha < 0 \ , \tag{120}$$

with appropriate $\mathcal{C}^0$ functions α which are independent of s_k. This affine inequality admits a smooth solution $s_k(x_{[k]})$.
(iii) Regarding the final statement of the lemma, recall that $\underline{M} < 0$ implies $M_{[k-1]}(0) < 0$ if $\phi_d(0) = 1$ in (60). The assumptions guarantee that $M_{[k]}(0) < 0$ is achievable with $\phi_{2k}(0) = \theta_{2k}(0) = 1$ and $s_k(0) = l_{k,k}$. Proposition 3 implies that (116) is solved with $\phi_{2k}(0) = \theta_{2k}(0) = 1$ and $s_k(0) = l_{k,k}$. Thus, $\nu_{2k}(0) = 1$ satisfies (118) at $x_{[k]} = 0$. It is also verified that

$$1 = \bar{\nu}_{2k}(0) = \min_{x_{[k]} \in \{x_{[k]} \,:\, 0 = \chi_{[k]}^T P_{[k]} \chi_{[k]}\}} \nu_{2k}(x_{[k]})$$

Proof of Lemma 5:
The claim is proved using the result (a) of Theorem 2 for linear Σ_0 as follows. Suppose that constant scalars

$$\hat{\phi}_{js} > 0, \quad \hat{\phi}_{jd} > 0, \quad 1 \leq j \leq 2n$$
$$P_k > 0, \quad 1 \leq k \leq n$$
$$d_{i,j}, \quad 2 \leq i \leq n, \ 1 \leq j \leq i-1$$

are given arbitrarily. Choose $\kappa = 0$ and $\phi_d = \phi_j$. Let $A = \underline{A}$, $B = \underline{B}$, $G = \underline{G}$, $C = \underline{C}$, $D = \underline{D}$, $H = \underline{H}$ and $\Psi = \bar{\Psi}(0)$. Since these matrices are independent of x, $M < 0$ can be solved with constant parameters $\{\phi_{2k-1} = \theta_{2k-1} > 0, \quad \phi_{2k} = \theta_{2k} > 0, \quad s_k\}$ following the procedure described in the proof of Theorem 2. This fact implies that $\underline{M} < 0$ is solved with $X = S^T P S$, $\underline{K} = [d_{n,1}, \ldots, d_{n,n-1}, s_n]S$ and

$$\underline{\phi}_{js} = \phi_j \hat{\phi}_{js} > 0,\ \underline{\phi}_{jd} = \phi_j \hat{\phi}_{jd} > 0,, \quad 1 \le j \le 2n$$

Hence, $u = \underline{K}x$ is a scaled $\mathcal{H}^\infty$ linear controller.

Proof of Theorem 3:

Let $\kappa = n$. According to Lemma 4, in the n-th step of the recursive procedure, we can choose a uniformly bounded $\mathcal{C}^0$ function $\phi_{2n}(\chi^T P \chi)$ such that

$$\begin{gathered} 0 < \beta_i \phi_{2n}(\chi^T P \chi) \le \phi_i(x_{[k]}), \quad x_{[k]} \in \mathcal{R}^k \\ \beta_i \phi_{2n}(0) = \phi_i(0) \end{gathered}$$

hold for some $\beta_i > 0$, $i = 1, 2, \ldots, 2n-1$. Lemma 3 and 4 also allow

$$\theta_i(0) = \phi_i(0), \quad i = 1, 2, \ldots, 2n-1$$

Then, the recursive procedure leads us to a controller (51) which satisfies

$$M = \begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB & \hat{S}^T \hat{C}^T \Psi\Phi \\ B^T T^T P & -\Theta & D^T \Psi \Phi \\ \Phi\Psi\hat{C}\hat{S} & \Phi\Psi D & -\Phi \end{bmatrix} < 0$$

for all $x \in \mathcal{R}^n$. Define a matrix-valued $\mathcal{C}^0$ function

$$\tilde{\Phi}(x) = \operatorname*{block-diag}_{j=1}^{2n} \beta_j \phi_{2n} \hat{\Phi}_j$$

where $\beta_{2n} = 1$. This matrix satisfies

$$0 < \tilde{\Phi}(x) \le \Phi(x), \quad \forall x \in \mathcal{R}^n \tag{121}$$

$$\tilde{\Phi}(0) = \Phi(0) = \Theta(0) \tag{122}$$

The matrix $\tilde{\Phi}$ also belongs to $\mathbf{\Phi}$ in (18). Since

$$M = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Phi\tilde{\Phi}^{-1} \end{bmatrix} \left(\begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB & \hat{S}^T \hat{C}^T \tilde{\Phi} \\ B^T T^T P & -\Theta & D^T \tilde{\Phi} \\ \tilde{\Phi}\hat{C}\hat{S} & \tilde{\Phi} D & -\tilde{\Phi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{\Phi} - \tilde{\Phi}\Phi^{-1}\tilde{\Phi} \end{bmatrix} \right) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tilde{\Phi}^{-1}\Phi \end{bmatrix}$$

holds, the inequality (121) implies

$$\begin{bmatrix} \hat{S}^T \hat{A}^T T^T P + PT\hat{A}\hat{S} & PTB\bar{W} & \hat{S}^T \hat{C}^T \Psi\tilde{\Phi} \\ \bar{W} B^T T^T P & -\tilde{\Phi} & \bar{W} D^T \Psi \tilde{\Phi} \\ \tilde{\Phi}\Psi\hat{C}\hat{S} & \tilde{\Phi}\Psi D \bar{W} & -\tilde{\Phi} \end{bmatrix} < 0 \tag{123}$$

$$\bar{W} = \Theta^{-1/2}\tilde{\Phi}^{1/2} = \operatorname*{block-diag}_{j=1}^{2n} \sqrt{\theta_j^{-1} \beta_j \phi_{2n} I} > 0$$

From (122), the matrix $\bar{W}(x)$ satisfies $\bar{W}(0) = I$. If we take ϕ_{2n} so that $\theta_j^{-1}\phi_{2n}$ is uniformly bounded in x for all j, $\bar{W}(x)$ becomes a uniformly bounded function. It is verified that (123) holds for all $x \in \mathcal{R}^n$ even if $\bar{W}(x)$ is replaced by any diagonal matrix $W(x)$ satisfying $0 < W(x) \leq \bar{W}(x)$. Define

$$\hat{W}(x) = \operatorname*{block\text{-}diag}_{j=1}^{2n} \sqrt{\beta_j \phi_{2n}} I$$

Then, Theorem 1 proves that $\hat{\Sigma}_P$ is globally globally uniformly asymptotically stabilized and achieves $\mathcal{L}_2$-gain less than or equal to τ for all admissible dynamic uncertainties. Furthermore, the control law (51) agrees locally with a scaled $\mathcal{H}^\infty$ linear controller as shown in Theorem 2 if we choose $\phi_i(0) = 1$ and $\beta_i = 1$ for $i = 1, 2, \ldots, 2n$.

Proof of Theorem 4:
Let $V_{[k]}$ be defined as (106) and (107). According to Lemma 2, the function $V_{[k]}$ is radially unbounded and positive definite with respect to $(\chi_{[k]}, x_{\Delta[k]})$. Since $\chi_{[k]} = S_{[k]} x_{[k]}$ is a diffeomorphism, there exist class $\mathcal{K}_\infty$ functions $\underline{\eta}(\cdot)$ and $\bar{\eta}(\cdot)$ such that

$$\underline{\eta}(\|x_{cl[k]}\|) \leq V_{[k]}(t, \chi_{[k]}, x_{\Delta[k]}) \leq \bar{\eta}(\|x_{cl[k]}\|), \quad \forall x_{cl[k]} = \begin{bmatrix} x_{[k]} \\ x_{\Delta[k]} \end{bmatrix} \in \mathcal{R}^{k+n_{\Delta[k]}},\ t \in [0, \infty) \quad (124)$$

Note also that $\hat{V}_{[k]}(\chi_{[k]}) = V_{[k]}(t, \chi_{[k]}, 0)$. From $\phi_k > 0$ and positive definiteness of $W_{\Delta k}$, it follows that

$$0 \leq \hat{V}_{[k]}(\chi_{[k]}) \leq V_{[k]}(t, \chi_{[k]}, x_{\Delta[k]}), \quad \forall \chi_{[k]} \in \mathcal{R}^k,\ x_{\Delta[k]} \in \mathcal{R}^{n_{\Delta[k]}},\ t \in [0, \infty)$$

Since ϕ_k is non-increasing, the above inequality implies

$$\phi_{k+1}(\hat{V}_{[k]}(\chi_{[k]})) \geq \phi_{k+1}(V_{[k]}(t, \chi_{[k]}, x_{\Delta[k]})) > 0, \quad \forall \chi_{[k]} \in \mathcal{R}^k,\ x_{\Delta[k]} \in \mathcal{R}^{n_{\Delta[k]}},\ t \in [0, \infty) \quad (125)$$

First, consider the case of $[r_1^T, \cdots, r_{2(n-1)}^T] \equiv 0$. Let k be an integer in $[1, n]$. Suppose that the time-derivative of $V_{[k-1]}$ along the trajectory $(x_{[k-1]}(t), x_{\Delta[k-1]}(t))$ of the closed-loop system (Σ_P, Σ_K) satisfies

$$\frac{d}{dt} V_{[k-1]}(t, \chi_{[k-1]}, x_{\Delta[k-1]}) \leq 0, \quad \forall \chi_{[k-1]} \in \mathcal{R}^{k-1},\ x_{\Delta[k-1]} \in \mathcal{R}^{n_{\Delta[k-1]}},\ t \in [0, \infty) \quad (126)$$

if $k \geq 2$. Using (125) and (126), the time-derivative of $V_{[k]}$ along the trajectory $(x_{[k]}(t), x_{\Delta[k]}(t))$ of (Σ_P, Σ_K) is obtained as

$$\begin{aligned} \frac{d}{dt} V_{[k]} &\leq \frac{1}{\phi_k(\hat{V}_{[k-1]}(\chi_{[k-1]}))} \frac{d}{dt} V_{[k-1]} + \frac{d}{dt} \left(\chi_k P_k \chi_k + \check{\phi}_{2k-1,d} W_{\Delta 2k-1} + \check{\phi}_{2k,d} W_{\Delta 2k} \right) \\ &\leq \frac{1}{\tilde{\phi}_k(\chi_{[k-1]})} N_{[k]}(t, \chi_{[k]}, x_{\Delta[k]}) \end{aligned}$$

Here, $N_{[l]}$ and $\tilde{\phi}_l$ for $1 \leq l \leq n$ are defined by

$$\begin{aligned} N_{[l]}(t, \chi_{[l]}, x_{\Delta[l]}) &= \sum_{i=1}^{l} \tilde{\phi}_i \frac{d}{dt} \left(\chi_i P_i \chi_i + \check{\phi}_{2i-1,d} W_{\Delta 2i-1} + \check{\phi}_{2i,d} W_{\Delta 2i} \right) \\ \tilde{\phi}_l(\chi_{[l-1]}) &= \prod_{j=1}^{l} \phi_j(\hat{V}_{[j-1]}(\chi_{[j-1]})) \end{aligned}$$

Due to (100), (101) and the definition of admissible uncertainties, the function $N_{[k]}$ satisfies

$$\begin{aligned} N_{[k]} &\le \sum_{i=1}^{k}\left\{\tilde{\phi}_i\frac{d}{dt}\chi_iP_i\chi_i+\left(\sum_{j=2i-1}^{2i}\begin{bmatrix} w_{jd}\\ z_{jd}\end{bmatrix}^T\begin{bmatrix} -\tilde{\phi}_i\check{\phi}_{jd}I & 0\\ 0 & \psi_{jd}\tilde{\phi}_i\check{\phi}_{jd}I\end{bmatrix}\begin{bmatrix} w_{jd}\\ z_{jd}\end{bmatrix}-\tilde{\phi}_i\check{\phi}_{jd}\beta_j\right)\right\} \\ &\le \sum_{i=1}^{k}\left\{\tilde{\phi}_i\left(\frac{d}{dt}\chi_iP_i\chi_i-\check{\phi}_{2i-1,d}\beta_{2i-1}-\check{\phi}_{2i,d}\beta_{2i}\right)\right\}+\begin{bmatrix} \bar{w}_{[k]}\\ \bar{z}_{[k]}\end{bmatrix}^T\begin{bmatrix} -\Theta_{[k]} & 0\\ 0 & \Psi_{[k]}^2\tilde{\Phi}_{[k]}\end{bmatrix}\begin{bmatrix} \bar{w}_{[k]}\\ \bar{z}_{[k]}\end{bmatrix} \\ \bar{w}_{[k]} &= \begin{bmatrix} \bar{w}_1\\ \bar{w}_2\\ \vdots\\ \bar{w}_{2k}\end{bmatrix}, \quad \bar{z}_{[k]}=\begin{bmatrix} \bar{z}_1\\ \bar{z}_2\\ \vdots\\ \bar{z}_{2k}\end{bmatrix}\end{aligned}$$

unless $k=n$. In the case of $k=n$, we obtain

$$\begin{aligned} N_{[n]} \le \sum_{i=1}^{n}\left\{\tilde{\phi}_i\left(\frac{d}{dt}\chi_iP_i\chi_i-\check{\phi}_{2i-1,d}\beta_{2i-1}-\check{\phi}_{2i,d}\beta_{2i}\right)\right\}+\begin{bmatrix} \bar{w}\\ \bar{z}\end{bmatrix}^T\begin{bmatrix} -\Theta & 0\\ 0 & \Psi^2\tilde{\Phi}\end{bmatrix}\begin{bmatrix} \bar{w}\\ \bar{z}\end{bmatrix} \\ +\tilde{\phi}_n\sum_{j=2n-1}^{2n}\left(r_j^Tr_j-\tau^{-2}e_j^Te_j\right)\end{aligned}$$

Owing to the triangular structure of A, B, $\hat{S}$ and T, we obtain

$$\begin{aligned} N_{[k]} &\le \begin{bmatrix} \chi_{[k]}\\ \bar{w}_{[k]}\end{bmatrix}^T\begin{bmatrix} \hat{S}_{[k]}^TA_{[k]}^TT_{[k]}^T\Xi_{[k]}+\Xi_{[k]}T_{[k]}A_{[k]}\hat{S}_{[k]} & \Xi_{[k]}T_{[k]}B_{[k]}\\ B_{[k]}^TT_{[k]}^T\Xi_{[k]} & 0\end{bmatrix}\begin{bmatrix} \chi_{[k]}\\ \bar{w}_{[k]}\end{bmatrix} \\ &\quad +\begin{bmatrix} \bar{w}_{[k]}\\ \bar{z}_{[k]}\end{bmatrix}^T\begin{bmatrix} -\Theta_{[k]} & 0\\ 0 & \Psi_{[k]}\tilde{\Phi}_{[k]}\end{bmatrix}\begin{bmatrix} \bar{w}_{[k]}\\ \bar{z}_{[k]}\end{bmatrix}-\sum_{i=1}^{k}\tilde{\phi}_i\left(\check{\phi}_{2i-1,d}\beta_{2i-1}+\check{\phi}_{2i,d}\beta_{2i}\right) \\ N_{[n]} &\le \begin{bmatrix} \chi\\ \bar{w}\end{bmatrix}^T\begin{bmatrix} \hat{S}^TA^TT^T\Xi+\Xi TA\hat{S} & \Xi TB\\ B^TT^T\Xi & 0\end{bmatrix}\begin{bmatrix} \chi\\ \bar{w}\end{bmatrix} \\ &\quad +\begin{bmatrix} \bar{w}\\ \bar{z}\end{bmatrix}^T\begin{bmatrix} -\Theta & 0\\ 0 & \Psi\tilde{\Phi}\end{bmatrix}\begin{bmatrix} \bar{w}\\ \bar{z}\end{bmatrix}-\sum_{i=1}^{n}\tilde{\phi}_i\left(\check{\phi}_{2i-1,d}\beta_{2i-1}+\check{\phi}_{2i,d}\beta_{2i}\right)+\tilde{\phi}_n\sum_{j=2n-1}^{2n}\left(r_j^Tr_j-\tau^{-2}e_j^Te_j\right)\end{aligned}$$

Recall that $M<0$ in (98) implies $M_{[l]}<0$ for any integer $l\in[1,n]$. Based on application of Schur complements formula to $M_{[k]}<0$, the inequality $M_{[k]}<0$ implies

$$N_{[k]}(t,\chi_{[k]},x_{\Delta[k]})\le-\rho_k(\chi_{[k]})-\sum_{i=1}^{k}\tilde{\phi}_i\left(\check{\phi}_{2i-1,d}\beta_{2i-1}+\check{\phi}_{2i,d}\beta_{2i}\right), \quad k<n$$

for a positive definite function $\rho_k(\cdot)$. Thus, the there exists a positive definite function $\bar{\rho}_k(\cdot,\cdot)$ for which

$$\frac{d}{dt}V_{[k]}(t,\chi_{[k]},x_{\Delta[k]})\le-\bar{\rho}_k(\chi_{[k]},x_{\Delta[k]}), \quad \forall\chi_{[k]}\in\mathcal{R}^k,\ x_{\Delta[k]}\in\mathcal{R}^{n_{\Delta[k]}},\ t\in[0,\infty)$$

holds under the assumption of (126). By induction from $k=1$ to $k=n$, we obtain

$$\frac{d}{dt}V_{[n]}(t,\chi,x)\le-\bar{\rho}_n(\chi,x_\Delta)+\sum_{j=2n-1}^{2n}\left(r_j^Tr_j-\tau^{-2}e_j^Te_j\right), \quad \forall\chi\in\mathcal{R}^n,\ x_\Delta\in\mathcal{R}^{n_\Delta},\ t\in[0,\infty) \tag{127}$$

for a positive definite function $\bar{\rho}_n(\cdot,\cdot)$. Hence, the stability follows from $r_{2n-1}\equiv 0$ and $r_{2n}\equiv 0$. Integration of the above inequality also proves $\mathcal{L}_2$-gain of the level τ between $[r_{2n-1}^T,r_{2n}^T]^T$ and

$[e_{2n-1}^T, e_{2n}^T]^T$. For the proof of the weighted $\mathcal{L}_2$-gain in the case of $[r_1^T, \cdots, r_{2(n-1)}^T] \neq 0$, the inequality (126) and (127) are replaced by

$$\frac{d}{dt}V_{[k-1]} - \sum_{i=1}^{k-1} \frac{1}{\phi_{i+1}(V_{[i]})\phi_{i+2}\cdots\phi_k(V_{[k-1]})} \sum_{j=2i-1}^{2i} (r_j^T r_j - \tau^{-2} e_j^T e_j) - \sum_{j=2k-1}^{2k} (r_j^T r_j - \tau^{-2} e_j^T e_j) \leq 0$$

$$\frac{d}{dt}V_{[n]} \leq -\bar{\rho}_n + \sum_{i=1}^{n-1} \frac{1}{\phi_{i+1}(V_{[i]})\phi_{i+2}\cdots\phi_n(V_{[n-1]})} \sum_{j=2i-1}^{2i} (r_j^T r_j - \tau^{-2} e_j^T e_j) + \sum_{j=2n-1}^{2n} \left(r_j^T r_j - \tau^{-2} e_j^T e_j\right)$$

respectively. Since $V_{[i]}$ satisfies (124) and ϕ_i is bounded from above, there exist a $\mathcal{C}^0$ function matrix $W(x, x_\Delta) > 0$ for which $\mathcal{L}_2$ disturbance attenuation of the level τ between r and $W(x, x_\Delta)e$ is achieved.